

6.1 Chains and Zorn's Lemma

Section 3.4 proved that injections both ways give a bijection, and left open whether, for any two sets, at least one injection exists. Section 4.3 ended with a question about partially ordered sets: if every chain has an upper bound, must there be a maximal element? One principle answers both questions. This section states it, Zorn's lemma, which answers the second question; explains why this volume does not prove it; and turns it into a method of proof, the maximality template, with two first uses. Section 6.2 uses the template to answer the first question.

6.1.1 Matching two sets, one pair at a time

Let A and B be sets, and suppose we want an injection from one of them into the other, in either direction. A natural plan is to **match** elements in pairs. Pick $a_0 \in A$ and $b_0 \in B$ and pair them. Then pick $a_1 \in A \setminus \{a_0\}$ and $b_1 \in B \setminus \{b_0\}$ and pair those. Continue until one side is used up. If A runs out first, every element of A has a partner, different elements have different partners, and the pairs form the graph of an injection $A \rightarrow B$. If B runs out first, the same pairs read backwards form an injection $B \rightarrow A$.

For finite sets the plan works, because each step uses one more element of each side, and a finite set is used up after as many steps as it has elements (Section 2.5). Take $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Pair 1 with x , then 2 with y . Now B is used up, and reading the pairs backwards gives $x \mapsto 1$, $y \mapsto 2$, an injection $B \rightarrow A$.

For $A = B = \mathbb{R}$ the plan breaks down. Suppose we have made one step for each natural number. The elements of A used so far are $a_0, a_1, a_2, \dots$, the image of the function $n \mapsto a_n$ on $\mathbb{N}$, so they form a countable set (Proposition 3.1.4). But $\mathbb{R}$ is uncountable (Theorem 3.3.4), so A is not used up; neither is B . The instruction “continue until one side is used up” has no last step, and the words “and so on” stand in for a step that is never stated: (N13) of Logic and Proof, Example (What is not allowed), “an ‘and so on’ needs its step”. (The infinitely many unnamed choices of a_n and b_n are a second problem, the one Section 3.2 met.)

Still, the attempt gets two things right, and they are the thesis of this section:

Every unfinished matching can be enlarged by one more pair, and a growing family of matchings, put together, is again a matching.

The first half is plain: if neither side is used up, an unused a and an unused b make one more pair. The rest of the section makes “growing family” exact and supplies a principle that turns these two facts into a finished matching. We return to the matching in Section 6.2.

6.1.2 Chains

Look at the matchings the plan produces, each a set of pairs. Of any two of them, one was built from the other by adding pairs, so one is a subset of the other: under $\subseteq$, any two are comparable. That is what “growing family” means, and it makes sense in any partially ordered set.

Definition 6.1.1. Chain

Let $(X, \preceq)$ be a partially ordered set. A subset $C \subseteq X$ is a **chain** if **any two** elements of C are comparable: for all $x, y \in C$, $x \preceq y$ or $y \preceq x$.

So a chain is a subset on which $\preceq$ is total (Definition 4.3.1); in a totally ordered set, every subset is a chain.

Examples.

- In $(\mathbb{N}, |)$, the set $\{1, 2, 4, 8, \dots\} = \{2^n : n \in \mathbb{N}\}$ is a chain: for $m \leq n$, $2^n = 2^{n-m} \cdot 2^m$, so $2^m \mid 2^n$. The set $\{2, 3\}$ is not, since $2 \nmid 3$ and $3 \nmid 2$.
- In $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$, the set $\{\emptyset, \{2\}, \{2, 3\}\}$ is a chain.
- Degenerate cases: $\emptyset$ is a chain, vacuously, and so is every one-element set $\{x\}$, by (PO1).

Quick check. Decide: (a) Is $\{\{1\}, \{1, 2\}, \{3\}\}$ a chain in $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$? (b) Is $\{0, 1, 5, 25\}$ a chain in $(\mathbb{N}, |)$? (c) Is $\{3, 6, 0\}$? (d) In $(\mathbb{R}, \leq)$, which subsets are chains?

Solution.

- (a) No: $\{1, 2\} \not\subseteq \{3\}$ and $\{3\} \not\subseteq \{1, 2\}$.
- (b) Yes. 1 divides every natural number, $25 = 5 \cdot 5$, and every natural number a divides $0 = 0 \cdot a$; so any two of the four are comparable.
- (c) Yes: $6 = 2 \cdot 3$, and $3 \mid 0$, $6 \mid 0$.
- (d) All of them, since $\leq$ is total on $\mathbb{R}$.

Why do chains matter for the matching? Take two pairs from the union of a growing family of matchings. Each pair comes from some matching of the family, and the two matchings are comparable, so both pairs lie in the larger of the two. Any condition that speaks about two pairs at a time can then be checked inside one matching. The same works for any finite number of elements, and that is the content of the next lemma.

The lemma speaks about a set of sets ordered by $\subseteq$. That is always a partially ordered set: each member of a set of sets $\mathcal{S}$ is a subset of $\bigcup \mathcal{S}$, and $\subseteq$ is a partial order on $\mathcal{P}(\bigcup \mathcal{S})$ (Example (Inclusion is a partial order, rarely total)), which stays one when restricted to the subset $\mathcal{S}$, since (PO1)–(PO3) are statements about all elements.

Lemma 6.1.1. Finitely Many Members of a Chain

- (a) Let C be a chain in a partially ordered set $(X, \preceq)$, let $n \geq 1$ and $x_1, \dots, x_n \in C$. Then some x_k satisfies $x_j \preceq x_k$ for $j = 1, \dots, n$.
- (b) Let $\mathcal{S}$ be a set of sets, partially ordered by $\subseteq$, and $\mathcal{C} \subseteq \mathcal{S}$ a chain. Then $C \subseteq \bigcup \mathcal{C}$ for every $C \in \mathcal{C}$; and if $n \geq 1$ and $z_1, \dots, z_n \in \bigcup \mathcal{C}$, then some single $C \in \mathcal{C}$ contains all of $z_1, \dots, z_n$.

Proof. (a) By induction on n (Theorem 2.2.1 with $n_0 = 1$). For $n = 1$, take $k = 1$: $x_1 \preceq x_1$ by (PO1). For the step, let $x_1, \dots, x_{n+1} \in C$, and by the induction hypothesis let x_k with $k \leq n$ be above $x_1, \dots, x_n$. Since C is a chain, $x_{n+1} \preceq x_k$ or $x_k \preceq x_{n+1}$. In the first case x_k is above all $n + 1$ elements. In the second, x_{n+1} is: $x_j \preceq x_k \preceq x_{n+1}$ gives $x_j \preceq x_{n+1}$ by (PO3) for $j \leq n$, and $x_{n+1} \preceq x_{n+1}$ by (PO1).

(b) The first claim is the definition of the union. For the second, induct on n (Theorem 2.2.1 with $n_0 = 1$). For $n = 1$: $z_1 \in \bigcup \mathcal{C}$ means that $z_1 \in C$ for some $C \in \mathcal{C}$, and one use of $\exists E$

(Logic and Proof, Definition 5.6.1) names such a C . For the step, by the induction hypothesis some $C' \in \mathcal{C}$ contains $z_1, \dots, z_n$, and one use of $\exists E$ gives $C'' \in \mathcal{C}$ with $z_{n+1} \in C''$. Since $\mathcal{C}$ is a chain, $C' \subseteq C''$ or $C'' \subseteq C'$, and the larger of the two contains all of $z_1, \dots, z_{n+1}$. ■

Each step of the induction in (b) makes one use of $\exists E$, so no choice is involved. Writing “use $\exists E$ n times” instead would be the “and so on” that (N13) flags; the induction is what supplies its step.

Quick check.

- (a) Is there a chain of subsets of $\mathbb{N}$, none of whose members contains both 0 and 1, whose union contains both? (b) Is there such a family of subsets that is not a chain?

Solution.

- (a) No. If 0 and 1 lie in the union of a chain, then by Lemma 6.1.1 (b) with $n = 2$ some member contains both.
 (b) Yes: $\{\{0\}, \{1\}\}$ has union $\{0, 1\}$, and neither member contains both. This is exactly what “chain” rules out.

6.1.3 Zorn's lemma

Return to the question of Section 4.3: in a partially ordered set in which every chain has an upper bound, is there a maximal element? A naive attempt: start at any p_0 . If p_0 is not maximal, there is $p_1 \succ p_0$; if p_1 is not maximal, there is $p_2 \succ p_1$; and so on. The p_n form a chain, which has an upper bound u ; if u is not maximal, continue above u , and so on. Each single step is legitimate. But the chain $p_0 \prec p_1 \prec \dots$ may have an upper bound that is again not maximal, the next chain may too, and “continue past every such bound” is neither a finite process nor one indexed by $\mathbb{N}$. It is (N13) once more: the “and so on” has no step. Instead of a proof, there is a principle, stated as its own result.

Theorem 6.1.2. Zorn's Lemma

Let $(P, \preceq)$ be a **non-empty** partially ordered set in which **every non-empty chain** has an upper bound in P . Then P has a maximal element.

We do not prove Theorem 6.1.2. It cannot be proved from the rules of logic and the working rules for sets of Chapter 1 alone: it is equivalent to the axiom of choice (Axiom 3.2.1), which, as Section 3.2 records, can be neither proved nor refuted from the other axioms. One half of the equivalence, that Zorn's lemma implies the axiom of choice, is the first application below (Theorem 6.1.3). The other half, which builds a maximal element from a choice function, is proved in *Axiomatic Set Theory*, Chapter 4, together with the other equivalents. As with the axiom, we use the lemma openly and cite it at each use. Zorn (1935) put the principle in this form, which suits algebra, where the objects to be built are sets ordered by inclusion.

Why each hypothesis is needed. Each of three small examples breaks one hypothesis and

has no maximal element.

- $(\mathbb{N}, \leq)$ is non-empty and total, and has no maximal element, since $n < n + 1$. The chain hypothesis fails: the chain $\mathbb{N}$ has no upper bound in $\mathbb{N}$.
- The empty partially ordered set satisfies the chain hypothesis vacuously, since it has no non-empty chain, and it has no maximal element. “Non-empty” is needed.
- The set of finite subsets of $\mathbb{N}$, ordered by $\subseteq$, is non-empty, as $\emptyset$ belongs to it, and has no maximal element. For finite F , $\mathbb{N}$ is not a subset of F , since a subset of a finite set is finite (Proposition 2.5.6) and $\mathbb{N}$ is infinite (Example (Hilbert’s Hotel)); for $m \in \mathbb{N} \setminus F$, the set $F \cup \{m\}$ is finite (Proposition 2.5.5) and strictly larger. By the contrapositive of Zorn’s lemma, some non-empty chain has no upper bound in this set, and here one is visible: $\{\{0, 1, \dots, n-1\} : n \in \mathbb{N}\}$, that is, $\emptyset, \{0\}, \{0, 1\}, \dots$. An upper bound would be a finite set containing every natural number, hence containing $\mathbb{N}$, which is impossible.

Warning. Maximal, not greatest, and not unique. Zorn’s lemma produces a maximal element, by no formula, and says nothing about how many there are. By Proposition 4.3.2 a greatest element would be maximal, but the converse fails (Example (Maximal elements that are not greatest)). In the applications below there are usually many maximal objects and no greatest one; a proof that treats “the” maximal element as unique, or as above everything, has gone wrong.

Quick check. Let $P = \{\{1\}, \{2\}, \{1, 3\}\}$, ordered by $\subseteq$. List the chains with at least two elements; check that every non-empty chain has an upper bound in P ; name the maximal elements.

Solution. The only comparable pair of different elements is $\{1\} \subseteq \{1, 3\}$, so the only chain with at least two elements is $\{\{1\}, \{1, 3\}\}$, with upper bound $\{1, 3\}$. A one-element chain $\{p\}$ has the upper bound p . The maximal elements are $\{2\}$ and $\{1, 3\}$: two of them, as the warning says, and no greatest element.

Remark (Every chain, or every non-empty chain). Some books ask for an upper bound of **every** chain, the empty one included. An upper bound of $\emptyset$ is any element of P (Definition 4.3.2), so that form contains “ $P \neq \emptyset$ ” as a special case. The two forms say the same thing.

6.1.4 The maximality template

Zorn’s lemma is almost always used in one pattern. Its first use proves the half of the equivalence with the axiom of choice that this volume can prove.

Theorem 6.1.3. Zorn’s Lemma Gives Choice

Assume Theorem 6.1.2. Let I be a set and $(A_i)_{i \in I}$ a family of **non-empty** sets. Then there is a function $c: I \rightarrow \bigcup_{i \in I} A_i$ with $c(i) \in A_i$ for every $i \in I$.

This is the conclusion of Axiom 3.2.1, so the axiom follows from Zorn's lemma.

Idea (Scratch work). A choice function can be built one index at a time, like the matching of the first subsection. Call a set of pairs (i, x) with $x \in A_i$, at most one pair for each i , a **partial choice function**: it is a graph (Definition 1.5.1) defined on part of I . Order these by $\subseteq$: the larger of two comparable ones agrees with the smaller and is defined at more indices. Write $U := \bigcup_{i \in I} A_i$.

Givens	Goal
$A_i \neq \emptyset$ for each $i \in I$	$c: I \rightarrow U$ with $c(i) \in A_i$ for all i
P : the $G \subseteq I \times U$ with (C1)	Zorn's hypotheses: $P \neq \emptyset$, and each
$(i, x) \in G \Rightarrow x \in A_i$ and (C2)	non-empty chain has an upper bound in P
$(i, x), (i, y) \in G \Rightarrow x = y$; order $\subseteq$	
$\emptyset \in P$; for a chain $\mathcal{C}$, try $\bigcup \mathcal{C}$: (C1) and (C2)	a maximal $M \in P$ is defined at every $i \in I$
mention at most two pairs (Lemma 6.1.1)	
M maximal; suppose i_0 is missed	$M \cup \{(i_0, x_0)\} \in P$, strictly larger: a contradiction

The last row is a proof by contradiction (Example (Method: choosing direct, contrapositive or contradiction)): a missed index gives a larger partial object. Choosing $x_0 \in A_{i_0}$ is **one** use of $\exists E$, which Section 3.2 showed is always allowed; Zorn's lemma does the infinitely many.

Proof. Let U and P be as in the table. Then $P \subseteq \mathcal{P}(I \times U)$ is a set, partially ordered by $\subseteq$.

P is non-empty: $\emptyset$ satisfies (C1) and (C2) vacuously.

Chains have upper bounds: let $\mathcal{C} \subseteq P$ be a non-empty chain and $H := \bigcup \mathcal{C} \subseteq I \times U$. If $(i, x) \in H$, it lies in some $G \in \mathcal{C}$, so $x \in A_i$ by (C1) for G . If $(i, x), (i, y) \in H$, some $G \in \mathcal{C}$ contains both (Lemma 6.1.1 (b) with $n = 2$), so $x = y$ by (C2) for G . So $H \in P$, and $G \subseteq H$ for every $G \in \mathcal{C}$: H is an upper bound of $\mathcal{C}$ in P .

Zorn: by Theorem 6.1.2, P has a maximal element M .

M is defined everywhere: suppose some $i_0 \in I$ has no pair $(i_0, x) \in M$. Since $A_{i_0} \neq \emptyset$, take $x_0 \in A_{i_0}$, and let $M' := M \cup \{(i_0, x_0)\}$. Then $M' \subseteq I \times U$, since $i_0 \in I$ and $x_0 \in A_{i_0} \subseteq U$; M' satisfies (C1), and it satisfies (C2), because the only pair of M' with first coordinate i_0 is (i_0, x_0) . So $M' \in P$, $M \subseteq M'$, and $M \neq M'$, as $(i_0, x_0) \notin M$; this contradicts the maximality of M . Hence for every $i \in I$ there is a pair $(i, x) \in M$, and by (C2) exactly one. By Definition 1.5.1, M is the graph of a function $c: I \rightarrow U$, and $c(i) \in A_i$ by (C1). ■

If $I = \emptyset$, then $P = \{\emptyset\}$, and the proof produces the empty function, which is correct. Of the proof's four steps, only the last is about choice functions; the first three run the same way in every use of the lemma, and that is worth recording.

Example. Method: the maximality template

Template. To show that a complete object exists: (1) **partial objects:** let P be the set of partial objects, of which the complete ones are the finished case, ordered by $\subseteq$ (extension); check $P \neq \emptyset$, usually with the empty or the given starting object. (2) **chains:** for a non-empty chain $\mathcal{C} \subseteq P$, show $\bigcup \mathcal{C} \in P$; each defining condition mentions finitely many elements, which lie in one member (Lemma 6.1.1). (3) **Zorn:** P has a maximal element M (Theorem 6.1.2). (4) **complete:** suppose M is not complete; use the missing piece to build a strictly larger partial object, a contradiction.

When to use it. The object can be built piece by piece, each condition on it checks finitely many pieces at a time, and the pieces cannot be listed.

The usual failure. The union of a chain is not a partial object, because a condition mentions infinitely many pieces (“finite”, “countable”); step (4) builds an object that is not strictly larger, or not in P ; and treating M as the greatest or the only complete object.

The second use runs the template with a shorter plan.

Example. A set of representatives

Let $\sim$ be an equivalence relation on a set X . Show that there is a set $S \subseteq X$ that meets every equivalence class in exactly one element.

Idea (Scratch work). By the method of Example (Method: the maximality template). A partial object should be a set that meets each class **at most** once, so that the complete ones meet each class exactly once. “At most once” reads: $s \sim t$ implies $s = t$, a condition on two elements at a time, so step (2) works. For step (4), a class $[x]$ that is missed gives the larger set $S \cup \{x\}$, and no choice is needed, since x is already in hand.

Solution. Let P be the set of $S \subseteq X$ such that for all $s, t \in S$, $s \sim t$ implies $s = t$; order it by $\subseteq$. Then $\emptyset \in P$. For a non-empty chain $\mathcal{C} \subseteq P$, any two elements s, t of $\bigcup \mathcal{C}$ lie in one member (Lemma 6.1.1 (b)), so $s \sim t$ gives $s = t$: $\bigcup \mathcal{C} \in P$, an upper bound of $\mathcal{C}$. By Theorem 6.1.2, take a maximal $M \in P$.

Suppose some class $[x]$ (Definition 4.2.2) contains no element of M . Then $x \notin M$, since $x \in [x]$ by (Eq1). And $M \cup \{x\} \in P$: let $s, t \in M \cup \{x\}$ with $s \sim t$. If both lie in M , then $s = t$. If $s = x$ and $t \in M$, then $t \sim x$ by (Eq2), so $t \in [x] \cap M$, which is impossible; the case $t = x$, $s \in M$ is the same with the letters exchanged; and if $s = t = x$, there is nothing to show. So $M \subsetneq M \cup \{x\} \in P$, contradicting maximality. Hence M meets every class, and by the condition defining P in at most one element: $S := M$ works.

The axiom of choice gives the same set in one line, by choosing one element from each class; the template’s advantage shows when the pieces interact, as in Section 6.2. On $\mathbb{R}$, the relation $x \sim y \iff x - y \in \mathbb{Q}$ is an equivalence relation (Section 4.2, after Example (The reals modulo the integers)). A set of representatives for it inside $[0, 1]$ is a set of real numbers that *Measure and Integration*, Chapter 1 shows has no length, and no formula for one is known.

Quick check. Let $X = \mathbb{Z}$ and $m \sim n \iff m - n$ is even. Describe the partial objects of Example (A set of representatives) that have two elements, and the maximal ones.

Solution. They are the sets $\{e, o\}$ with e even and o odd: two even or two odd integers are related, since their difference is even, so a partial object holds at most one of each parity. Conversely, $\{e, o\}$ is a partial object: with $e = 2a$ and $o = 2b + 1$, $e - o = 2(a - b - 1) + 1$ is odd, hence not even (no integer is both, Theorem 1.1.2), so $e \not\sim o$. These are exactly the maximal elements, since any third integer has the parity of e or of o (every integer is even or odd, Theorem 1.1.2), and $\emptyset$ and the one-element sets are not maximal, since each lies strictly inside some $\{e, o\}$. There are many maximal elements, such as $\{0, 1\}$ and $\{4, -7\}$, and no greatest one.

Warning. **Check that the union is a partial object.** Here is an almost-correct proof that $\mathbb{N}$ has a maximal finite subset. “Let P be the set of finite subsets of $\mathbb{N}$, ordered by $\subseteq$. A chain has its union as an upper bound. By Zorn’s lemma, P has a maximal element.” The conclusion is false, by the third example after Theorem 6.1.2. The failing line is the second: the union is an upper bound in $\mathcal{P}(\mathbb{N})$, but it need not be finite, so it need not lie in P . “Finite” is a condition on all the elements at once, and Lemma 6.1.1 handles only finitely many.

Section 6.2 applies the template twice more, first to the matching that opened this section.

6.1.5 Exercises

6.1.5.1 A. Check your understanding

Exercise. A1

These questions are about the proof of Theorem 6.1.3.

- (a) Where is “ $A_i \neq \emptyset$ ” used, and which step fails without it?
- (b) Why does the chain step need Lemma 6.1.1 only for $n = 2$?
- (c) Run the proof on $I = \{1, 2\}$, $A_1 = \{0\}$, $A_2 = \{5, 6\}$: list P , name its maximal elements, and decide whether P has a greatest element.
- (d) Which of the two hypotheses of Theorem 6.1.2 does the empty partially ordered set satisfy?

Solution.

- (a) In the last step: $x_0 \in A_{i_0}$ exists only because $A_{i_0} \neq \emptyset$. Without it, M could miss i_0 and still be maximal, and indeed no choice function exists if some A_i is empty.
- (b) Condition (C1) mentions one pair and (C2) mentions two; no condition defining P mentions more.
- (c) $U = \{0, 5, 6\}$, and

$$P = \{\emptyset, \{(1, 0)\}, \{(2, 5)\}, \{(2, 6)\}, \{(1, 0), (2, 5)\}, \{(1, 0), (2, 6)\}\}.$$

The maximal elements are the last two, each the graph of a choice function. There is no greatest element: the two maximal elements are incomparable, and a greatest element would be maximal (Proposition 4.3.2 (a)), so it would be one of them and contain the

other.

- (d) The chain hypothesis, vacuously; not non-emptiness.

6.1.5.2 B. Practice

Exercise. B1: Maximal chains

Let $(X, \preceq)$ be a partially ordered set and $C_0 \subseteq X$ a chain. Prove that there is a chain $C \supseteq C_0$ that is **maximal**: no chain contains C properly.

Solution (Hint). The partial objects are the chains containing C_0 .

Solution. Let P be the set of chains $C \subseteq X$ with $C_0 \subseteq C$, ordered by $\subseteq$; then $C_0 \in P$. Let $\mathcal{K} \subseteq P$ be a non-empty chain and $H := \bigcup \mathcal{K}$. Then $H \supseteq C_0$, since $\mathcal{K}$ has a member and each member contains C_0 . Any $x, y \in H$ lie in one member $K \in \mathcal{K}$ (Lemma 6.1.1 (b)), which is a chain, so x and y are comparable. So $H \in P$, and it is an upper bound of $\mathcal{K}$. By Theorem 6.1.2, P has a maximal element C . If a chain $D \supsetneq C$ existed, then $D \supseteq C_0$, so $D \in P$ and $C \subsetneq D$, contradicting maximality. So C is a maximal chain containing C_0 .

Exercise. B2: Name the move, then complete it

Here is the frame of a proof that for every set $\mathcal{F}$ of non-empty sets there is a subfamily $\mathcal{D} \subseteq \mathcal{F}$ of pairwise disjoint sets such that every member of $\mathcal{F}$ meets some member of $\mathcal{D}$.

Let P be the set of subfamilies $\mathcal{E} \subseteq \mathcal{F}$ whose members are pairwise disjoint ($E \cap E' = \emptyset$ for $E \neq E'$ in $\mathcal{E}$), ordered by $\subseteq$. Then $\emptyset \in P$. For a non-empty chain of such subfamilies, two members $E \neq E'$ of its union lie in one subfamily of the chain, so $E \cap E' = \emptyset$; the union lies in P and is an upper bound. By Zorn's lemma, P has a maximal element $\mathcal{D}$.

⋮

... Hence every member of $\mathcal{F}$ meets some member of $\mathcal{D}$.

Name the move, then complete it.

Solution. The move is step (4) of Example (Method: the maximality template): a missing piece gives a strictly larger partial object. Suppose $F \in \mathcal{F}$ meets no member of $\mathcal{D}$. Then $F \notin \mathcal{D}$: otherwise F would meet itself, as $F \cap F = F \neq \emptyset$. And $\mathcal{D} \cup \{F\}$ is pairwise disjoint, since $F \cap D = \emptyset$ for every $D \in \mathcal{D}$ by assumption. So $\mathcal{D} \subsetneq \mathcal{D} \cup \{F\} \in P$, against the maximality of $\mathcal{D}$. The check $F \notin \mathcal{D}$ is what makes the new family strictly larger, the failure the method box warns about. Hence every member of $\mathcal{F}$ meets some member of $\mathcal{D}$.

Exercise. B3: A small poset of partial injections

Let $A = \{1, 2\}$ and $B = \{b\}$. List the relations $G \subseteq A \times B$ in which each element of A and each element of B occurs in at most one pair, ordered by $\subseteq$. Name the maximal elements. For each, is every element of A paired? Every element of B ?

Solution. They are $\emptyset$, $\{(1, b)\}$ and $\{(2, b)\}$; the only other subset of $A \times B$, $\{(1, b), (2, b)\}$, fails because b occurs twice. The maximal elements are $\{(1, b)\}$ and $\{(2, b)\}$. Neither pairs every element of A ; each pairs every element of B , and read backwards it is an injection $B \rightarrow A$. (Section 6.2 shows that this always happens: a maximal partial injection uses up one side.)

Exercise. B4: Prove or disprove

Prove each true statement and refute each false one.

- (a) If every chain of a non-empty partially ordered set P has an upper bound in P , then P has a greatest element.
- (b) Every non-empty finite partially ordered set has a maximal element.
- (c) The set of subsets of $\mathbb{N}$ that contain no two consecutive numbers, ordered by $\subseteq$, has a greatest element.
- (d) If P is a non-empty partially ordered set in which every non-empty chain has an upper bound, then every $p \in P$ lies below some maximal element of P .
- (e) The set of countable subsets of $\mathbb{R}$, ordered by $\subseteq$, has a maximal element.

Hint for (d): apply Zorn's lemma to the elements above p .

Solution.

- (a) False. Take $P = \{\{1\}, \{2\}\}$ under $\subseteq$. Its chains are $\emptyset$, $\{\{1\}\}$ and $\{\{2\}\}$, each with an upper bound in P . But P has no greatest element, since $\{1\}$ and $\{2\}$ are incomparable.
- (b) True. Let P be non-empty and finite, and let C be a non-empty chain of P . Then C is finite (Proposition 2.5.6); list it as $x_1, \dots, x_n$ with $n \geq 1$. By Lemma 6.1.1 (a), one x_k is above all of them, an upper bound of C in P . By Theorem 6.1.2, P has a maximal element. (Section 4.3, Exercise R4 proves this without Zorn's lemma, by induction.)
- (c) False. The set E of even numbers and the set O of odd numbers both contain no two consecutive numbers. A greatest element G would contain E and O , so $0, 1 \in G$, which are consecutive. (In fact E and O are both maximal: adding any $n \notin E$ to E puts n and $n+1$ in it (n is odd by Theorem 1.1.2, so $n+1$ is even), and adding any $n \notin O$ to O does the same. So Zorn's lemma, if applied here, gives a maximal element, never a greatest one, as the warning after Theorem 6.1.2 says.)
- (d) True. Let $p \in P$, and let $Q := \{q \in P : p \preceq q\}$, ordered by $\preceq$; then $p \in Q$. A non-empty chain $\mathcal{K}$ of Q is a chain of P , so it has an upper bound $u \in P$; for any $c \in \mathcal{K}$, $p \preceq c \preceq u$, so $u \in Q$ by (PO3). By Theorem 6.1.2, Q has a maximal element m , and $p \preceq m$. If $m \prec q$ for some $q \in P$, then $p \preceq q$ by (PO3), so $q \in Q$, against the maximality of m in Q . So m is maximal in P .
- (e) False. Let $C \subseteq \mathbb{R}$ be countable. Then $C \neq \mathbb{R}$, since $\mathbb{R}$ is uncountable (Theorem 3.3.4); for $x \in \mathbb{R} \setminus C$, the set $C \cup \{x\}$ is countable (Proposition 3.1.5) and strictly larger. So no countable set is maximal. (By the contrapositive of Zorn's lemma, some non-empty chain of countable sets has no countable upper bound; the answer does not need one.)

6.1.5.3 C. Going deeper

Exercise. C1: Find the error

Claim: every chain C_0 of a partially ordered set X lies in a maximal chain.

“Proof”: Let P be the set of chains containing C_0 , ordered by $\subseteq$; then $C_0 \in P$. Let $\mathcal{K}$ be a non-empty chain in P . Every $K \in \mathcal{K}$ is a subset of $\bigcup \mathcal{K}$, so $\bigcup \mathcal{K}$ is an upper bound of $\mathcal{K}$. By Zorn's lemma, P has a maximal element C . A chain containing C contains C_0 , so it lies in P ; hence C is a maximal chain.

Is the conclusion true? Does the argument prove it?

Solution. The conclusion is true: it is Exercise B1.

The argument does not prove it. It never checks that $\bigcup \mathcal{K}$ is a **chain**, that is, that it lies in P ; the upper bound in [Theorem 6.1.2](#) must belong to P . The step's general form, “the union of a chain of partial objects lies in P ”, is false: among the finite subsets of $\mathbb{N}$, the chain $\{\{0, \dots, n-1\} : n \in \mathbb{N}\}$ has union $\mathbb{N}$, which is not finite (the warning before these exercises). Here the conclusion of the step happens to be true, which is exactly why it must be proved, not assumed. The repair is the chain step of B1: two elements of $\bigcup \mathcal{K}$ lie in one member $K \in \mathcal{K}$ ([Lemma 6.1.1](#) (b)), which is a chain, so they are comparable.

Marking scheme. (10 points) - 2: the conclusion is true, with B1 cited. - 3: the missing check named: $\bigcup \mathcal{K} \in P$. - 3: a counterexample to the general inference “the union of a chain of partial objects lies in P ” (in any partially ordered set), with the remark that here the step's conclusion is true but unproved. - 2: the repair via [Lemma 6.1.1](#). Fatal errors (0 for the step): calling the conclusion false; “repairing” the proof by assuming that a union of chains is always a chain.

Exercise. C2: Exit check: when no choice is needed

- (a) Let $\sim$ be an equivalence relation on $\mathbb{N}$. Without Zorn's lemma or the axiom of choice, give a set S meeting every class exactly once, and prove that it does.
- (b) Do the same for an equivalence relation on a set X with an injection $j: X \rightarrow \mathbb{N}$.
- (c) Which ingredient of [Example \(A set of representatives\)](#) does your answer to (a) make unnecessary, and why?

Solution (Hint 1). Each class is a non-empty set of natural numbers. Which element of it can be named without choosing?

Solution.

- (a) Let $S := \{m_x : x \in \mathbb{N}\}$, where m_x is the least element of $[x]$. It exists by [Theorem 2.1.3](#), since $[x] \subseteq \mathbb{N}$ contains x and so is non-empty, and it is unique ([Proposition 4.3.1](#)), so the rule names one value. *Every class is met:* $m_x \in [x] \cap S$. *At most once:* let $m_x, m_y \in S$ lie in the same class $[z]$. Then $m_x \in [x] \cap [z]$, so $[x] = [z]$, and likewise $[y] = [z]$, since different classes are disjoint ([Theorem 4.2.1](#)). So $[x] = [y]$, these classes have the same

least element, and $m_x = m_y$. Hence S meets each class $[x]$ in m_x and nowhere else.

- (b) For $x \in X$, the set $j([x]) \subseteq \mathbb{N}$ is non-empty, so it has a least element k_x (Theorem 2.1.3), and since j is injective exactly one $s_x \in [x]$ has $j(s_x) = k_x$. Let $S := \{s_x : x \in X\}$. The argument of (a) applies word for word, with s_x for m_x : equal classes give equal k_x , hence equal s_x .
- (c) Zorn's lemma, that is, the whole argument with P , chains and a maximal M : a rule names each representative, so nothing needs to be chosen (Example (Method: name the choice)).

Marking scheme. (10 points) - 3: (a) each class has a well-defined representative (well-ordering, and the rule names one value). - 3: (a) every class met, and met at most once (justified). - 2: (b) the least value of j , and injectivity naming one element. - 2: (c) Zorn's lemma named as the unnecessary ingredient, because a rule names each representative. Fatal errors (0 for the step): "choose an element of each class" in (a) or (b), which is the axiom of choice.