

The two earlier volumes used numbers as familiar objects. Their arithmetic and order were school facts, named where they were used and never proved. This volume proves them. It starts from a few axioms about sets and builds the natural numbers, then the integers, the rationals and the real numbers, each out of the one before. At the end the real numbers form a complete ordered field, and any two complete ordered fields are the same up to exactly one isomorphism.

Who it is for. The reader we have in mind has worked through *Logic and Proof* and *Sets, Relations, and Cardinality*, or has their habits from elsewhere. That means negating a quantified statement, running an induction, and checking that a rule on equivalence classes does not depend on the representative chosen. The last of these is used in almost every chapter. The chapters point back to both volumes where they use a definition or a rule, so a reader who is unsure of a step can look it up.

What it covers.

Chapter	Title	What it does
1	The ZF Fragment and the von Neumann Natural Numbers	Seven axioms about sets, ordered pairs built from sets, the natural numbers as $0 = \emptyset$ and $n + 1 = n \cup \{n\}$, and Peano's axioms, recursion, arithmetic and order proved for them
2	Building the Integers and the Rationals from Equivalence Classes	$\mathbb{Z}$ and $\mathbb{Q}$ as quotient sets of pairs, well-defined operations, $\mathbb{Z}$ an ordered ring and $\mathbb{Q}$ an Archimedean ordered field, and the gap in $\mathbb{Q}$ where $\sqrt{2}$ should be
3	Divisibility, Division with Remainder, and Modular Arithmetic	Division with remainder, Bézout's theorem and the Euclidean algorithm, unique prime factorization, quotient rings by an ideal, $\mathbb{Z}/n\mathbb{Z}$, and the Chinese remainder theorem
4	Ordered Fields and Sequences in an Ordered Field	Limits and Cauchy sequences in an arbitrary ordered field, the Archimedean property, and a Cauchy sequence of rationals with no rational limit
5	Constructing the Real Numbers from Cauchy Sequences	$\mathbb{R}$ as classes of rational Cauchy sequences, proved an ordered field in which every Cauchy sequence converges and every bounded set has a supremum

Chapter	Title	What it does
6	The Complete Ordered Field Is Unique	Exactly one order-preserving isomorphism between any two complete ordered fields, two fields that show what each hypothesis buys, and what <i>Analysis on the Real Line</i> takes from this volume
7	Constructing the Real Numbers by Dedekind Cuts (optional)	$\mathbb{R}$ as sets of rationals closed downward, a second complete ordered field, and the isomorphism that links it to Chapter 5's

Chapters 1 and 2 build the numbers that algebra needs, and Chapter 3 does their arithmetic. Chapters 4 and 5 fill the gap that Chapter 2 found in $\mathbb{Q}$: first the rules for sequences, proved without the number they aim at, then the numbers made from the sequences themselves. Chapter 6 shows that the construction does not matter, and Chapter 7 checks this on a second construction.

How to read it. The main line is Chapters 1–6; read them in order, since each uses the ones before it. Chapter 7 is optional, and nothing outside it depends on it. Read it for a second construction of $\mathbb{R}$, and to see the uniqueness theorem of Chapter 6 at work on two concrete models. Every section has Quick checks with folded answers; answer each one before you unfold it. The exercises of a section come in three groups, A to check understanding, B for practice and C to go deeper, and the last section of each chapter adds R items that reach back to earlier chapters. A chapter's opening page lists what it needs, with links, and the time it takes. Second-reading folds, in Chapters 5–7, hold edge cases and longer justifications. They can stay closed on a first reading, but they are not optional, since later proofs may use what they contain. Results marked optional can be skipped, and nothing outside them depends on them.

Time. The chapter openings add up to about 150 hours on the core track for the main line, Chapters 1–6, and about 210 hours for everything in it. Chapter 7 adds about 25 hours on the core track and about 30 for everything.

What comes next. *Analysis on the Real Line* does not need this volume. It starts from the axioms of a complete ordered field, the axioms this volume proves for its $\mathbb{R}$, and Chapter 6 explains why it does not matter which model of them one has in mind. A reader who wants to know that those axioms describe something real, and only one thing, will find both answers here. *Algebraic Preliminaries* builds on Chapters 3, 5 and 6: quotient rings and $\mathbb{Z}/n\mathbb{Z}$, the real numbers as a complete ordered field, and square roots in it, together with the ring and field rules of Chapters 2 and 4 that come with them.