

7.4 Multiplication

Like every section of this chapter, this one is optional: nothing outside Chapter 7 depends on it. Section 7.3 made the cuts an ordered group under addition (Theorem 7.3.4). This section adds a product and finishes the field axioms. On the Cauchy road of Chapter 5, multiplication was termwise and cost little; on this road it is the hard part. The set of all products of elements is useless, so we define the product of two **positive** cuts first, prove the laws there, and then extend to all cuts by signs, once, through one lemma (Lemma 7.4.4). The reciprocal needs the walk up by ε again, in a multiplicative form. The section ends with the theorem that the cuts form an ordered field.

As in Section 7.3, letters p, q, r, s, t, x stand for rationals, and the rules of the ordered field $\mathbb{Q}$ (Theorem 2.5.1, Proposition 2.3.5, Proposition 2.5.2) are used for them without comment.

7.4.1 Why not the set of products

For sums, the set of all sums worked. The analogous set for products,

$$\{pq : p \in \alpha, q \in \beta\},$$

fails in two different ways.

It can be everything. Suppose $0 \in \alpha$ and $0 \in \beta$, as for $\alpha = \beta = D_1$. Then both cuts contain every negative rational (C2), and α contains some $a > 0$ (C3). Every rational is a product: for $x > 0$, $x = (-1)(-x)$ with $-1 \in \alpha$, $-x \in \beta$; $0 = 0 \cdot 0$; and for $x < 0$, $x = a \cdot (x/a)$ with $x/a < 0$, so $x/a \in \beta$. So the set is $\mathbb{Q}$, which fails (C1). The culprit is that negatives times negatives are unbounded above.

It can be upside down. For $\alpha = \beta = D_{-1}$, every product pq with $p, q < -1$ is $(-p)(-q) > 1$, and every $x > 1$ is such a product: $x = pq$ with $p := -(1+x)/2 < -1$ and $q := -2x/(1+x) < -1$. So the set is $\{x \in \mathbb{Q} : x > 1\}$. It contains 2 but not $0 < 2$, so it fails (C2): it is closed upward.

Both failures come from negative elements, and every cut has them. For a cut $\alpha > D_0$, the positive elements carry all the information, since the others are exactly the rationals $p \leq 0$: by Proposition 7.1.2, $\alpha > D_0$ means $0 \in \alpha$, so every $p \leq 0$ is in α by (C2), and α has a positive element by (C3). So for positive cuts we multiply positive elements only, then put back everything below.

Warning. The product of two cuts is **not** the set of products of their elements, even for positive cuts: D_1 and D_1 give $\mathbb{Q}$, as shown above. Every proof in this section works with the definitions below, never with that set.

7.4.2 Positive cuts

Definition 7.4.1. Product of Positive Cuts

Let α and β be cuts with $\alpha > D_0$ and $\beta > D_0$ (equivalently, $0 \in \alpha$ and $0 \in \beta$, by Proposi-

tion 7.1.2). Their **product** is

$$\alpha\beta := \{p \in \mathbb{Q} : p \leq rs \text{ for some } r \in \alpha \text{ and } s \in \beta \\ \text{with } r > 0 \text{ and } s > 0\}.$$

In words: take the products of **positive** elements, and then every rational at or below one of them. The second step is what makes $\alpha\beta$ closed downward; it also puts every $p \leq 0$ in $\alpha\beta$, since positive $r \in \alpha$ and $s \in \beta$ exist and $p \leq 0 < rs$. For instance, in D_2D_3 every product rs of positive $r < 2$, $s < 3$ is below 6, so $D_2D_3 \subseteq D_6$; Exercise B1 proves equality.

Quick check. Let $\alpha, \beta > D_0$. Suppose the definition said “ $p = rs$ ” instead of “ $p \leq rs$ ”. Which set would it give, and which cut condition would that set fail?

Solution. It would give the set of products rs of positive elements, a set of positive rationals. It is non-empty, but it fails (C2): it contains a positive rs and not $0 < rs$.

Proposition 7.4.1. Positive Cuts under Multiplication

Let α, β, γ be cuts with $\alpha, \beta, \gamma > D_0$. Then:

- (a) $\alpha\beta$ is a cut, and $\alpha\beta > D_0$;
- (b) $\alpha\beta = \beta\alpha$;
- (c) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
- (d) $\alpha D_1 = \alpha$.

Idea. Each cut condition for $\alpha\beta$ is inherited from α or β , as for sums: an outside point of each factor bounds the products, and moving up in α moves the product up. The laws (b) and (c) are (F6) and (F5) of $\mathbb{Q}$, applied to positive elements. For (d), multiplying by an element of D_1 pushes down, so one inclusion is (C2); the other writes p as a larger element of α times a factor below 1.

Proof. Throughout, α, β, γ contain 0 and a positive element (by Proposition 7.1.2 and (C3)), and every element outside one of them is positive (Lemma 7.1.1, against 0).

- (a) (C1): $0 \in \alpha\beta$, as noted after the definition. By (C1) there are $a \notin \alpha$ and $b \notin \beta$; then $a, b > 0$. For positive $r \in \alpha$, $s \in \beta$, separation gives $r < a$ and $s < b$, so $rs < as < ab$ by Proposition 2.3.5 (c), since $s > 0$ and $a > 0$. So every $p \in \alpha\beta$ has $p \leq rs < ab$ for suitable r, s , and $ab \notin \alpha\beta$. (C2): if $p \leq rs$ and $q < p$, then $q \leq rs$, with the same r, s . (C3): let $p \in \alpha\beta$, with $p \leq rs$ for positive $r \in \alpha$, $s \in \beta$. By (C3) for α there is $r' \in \alpha$ with $r' > r$; then $r' > 0$, so $r's \in \alpha\beta$, and $r's > rs \geq p$ by Proposition 2.3.5 (c). So $\alpha\beta$ is a cut, and since $0 \in \alpha\beta$, $\alpha\beta > D_0$ by Proposition 7.1.2.
- (b) The condition “ $p \leq rs$ for some positive $r \in \alpha$, $s \in \beta$ ” is the same as “ $p \leq sr$ for some positive $s \in \beta$, $r \in \alpha$ ”, by (F6) in $\mathbb{Q}$. So the two sets are equal.
- (c) Let T be the set of rationals p with $p \leq rst$ for some positive $r \in \alpha$, $s \in \beta$, $t \in \gamma$. We show $(\alpha\beta)\gamma = T$. ($\subseteq$) Let $p \leq ut$ with $u \in \alpha\beta$, $t \in \gamma$ positive. Then $u \leq rs$ for some

positive $r \in \alpha$, $s \in \beta$, and $ut \leq (rs)t$ since $t > 0$; so $p \leq rst$ and $p \in T$. ($\supseteq$) If $p \leq (rs)t$ with r, s, t as in T , then $u := rs$ is positive and in $\alpha\beta$ (as $rs \leq rs$), so $p \in (\alpha\beta)\gamma$. The same argument, grouping $r(st)$, gives $\alpha(\beta\gamma) = T$, using (F5) in $\mathbb{Q}$.

- (d) ($\subseteq$) Let $p \leq rs$ with positive $r \in \alpha$ and $s \in D_1$, so $0 < s < 1$. Then $rs < r$, so $p < r$ and $p \in \alpha$ by (C2). ($\supseteq$) Let $p \in \alpha$. If $p \leq 0$, then $p \in \alpha D_1$ as noted after the definition. If $p > 0$, take $p' \in \alpha$ with $p' > p$ by (C3). Then $0 < p/p' < 1$, so $p/p' \in D_1$ is positive, and $p = p' \cdot (p/p') \in \alpha D_1$.

■

The proof of (d) used (C3) in the same place as the proof that $\alpha + D_0 = \alpha$ (Proposition 7.3.2): to reach p from inside, one needs an element of α strictly above it.

The reciprocal will need, as the negative did, a point inside a cut and a point outside that are close. For products “close” means “close in ratio”, and the walk up by ε gives that too.

Lemma 7.4.2. The Walk Up, Multiplicatively

Let α be a cut with $\alpha > D_0$, and let $\mu \in \mathbb{Q}$ with $\mu > 1$. Then there is $r \in \alpha$ with $r > 0$ and $\mu r \notin \alpha$.

Idea. Walk up from a positive point a in steps of $\varepsilon := (\mu - 1)a$. The last point r inside is at least a , so the step ε is at most $(\mu - 1)r$, and $\mu r = r + (\mu - 1)r$ is at least the first point outside.

Proof. By Proposition 7.1.2, $0 \in \alpha$, and by (C3) there is $a \in \alpha$ with $a > 0$; fix one $[\exists E]$ (Logic and Proof, Definition 5.6.1). Let $\varepsilon := (\mu - 1)a$; it is positive, as a product of positives. By Lemma 7.1.3 there is $n \in \mathbb{N}$ with $r := a + n\varepsilon \in \alpha$ and $r + \varepsilon \notin \alpha$. Since $n\varepsilon \geq 0$, $r \geq a > 0$. Then

$$\mu r - (r + \varepsilon) = (\mu - 1)r - (\mu - 1)a = (\mu - 1)(r - a) \geq 0,$$

so $\mu r \geq r + \varepsilon$. As $r + \varepsilon \notin \alpha$ and the outside of a cut is closed upward (Lemma 7.1.1), $\mu r \notin \alpha$.

■

The last law for positive cuts is distributivity. The sum $\beta + \gamma$ of two positive cuts contains $0 = 0 + 0$, since $0 \in \beta$ and $0 \in \gamma$, so it is positive (both steps by Proposition 7.1.2 (a)), and $\alpha(\beta + \gamma)$ is defined.

Lemma 7.4.3. Distributivity for Positive Cuts

For all cuts $\alpha, \beta, \gamma > D_0$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Idea. An element of $\beta + \gamma$ is a sum $s + t$, and $r(s + t) = rs + rt$. The only snag is that s or t may be negative, and the definition of $\alpha\beta$ wants positive factors; raising them to positive elements only increases $r(s + t)$, which is harmless because both sides are closed downward. For the other inclusion, two different first factors r_1, r_2 are replaced by the larger.

Proof. Let $\alpha, \beta, \gamma > D_0$. By Theorem 7.3.1 and Proposition 7.4.1 (a), both sides are cuts.

($\subseteq$) Let $p \in \alpha(\beta + \gamma)$: $p \leq ru$ for some positive $r \in \alpha$ and positive $u \in \beta + \gamma$, and $u = s + t$ with $s \in \beta$, $t \in \gamma$. Let $\tilde{s} := s$ if $s > 0$, and otherwise let $\tilde{s}$ be a positive element of β (one exists by (C3), since $0 \in \beta$). Then $\tilde{s} \in \beta$, $\tilde{s} > 0$ and $\tilde{s} \geq s$. Choose $\tilde{t} \in \gamma$ in the same way. Since $r > 0$,

$$p \leq r(s + t) = rs + rt \leq r\tilde{s} + r\tilde{t},$$

and $r\tilde{s} \in \alpha\beta$, $r\tilde{t} \in \alpha\gamma$. So $r\tilde{s} + r\tilde{t} \in \alpha\beta + \alpha\gamma$, and p is in that cut, being equal to this element or below it (C2).

($\supseteq$) Let $x \in \alpha\beta$ and $y \in \alpha\gamma$: $x \leq r_1s$ and $y \leq r_2t$ with positive $r_1, r_2 \in \alpha$, $s \in \beta$, $t \in \gamma$. Let $r := \max\{r_1, r_2\}$, one of r_1, r_2 , so $r \in \alpha$ and $r > 0$. Then

$$x + y \leq r_1s + r_2t \leq rs + rt = r(s + t),$$

and $s + t \in \beta + \gamma$ is positive. So $x + y \in \alpha(\beta + \gamma)$. ■

7.4.3 Signs

Every cut is positive, zero or negative (Theorem 7.2.1), and a negative cut has a positive negative (Theorem 7.3.4 (f)). So the product of positive cuts determines all others, as in school: multiply the sizes and fix the sign.

Definition 7.4.2. Absolute Value, Sign and Product of Cuts

For a cut α , its **absolute value** and **sign** are

$$|\alpha| := \begin{cases} \alpha & \text{if } \alpha \geq D_0, \\ -\alpha & \text{if } \alpha < D_0, \end{cases} \quad \text{sgn } \alpha := \begin{cases} 1 & \text{if } \alpha > D_0, \\ 0 & \text{if } \alpha = D_0, \\ -1 & \text{if } \alpha < D_0. \end{cases}$$

For cuts α and β , their **product** is

$$\alpha\beta := \begin{cases} D_0 & \text{if } \alpha = D_0 \text{ or } \beta = D_0, \\ |\alpha||\beta| & \text{if } \text{sgn } \alpha = \text{sgn } \beta \neq 0, \\ -(|\alpha||\beta|) & \text{if } \text{sgn } \alpha \neq \text{sgn } \beta \text{ and neither is } 0, \end{cases}$$

where $|\alpha||\beta|$ is the product of positive cuts of Definition 7.4.1.

The cases do not overlap and cover every cut and every pair, by Theorem 7.2.1. For $\alpha \neq D_0$, $|\alpha| > D_0$ (Theorem 7.3.4 (f)), so in the last two cases $|\alpha||\beta|$ is defined, and it is a cut (Proposition 7.4.1 (a)). So $\alpha\beta$ is a cut in every case: D_0 , a product of positive cuts, or the negative of one (Definition 7.3.2, with the check after it). The product is a function $\mathbb{R}_D \times \mathbb{R}_D \rightarrow \mathbb{R}_D$. The new product agrees with the old one where both are defined: for $\alpha, \beta > D_0$ the signs agree, $|\alpha| = \alpha$, $|\beta| = \beta$, and the second case returns $\alpha\beta$ of Definition 7.4.1. So writing $\alpha\beta$ for both is safe. The degenerate cases: $|D_0| = D_0$, $\text{sgn } D_0 = 0$, and $\alpha D_0 = D_0 \alpha = D_0$ for every cut α . Two small examples. Since $-1 < 0$, $D_{-1} < D_0$ (Section 7.2, “Two translations”), so $\text{sgn } D_{-1} = -1$ and $|D_{-1}| = -D_{-1} = D_1$, because $p \in -D_{-1}$ exactly when $-p - r \geq -1$ for some $r > 0$, that is,

when $p < 1$; and so $D_{-1}D_{-1} = D_1D_1 = D_1$, by [Proposition 7.4.1](#) (d).

Warning. The absolute value $|\alpha|$ is defined here before we know that $\mathbb{R}_D$ is an ordered field. Once [Theorem 7.4.7](#) below has shown that it is, $|\alpha|$ is the absolute value of that ordered field in the sense of Definition 4.1.1, by the same formula, so the two readings never disagree. Before that theorem, no rule of Proposition 4.1.3 may be used for cuts; the rules we need are proved in the next lemma.

The next lemma is where the reduction to positive cuts pays off. That reduction is the last of the chapter's named moves.

Reduce to the positive case: define and prove for positive cuts, then extend to all cuts by signs once, through one lemma, never axiom by axiom.

The lemma is proved once, by the definition's cases, and the later sign laws (distributivity, the reciprocal of a negative cut, the field axioms in [Theorem 7.4.7](#)) cite it instead of repeating the case analysis.

Lemma 7.4.4. Sign Rules for Cuts

For all cuts α, β, γ :

- (a) $\alpha = \text{sgn}(\alpha)|\alpha|$, reading $1 \cdot \xi := \xi$, $0 \cdot \xi := D_0$ and $(-1) \cdot \xi := -\xi$; so two cuts with the same sign and the same absolute value are equal;
- (b) $|\alpha\beta| = |\alpha||\beta|$ and $\text{sgn}(\alpha\beta) = \text{sgn } \alpha \text{sgn } \beta$;
- (c) $|- \alpha| = |\alpha|$ and $\text{sgn}(-\alpha) = -\text{sgn } \alpha$;
- (d) $(-\alpha)\beta = -(\alpha\beta) = \alpha(-\beta)$;
- (e) $\alpha\beta = \beta\alpha$;
- (f) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
- (g) $\alpha D_1 = \alpha$.

In words: a cut is its sign times its size; sizes multiply as positive cuts do, and signs multiply as the integers $-1, 0, 1$ do.

Idea. Clause (a) says a cut is determined by two pieces of data, its sign and its absolute value. Clauses (b) and (c) compute both pieces for a product and a negative. After that, each of (d)–(g) is an equation between two cuts whose signs and absolute values are visibly equal: the signs by arithmetic in $\{-1, 0, 1\}$, the absolute values by [Proposition 7.4.1](#) once a factor D_0 is set aside. No axiom is checked over the sign combinations.

Proof. We use Theorem 7.3.4 (e), (f) freely: $-(-\xi) = \xi$, $-D_0 = D_0$, and $\xi > D_0 \iff -\xi < D_0$.

- (a) If $\alpha > D_0$, then $\text{sgn}(\alpha)|\alpha| = 1 \cdot \alpha = \alpha$. If $\alpha = D_0$, it is $0 \cdot D_0 = D_0$. If $\alpha < D_0$, it is $(-1) \cdot (-\alpha) = -(-\alpha) = \alpha$. So two cuts with equal signs and equal absolute values are both equal to $\text{sgn}(\alpha)|\alpha|$.
- (b) If $\alpha = D_0$ or $\beta = D_0$, then $\alpha\beta = D_0$, so $|\alpha\beta| = D_0$ and $\text{sgn}(\alpha\beta) = 0$; and $|\alpha||\beta| = D_0$,

since one factor is $|D_0| = D_0$, and $\text{sgn } \alpha \text{sgn } \beta = 0$. Otherwise let $\pi := |\alpha||\beta|$, a product of positive cuts, so $\pi > D_0$ by [Proposition 7.4.1](#) (a). If the signs agree, $\alpha\beta = \pi$, with absolute value π and sign $1 = \text{sgn } \alpha \text{sgn } \beta$. If they differ, $\alpha\beta = -\pi < D_0$, with absolute value $-(-\pi) = \pi$ and sign $-1 = \text{sgn } \alpha \text{sgn } \beta$.

- (c) If $\alpha > D_0$, then $-\alpha < D_0$, so $|-\alpha| = -(-\alpha) = \alpha = |\alpha|$, and the signs are -1 and 1 . If $\alpha = D_0$, then $-\alpha = D_0$. If $\alpha < D_0$, then $-\alpha > D_0$, so $|-\alpha| = -\alpha = |\alpha|$, and the signs are 1 and -1 .
- (d) By (b) and (c), $(-\alpha)\beta$ has absolute value $|-\alpha||\beta| = |\alpha||\beta|$ and sign $-\text{sgn } \alpha \text{sgn } \beta$. By (c) and (b), $-(\alpha\beta)$ has absolute value $|\alpha\beta| = |\alpha||\beta|$ and sign $-\text{sgn}(\alpha\beta) = -\text{sgn } \alpha \text{sgn } \beta$. By (a) they are equal. The same computation, with β negated instead of α , gives $\alpha(-\beta) = -(\alpha\beta)$.
- (e) The signs of $\alpha\beta$ and $\beta\alpha$ are $\text{sgn } \alpha \text{sgn } \beta = \text{sgn } \beta \text{sgn } \alpha$. Their absolute values are $|\alpha||\beta|$ and $|\beta||\alpha|$, by (b): both are D_0 if α or β is D_0 , and otherwise they are equal by [Proposition 7.4.1](#) (b). By (a), $\alpha\beta = \beta\alpha$.
- (f) By (b), twice, the absolute values of the two sides are $(|\alpha||\beta|)|\gamma|$ and $|\alpha|(|\beta||\gamma|)$. If one of α, β, γ is D_0 , both are D_0 , since a product with a factor D_0 is D_0 ; otherwise they are equal by [Proposition 7.4.1](#) (c). The signs are $(\text{sgn } \alpha \text{sgn } \beta) \text{sgn } \gamma = \text{sgn } \alpha (\text{sgn } \beta \text{sgn } \gamma)$. By (a), the two sides are equal.
- (g) $D_1 > D_0$, since $0 \in D_1$ ([Proposition 7.1.2](#)), so $|D_1| = D_1$ and $\text{sgn } D_1 = 1$. By (b), αD_1 has sign $\text{sgn } \alpha$ and absolute value $|\alpha|D_1$, which is $D_0 = |\alpha|$ if $\alpha = D_0$, and $|\alpha|$ by [Proposition 7.4.1](#) (d) otherwise. By (a), $\alpha D_1 = \alpha$.

■

Quick check. Compute $D_{-2} \cdot D_3$ through the cases of [Definition 7.4.2](#). Which case applies, and what are $|D_{-2}|$ and $\text{sgn } D_{-2}$? Express the product through $D_2 D_3$.

Solution. Since $0 \notin D_{-2}$, $D_{-2} > D_0$ fails ([Proposition 7.1.2](#)), and $D_{-2} \neq D_0$ (as $-1 \in D_0 \setminus D_{-2}$); so $D_{-2} < D_0$ and $\text{sgn } D_{-2} = -1$, while $\text{sgn } D_3 = 1$ ($0 \in D_3$). The signs differ, so the third case applies: $D_{-2} D_3 = -(|D_{-2}||D_3|)$. Here $|D_{-2}| = -D_{-2} = D_2$, since $p \in -D_{-2}$ exactly when $-p - r \geq -2$ for some $r > 0$, that is, when $p < 2$; and $|D_3| = D_3$. So $D_{-2} D_3 = -(D_2 D_3)$. Since $D_2 D_3 = D_6$ ([Exercise B1](#)) and $-D_6 = D_{-6}$ (by the same computation), the product is D_{-6} , as $(-2) \cdot 3 = -6$ suggests. [Section 7.5](#) proves $D_p D_q = D_{pq}$ for all rationals.

Distributivity for all cuts is the one law where the reduction to positive cuts needs an idea, because $\beta + \gamma$ may have a sign different from both β and γ . The device is to move the negative summand to the other side, where it becomes positive.

Lemma 7.4.5. Distributivity for All Cuts

For all cuts α, β, γ , $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Idea. Fix $\alpha > D_0$ and write $f(\xi) := \alpha\xi$. We know two things about f : it is additive on cuts $> D_0$ ([Lemma 7.4.3](#)), and trivially when one summand is D_0 , and it commutes with negation

(Lemma 7.4.4 (d)). If β and γ have the same sign, negate both if necessary. If $\beta \geq D_0 \geq \gamma$, one of β and $-\gamma$ is a sum of two cuts $\geq D_0$: $\beta = (\beta + \gamma) + (-\gamma)$ when $\beta + \gamma \geq D_0$, and $-\gamma = (-(\beta + \gamma)) + \beta$ otherwise. Apply additivity to that sum and solve for $f(\beta + \gamma)$. For $\alpha < D_0$, pull the sign out of α .

Before reading on: which case uses that $-\gamma \geq D_0$?

Proof. Throughout we use Theorem 7.3.4 and Lemma 7.4.4 (d) as in the Idea. If $\alpha = D_0$, both sides are D_0 (Definition 7.4.2, Proposition 7.3.2).

Let $\alpha > D_0$ and $f(\xi) := \alpha\xi$ for every cut ξ . Two facts:

- (A) $f(\xi + \eta) = f(\xi) + f(\eta)$ whenever $\xi, \eta \geq D_0$. If $\xi, \eta > D_0$, this is Lemma 7.4.3. If $\xi = D_0$, both sides are $f(\eta)$, since $f(D_0) = D_0$ and $D_0 + \eta = \eta$; likewise if $\eta = D_0$.
- (N) $f(-\xi) = -f(\xi)$ for every cut ξ , by Lemma 7.4.4 (d).

By totality (Theorem 7.2.1), each of β and γ is $\geq D_0$ or $\leq D_0$, so one of the following cases applies.

Case 1: $\beta, \gamma \geq D_0$. This is (A).

Case 2: $\beta, \gamma \leq D_0$. Then $-\beta, -\gamma \geq D_0$ (Theorem 7.3.4 (f)), and $-(\beta + \gamma) = (-\beta) + (-\gamma)$ (Theorem 7.3.4 (e)). By (N), (A) and (N),

$$\begin{aligned} f(\beta + \gamma) &= -f((-\beta) + (-\gamma)) = -(f(-\beta) + f(-\gamma)) \\ &= -((-f(\beta)) + (-f(\gamma))) = f(\beta) + f(\gamma), \end{aligned}$$

where the last step is Theorem 7.3.4 (e) twice.

Case 3: $\beta \geq D_0 \geq \gamma$ and $\beta + \gamma \geq D_0$. Then $\beta = (\beta + \gamma) + (-\gamma)$, with both summands $\geq D_0$. By (A) and (N), $f(\beta) = f(\beta + \gamma) + f(-\gamma) = f(\beta + \gamma) + (-f(\gamma))$. Adding $f(\gamma)$ to both sides gives $f(\beta) + f(\gamma) = f(\beta + \gamma)$.

Case 4: $\beta \geq D_0 \geq \gamma$ and $\beta + \gamma < D_0$. Then $-\gamma = (-(\beta + \gamma)) + \beta$, since $-(\beta + \gamma) + \beta = (-\beta) + (-\gamma) + \beta = -\gamma$; both summands are $\geq D_0$. By (N) and (A), $-f(\gamma) = -f(\beta + \gamma) + f(\beta)$. Adding $f(\beta + \gamma) + f(\gamma)$ to both sides gives $f(\beta + \gamma) = f(\beta) + f(\gamma)$.

Case 5: $\gamma \geq D_0 \geq \beta$. Exchange β and γ in Cases 3 and 4; both sides of the claim are symmetric in β and γ (Theorem 7.3.4 (a)).

This proves the lemma for $\alpha > D_0$. Let $\alpha < D_0$. Then $-\alpha > D_0$, and by Lemma 7.4.4 (d), $\alpha\xi = -((- \alpha)\xi)$ for every cut ξ . So, by the case just proved (applied to $-\alpha$),

$$\begin{aligned} \alpha(\beta + \gamma) &= -((- \alpha)(\beta + \gamma)) = -((- \alpha)\beta + (- \alpha)\gamma) \\ &= -((- \alpha)\beta) + -((- \alpha)\gamma) = \alpha\beta + \alpha\gamma, \end{aligned}$$

where the third equality is Theorem 7.3.4 (e). ■

The answer to the prompt. Case 3 uses $-\gamma \geq D_0$ to apply (A) to $(\beta + \gamma) + (-\gamma)$, and Case 2 uses it, with $-\beta \geq D_0$, to apply (A) to $(-\beta) + (-\gamma)$. Case 4 does not: there $-\gamma$ is the sum, and the summands are $-(\beta + \gamma)$ and β .

7.4.4 Reciprocals

For the reciprocal we repeat the idea of the negative, with division in place of subtraction. For $\alpha > D_0$, the reciprocal should consist of the rationals $p \leq 0$ together with the positive rationals below $1/q$ for some rational q outside α . The analogue of the margin is a **strict** inequality.

The reciprocal of a positive cut: the non-positive rationals, and the positive rationals strictly below the reciprocal of some point outside.

Definition 7.4.3. Reciprocal of a Cut

Let α be a cut. If $\alpha > D_0$, its **reciprocal** is the set

$$\begin{aligned} \alpha^{-1} := & \{p \in \mathbb{Q} : p \leq 0\} \\ & \cup \{p \in \mathbb{Q} : p > 0 \text{ and } pq < 1 \text{ for some } q \in \mathbb{Q} \setminus \alpha\}. \end{aligned}$$

If $\alpha < D_0$, its reciprocal is $\alpha^{-1} := -((- \alpha)^{-1})$. The cut D_0 has no reciprocal. The quantifier in the set for $\alpha > D_0$ is **some**: one point q outside α suffices, and it may depend on p .

Every $q \notin \alpha$ is positive when $\alpha > D_0$ (Lemma 7.1.1 against $0 \in \alpha$), so for $p > 0$ the condition $pq < 1$ says $p < 1/q$. For $\alpha < D_0$, the cut $- \alpha$ is positive (Theorem 7.3.4 (f)), so the definition for $\alpha < D_0$ refers to the one for $\alpha > D_0$.

Example. $D_2^{-1} = D_{1/2}$. The outside of D_2 is $\{q : q \geq 2\}$. A positive $p < 1/2$ is in D_2^{-1} , with $q = 2$: $2p < 1$. A positive $p \geq 1/2$ is not, since every $q \geq 2$ gives $pq \geq 1$. With the non-positive rationals, $D_2^{-1} = \{p : p < 1/2\}$.

A non-example by minimal change. Replace “ $pq < 1$ ” by “ $pq \leq 1$ ”. For D_2 the set becomes $\{p : p \leq 1/2\}$, which has a largest element. This is the multiplicative mirror of Example (The Negatives of the Outside), and Exercise C1 shows that, as there, it fails exactly for the rational cuts.

Theorem 7.4.6. Reciprocals of Cuts

Let α be a cut with $\alpha \neq D_0$. Then α^{-1} is a cut, $\alpha^{-1} > D_0$ if $\alpha > D_0$, $\alpha^{-1} < D_0$ if $\alpha < D_0$, and

$$\alpha\alpha^{-1} = D_1.$$

Idea. For $\alpha > D_0$, the cut conditions are checked as for the negative; the strict inequality $pq < 1$ leaves room above p , as the margin did. The inclusion $\alpha\alpha^{-1} \subseteq D_1$ is separation: $r < q$ for $r \in \alpha$, $q \notin \alpha$, so $rs < qs < 1$.

For $\supseteq$, work backwards. Given $0 < p < 1$, we want positive $r \in \alpha$ and $s \in \alpha^{-1}$ with $rs = p$. So $s = p/r$, and “ $s \in \alpha^{-1}$ ” asks for some $q \notin \alpha$ with $(p/r)q < 1$, that is, $q < r/p$. So we need **a point r inside and a point outside below r/p** : a point inside and a point outside whose **ratio** is less than $1/p$. The multiplicative walk (Lemma 7.4.2) supplies the ratio μ , and μ must be chosen strictly between 1 and $1/p$, for the same reason that ε was $-s/2$ and not $-s$: we take the midpoint $\mu := (1 + 1/p)/2$. For $\alpha < D_0$, the sign rules move everything to $- \alpha$.

Exercise B2 asks for the last step of the proof; the fold below has the proof in full.

○ **Second reading** · *The proof of the reciprocal theorem (try Exercise B2 first)*

Proof. Let $\alpha > D_0$. Then $0 \in \alpha$ (Proposition 7.1.2), so every $q \notin \alpha$ is positive (Lemma 7.1.1), and α has a positive element (C3).

(C1) $0 \in \alpha^{-1}$. Let $a \in \alpha$ with $a > 0$. Then $1/a \notin \alpha^{-1}$: it is positive, and every $q \notin \alpha$ has $q > a$ (Lemma 7.1.1), so $(1/a)q > (1/a)a = 1$.

(C2) Let $p \in \alpha^{-1}$ and $p' < p$. If $p' \leq 0$, then $p' \in \alpha^{-1}$. Otherwise $0 < p' < p$, so $p > 0$ and $pq < 1$ for some $q \notin \alpha$; as $q > 0$, $p'q < pq < 1$, and $p' \in \alpha^{-1}$ with the same q .

(C3) Let $p \in \alpha^{-1}$. If $p \leq 0$, take any $q \notin \alpha$ (C1); then $1/(2q) > 0 \geq p$, and $(1/(2q))q = 1/2 < 1$, so $1/(2q) \in \alpha^{-1}$. If $p > 0$ with $pq < 1$, $q \notin \alpha$, then $p < 1/q$; the midpoint $p'' := (p + 1/q)/2$ has $p < p'' < 1/q$ (Proposition 4.1.2 (b)), so $p'' > 0$ and $p''q < 1$, and $p'' \in \alpha^{-1}$.

So α^{-1} is a cut, and $\alpha^{-1} > D_0$ because $0 \in \alpha^{-1}$ (Proposition 7.1.2). The product $\alpha\alpha^{-1}$ is therefore given by Definition 7.4.1.

($\subseteq$) Let $x \in \alpha\alpha^{-1}$: $x \leq rs$ with positive $r \in \alpha$ and $s \in \alpha^{-1}$, and $sq < 1$ for some $q \notin \alpha$. By Lemma 7.1.1, $r < q$, so $rs < qs < 1$ since $s > 0$. Hence $x < 1$ and $x \in D_1$.

($\supseteq$) Let $p \in D_1$, so $p < 1$. If $p \leq 0$, then $p \in \alpha\alpha^{-1}$, since positive elements of α and of α^{-1} exist and their product is above p . Let $0 < p < 1$, and put

$$\mu := \frac{1}{2}\left(1 + \frac{1}{p}\right);$$

it depends only on p . Since $1/p > 1$ (Proposition 2.5.2 (b)), μ is the midpoint of $1 < 1/p$, so $1 < \mu < 1/p$. By Lemma 7.4.2 there is $r \in \alpha$ with $r > 0$ and $\mu r \notin \alpha$; fix one $[\exists E]$. Put $s := p/r > 0$. Then

$$s \cdot (\mu r) = p\mu < p \cdot \frac{1}{p} = 1,$$

and $\mu r \notin \alpha$, so $s \in \alpha^{-1}$ $[\exists I]$ (Logic and Proof, Definition 5.5.1), with witness $q := \mu r$. Since $p = rs \leq rs$, $p \in \alpha\alpha^{-1}$ $[\exists I]$. As p was arbitrary, $D_1 \subseteq \alpha\alpha^{-1}$ $[\forall I]$ (Logic and Proof, Definition 5.4.1).

Let $\alpha < D_0$. Then $-\alpha > D_0$, so by the first part $\beta := (-\alpha)^{-1}$ is a cut with $\beta > D_0$ and $(-\alpha)\beta = D_1$. Then $\alpha^{-1} = -\beta$ is a cut (Definition 7.3.2), and $-\beta < D_0$ (Theorem 7.3.4 (f)). By Lemma 7.4.4 (d), used twice,

$$\alpha\alpha^{-1} = \alpha(-\beta) = -(\alpha\beta) = (-\alpha)\beta = D_1.$$

■

The walk up by ε has now produced both inverses: the additive one through Lemma 7.1.3 directly, the multiplicative one through its multiplicative form. No other law of this section constructs a point by walking up.

7.4.5 The cuts form an ordered field

Theorem 7.4.7. The Cuts Form an Ordered Field

With the addition of Definition 7.3.1, the multiplication of Definition 7.4.2 and the order of Definition 7.2.1, $\mathbb{R}_D$ is an ordered field (Definition 2.3.2), with zero D_0 and one D_1 .

Proof. Let α, β, γ be cuts. Both operations map $\mathbb{R}_D \times \mathbb{R}_D$ into $\mathbb{R}_D$: sums by Theorem 7.3.1, products as noted after Definition 7.4.2. The table names, for each clause of Definition 2.3.1 and Definition 2.3.2, the line that proves it.

Clause	Statement for cuts	Proved by
(F1), (F2)	$+$ commutative, associative	Theorem 7.3.1
(F3)	$\alpha + D_0 = \alpha$	Proposition 7.3.2
(F4)	$\alpha + (-\alpha) = D_0$	Theorem 7.3.3
(F5)	$(\alpha\beta)\gamma = \alpha(\beta\gamma)$	Lemma 7.4.4 (f)
(F6)	$\alpha\beta = \beta\alpha$	Lemma 7.4.4 (e)
(F7)	$\alpha D_1 = \alpha$	Lemma 7.4.4 (g)
(F8)	$\alpha\alpha^{-1} = D_1$ for $\alpha \neq D_0$	Theorem 7.4.6
(F9)	$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$	Lemma 7.4.5
$1 \neq 0$	$D_1 \neq D_0$	$0 \in D_1 \setminus D_0$
(O1)	exactly one of $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$	Theorem 7.2.1
(O2)	$\alpha < \beta < \gamma \Rightarrow \alpha < \gamma$	below
(O3)	$\alpha < \beta \Rightarrow \alpha + \gamma < \beta + \gamma$	Theorem 7.3.4 (d)
(O4)	$D_0 < \alpha, \beta \Rightarrow D_0 < \alpha\beta$	Proposition 7.4.1 (a)

For (O2): if $\alpha \subsetneq \beta \subsetneq \gamma$, then $\alpha \subseteq \gamma$ (Sets, Relations, and Cardinality, Proposition 1.1.2), and $\alpha = \gamma$ would give $\beta \subseteq \gamma = \alpha \subseteq \beta$, so $\alpha = \beta$ (Sets, Relations, and Cardinality, Proposition 1.1.1), against $\alpha < \beta$. For (O4): for $\alpha, \beta > D_0$ the product of Definition 7.4.2 is that of Definition 7.4.1, as noted after the definition. Every clause is covered, so $\mathbb{R}_D$ is an ordered field. ■

Nothing in the table is left to an exercise. Two rows are proved by walking up: (F4), through Lemma 7.1.3, and (F8), through Lemma 7.4.2. Several others, such as (O3), (F5)–(F7) and (F9), lean on (F4) through the cancellation law and $-(-\alpha) = \alpha$ of Theorem 7.3.4 (e). The order rows cost almost nothing, because the order is inclusion; on the Cauchy road of Chapter 5 the order took a section of its own. Section 7.5 puts $\mathbb{Q}$ inside $\mathbb{R}_D$ and adds completeness, and Section 7.6 compares the result with Chapter 5's field.

7.4.6 Exercises

7.4.6.1 A. Check your understanding

Exercise. A1

- (a) Why is the set $\{pq : p \in \alpha, q \in \beta\}$ useless as a product for $\alpha = \beta = D_1$?
- (b) In the table of the proof of [Theorem 7.4.7](#), which rows are proved by a lemma that walks up through a cut, [Lemma 7.1.3](#) or [Lemma 7.4.2](#)? Which further rows depend on such a row?
- (c) Compute $|D_{-1}|$, $\text{sgn } D_{-1}$, $|D_{1/2}|$, $\text{sgn } D_{1/2}$ and $D_{-1}D_{1/2}$ from the definitions.
- (d) True or false: if $\alpha > D_0$, then α^{-1} contains every negative rational. Give a reason.

Solution.

- (a) It is all of $\mathbb{Q}$, so it fails (C1): $x > 0$ is $(-1)(-x)$, $0 = 0 \cdot 0$, and $x < 0$ is $(1/2)(2x)$, with every factor in D_1 .
- (b) (F4), the additive inverse: [Theorem 7.3.3](#) uses [Lemma 7.1.3](#). (F8), the multiplicative inverse: [Theorem 7.4.6](#) uses [Lemma 7.4.2](#), whose proof uses [Lemma 7.1.3](#). Further rows depend on (F4): the strict form of (O3) uses cancellation, and (F5)–(F7) and (F9) use [Lemma 7.4.4](#) and [Lemma 7.4.5](#), which use $-(-\alpha) = \alpha$, cancellation, or both ([Theorem 7.3.4](#) (e)), all proved from $\alpha + (-\alpha) = D_0$. The rows (F1)–(F3), $1 \neq 0$, (O1), (O2) and (O4) use neither lemma.
- (c) $D_{-1} < D_0$: $D_{-1} > D_0$ fails since $0 \notin D_{-1}$, and $D_{-1} \neq D_0$ since $-1/2 \in D_0 \setminus D_{-1}$. So $\text{sgn } D_{-1} = -1$ and $|D_{-1}| = -D_{-1} = D_1$ ([Exercise \(B1\)](#) with $q = -1$). Since $0 \in D_{1/2}$, $D_{1/2} > D_0$, so $\text{sgn } D_{1/2} = 1$ and $|D_{1/2}| = D_{1/2}$. The signs differ, so

$$D_{-1}D_{1/2} = -(D_1D_{1/2}) = -(D_{1/2}D_1) = -D_{1/2} = D_{-1/2},$$

by [Proposition 7.4.1](#) (b) and (d), and [Exercise \(B1\)](#) with $q = 1/2$.

- (d) True: by [Definition 7.4.3](#), α^{-1} contains every $p \leq 0$, with no condition.

7.4.6.2 B. Practice**Exercise. B1**

Prove from [Definition 7.4.1](#) that $D_2D_3 = D_6$.

Hint: for $0 < x < 6$, scale 2 and 3 down by suitable factors.

Solution. $D_2, D_3 > D_0$, since $0 \in D_2$ and $0 \in D_3$.

($\subseteq$) Let $p \leq rs$ with $0 < r < 2$, $0 < s < 3$. Then $rs < 2s < 6$ ([Proposition 2.3.5](#) (c), twice), so $p < 6$.

($\supseteq$) Let $x < 6$. If $x \leq 0$, then $x \leq 1 \cdot 1$ with $1 \in D_2$ and $1 \in D_3$. Let $0 < x < 6$, and put

$$\lambda := \frac{1}{2}\left(1 + \frac{x}{6}\right), \quad r := 2\lambda, \quad s := \frac{x}{2\lambda}.$$

Since $0 < x/6 < 1$, λ is the midpoint of $x/6 < 1$, so $x/6 < \lambda < 1$. Hence $0 < r < 2$, so $r \in D_2$. Also $s > 0$, and $s < 3$ because $x < 6\lambda$; so $s \in D_3$. Finally $rs = x$, so $x \in D_2D_3$. Hence $D_2D_3 = D_6$.

Exercise. B2: The last step of the reciprocal

Let α be a cut with $\alpha > D_0$, and let $p \in \mathbb{Q}$ with $0 < p < 1$. We show that $p \in \alpha\alpha^{-1}$.

- (1) Let $\mu := (1 + 1/p)/2$. Since $1/p > 1$, μ is the midpoint of 1 and $1/p$, so $1 < \mu < 1/p$.
- (2) By [Lemma 7.4.2](#) there is $r \in \alpha$ with $r > 0$ and $\mu r \notin \alpha$.
- (3) Hence $p \in \alpha\alpha^{-1}$.

Name the move, then complete it: write out step (3).

Solution. The move is the **walk up by ε** , in its multiplicative form: step (2) walks up through α until the next point is outside, which gives a point inside and a point outside whose ratio is μ , less than $1/p$.

- (3) Put $s := p/r$; it is positive. Then

$$s \cdot (\mu r) = p\mu < p \cdot \frac{1}{p} = 1,$$

using $\mu < 1/p$ and $p > 0$. Since $\mu r \notin \alpha$, the rational $q := \mu r$ witnesses $s \in \alpha^{-1}$ ([Definition 7.4.3](#)). Now $r \in \alpha$ and $s \in \alpha^{-1}$ are positive and $p = rs \leq rs$, so $p \in \alpha\alpha^{-1}$ by [Definition 7.4.1](#).

Why μ and not $1/p$ itself: with $\mu = 1/p$, step (2) gives r with $r/p \notin \alpha$, and $s = p/r$ would need some $q \notin \alpha$ with $(p/r)q < 1$, that is, $q < r/p$. The point r/p itself does not qualify, and there may be no smaller one outside. The strict room between μ and $1/p$ plays the part of the margin.

Exercise. B3

Prove that $(-\alpha)(-\beta) = \alpha\beta$ for all cuts α and β .

Solution. By [Lemma 7.4.4](#) (d), applied first with $-\alpha$ in front and then to $\alpha(-\beta)$,

$$(-\alpha)(-\beta) = -(\alpha(-\beta)) = -(-(\alpha\beta)) = \alpha\beta,$$

where the last step is $-(-\xi) = \xi$ (Theorem 7.3.4 (e)) with $\xi = \alpha\beta$.

7.4.6.3 C. Going deeper**Exercise. C1: The reciprocal without strictness**

For a cut $\alpha > D_0$, let

$$\begin{aligned} \rho(\alpha) := & \{p \in \mathbb{Q} : p \leq 0\} \\ & \cup \{p \in \mathbb{Q} : p > 0 \text{ and } pq \leq 1 \text{ for some } q \in \mathbb{Q} \setminus \alpha\}. \end{aligned}$$

Prove that $\rho(\alpha)$ is a cut if and only if α is not a rational cut, that is, $\alpha \neq D_a$ for every $a \in \mathbb{Q}$.

Solution (Hint 1). Compute $\rho(D_a)$ first. Which element of the outside of D_a does all the work?

Solution (Hint 2). If α is not rational, the outside $\mathbb{Q} \setminus \alpha$ has no least element: a least element m would force $\alpha = D_m$.

Solution. Let $\alpha > D_0$. Then $0 \in \alpha$, every $q \notin \alpha$ is positive (Lemma 7.1.1), and α has a positive element.

If $\alpha = D_a$, then $\rho(\alpha)$ is not a cut. Here $a > 0$, since $0 \in D_a$, and the outside is $\{q : q \geq a\}$. For $p > 0$: if $pq \leq 1$ with $q \geq a$, then $pa \leq pq \leq 1$; conversely $pa \leq 1$ gives the witness $q = a$. So $\rho(D_a) = \{p : p \leq 0\} \cup \{p > 0 : p \leq 1/a\} = \{p : p \leq 1/a\}$. It has the largest element $1/a$, so (C3) fails.

If α is not rational, then $\rho(\alpha)$ is a cut. First, $\mathbb{Q} \setminus \alpha$ has no least element. Suppose m were one. Every $r < m$ is then in α (otherwise r is a smaller element of the outside), and every $r \in \alpha$ has $r < m$ (Lemma 7.1.1). So $\alpha = D_m$, against the hypothesis.

(C1) $0 \in \rho(\alpha)$. For a positive $a \in \alpha$, $1/a \notin \rho(\alpha)$: every $q \notin \alpha$ has $q > a$, so $(1/a)q > 1$.

(C2) As for α^{-1} : if $0 < p' < p$ and $pq \leq 1$ with $q > 0$, then $p'q < pq \leq 1$; and every $p' \leq 0$ is in $\rho(\alpha)$.

(C3) If $p \leq 0$, then for any $q \notin \alpha$, $1/(2q)$ is in $\rho(\alpha)$ and larger. Let $p > 0$ with $pq \leq 1$, $q \notin \alpha$. Since q is not least in the outside, there is $q' \notin \alpha$ with $q' < q$; then $q' > 0$ and $pq' < pq \leq 1$, so $p < 1/q'$. The midpoint $p'' := (p + 1/q')/2$ has $p < p'' < 1/q'$, so $p''q' < 1$, and $p'' \in \rho(\alpha)$ is larger than p .

So $\rho(\alpha)$ is a cut exactly when α is not rational. For such α , $\rho(\alpha) = \alpha^{-1}$: the argument for (C3) shows that $pq \leq 1$ with $q \notin \alpha$ gives $pq' < 1$ with $q' \notin \alpha$, and the converse is immediate. As with the negatives of the outside in Section 7.3, the non-strict recipe works for the cut σ of Example (The cut of the gap), which is not a rational cut (Exercise C1 of Section 7.1), and fails on the simplest cuts; the strict inequality in Definition 7.4.3 is the multiplicative margin.

Marking scheme. (10 points) - 3: shows that $\rho(D_a)$ has the largest element $1/a$ ($1/a \in \rho(D_a)$ and every element is $\leq 1/a$; the full equality $\rho(D_a) = \{p : p \leq 1/a\}$ also earns this), and names (C3) as the failing condition. - 2: proves that the outside of a non-rational cut has no least element. - 1: (C1) and (C2) for $\rho(\alpha)$. - 3: (C3) for $\rho(\alpha)$, using a smaller outside point q' to make the inequality strict. - 1: states the equivalence with both directions. Second route: showing $\rho(\alpha) = \alpha^{-1}$ for non-rational α ($\subseteq$ by a smaller outside point q' making $pq' < 1$, $\supseteq$ since $<$ implies $\leq$) and citing Theorem 7.4.6 earns the 1 point for (C1) and (C2) and the 3 points for (C3). The 2 points for the no-least-element step still require that step. Fatal errors (0 for the step): assuming $q > 0$ without separation against $0 \in \alpha$; “a smaller outside point exists” asserted for every cut.