

Constructing the Real Numbers by Dedekind Cuts

This chapter is optional: nothing outside it depends on it, and the volume's main line ended with Chapter 6. Chapter 5 built a real number as a class of Cauchy sequences of rationals, many rational approximations identified when they agree in the limit. Dedekind's idea, published in 1872, is shorter to state: a real number is the set of all rationals **below** it. The gap that Section 2.6 found in $\mathbb{Q}$ splits the rationals into a lower piece, those that are negative or have square below 2, and the rest, and Dedekind takes the lower piece **as** the square root of 2. The chapter makes this precise. It says which sets of rationals qualify (the cuts), orders them by inclusion, adds and multiplies them, and proves that they form a complete ordered field $\mathbb{R}_D$ in which the rationals sit as the rational cuts and the cut of the gap squares to 2.

The question behind the chapter is: *if two people build the real numbers in different ways, in what sense do they have the same numbers?* On this road the costs fall the other way round from Chapter 5. The least upper bound of a bounded set of cuts is its union, checked in a few lines with no arithmetic at all, while negatives need a margin and multiplication needs a detour through positive cuts and signs; on the Cauchy road the arithmetic was termwise and completeness took two sections. The last section applies the uniqueness theorem of Chapter 6 to the two models. There is exactly one order-preserving field isomorphism from Chapter 5's field to the cuts, and it can be written down: it sends a Cauchy real to the set of rationals below it. So the two roads give the same real numbers, and Dedekind's definition is what the Cauchy model sees when it looks at one of its elements through the rationals.

What you need. Chapter 2 above all, and Chapter 4; for Sections 7.5 and 7.6, Chapters 5 and 6; *Sets, Relations, and Cardinality*, Chapters 1 and 2, chiefly Section 2.6; and *Logic and Proof*, Chapters 5, 6 and 8. From Chapter 2: $\mathbb{Q}$ as an ordered field, with the axioms (F1)–(F9) and (O1)–(O4) and their rules (Theorem 2.5.1, Definition 2.3.1, Definition 2.3.2, Proposition 2.3.2, Proposition 2.3.5, Proposition 2.5.2); $\mathbb{Q}$ Archimedean (Theorem 2.5.3, Corollary 2.5.4); least elements and discreteness of the integers, for the walk up by ε (Proposition 2.3.6, Theorem 2.3.4); and the gap at $\sqrt{2}$ (Theorem 2.6.1, Theorem 2.6.2). From Chapter 4: halves and fractions in an ordered field and the absolute value (Proposition 4.1.2, Lemma 4.1.1, Definition 4.1.1); the one copy ι_F of $\mathbb{Q}$ in an ordered field, with its uniqueness clause (Theorem 4.2.2); the Archimedean property and density of $\mathbb{Q}$ (Definition 4.2.2, Theorem 4.2.5); and, for exercises of Section 7.6, limits and Cauchy sequences (Definition 4.3.2, Definition 4.5.1). From Chapter 5: complete ordered fields (Definition 5.6.1), the Cauchy model and the theorem that it is one (Definition 5.2.1, Theorem 5.6.3), its copy of $\mathbb{Q}$ (Theorem 5.4.1), and the Newton class whose square is 2 (Example (The Class of the Newton Sequence)). From Chapter 6: order-preserving field maps and the rationals below an element (Definition 6.1.1, Definition 6.1.2, Proposition 6.1.1), and the rationals below a sum (Lemma 6.2.1), for the remark after the proof in Section 7.6 and Exercise B1; the uniqueness theorem (Theorem 6.3.1), which Section 7.6 applies; that complete ordered fields are Archimedean (Theorem 6.1.2), for the second proof in Section 7.5; and the embedding of an Archimedean field into a complete one, with its property $(\star)$ and uniqueness clause (Theorem 6.2.3), together with the fold after Corollary 6.3.2 on inverses of isomorphisms, for the inverse map in Section 7.6. Chapter 1 supplies the axioms that make the set of all cuts a set (Axiom 1.1.1). From *Sets, Relations, and Cardinality*: power sets, unions of families and inclusion, Sections 1.1 and 1.2 (Sets, Relations, and Cardinality, Definition 1.2.2, Sets, Relations, and Cardinality, Definition 1.2.4, Sets, Relations, and Cardinality, Proposition 1.1.1, Sets, Relations, and Cardinality, Proposition 1.1.2); sums of sets of numbers, Section 1.6 (Sets, Relations, and Cardinality, Example (Reflections, scalings and sums as images)); partial orders, upper bounds

and suprema, Section 2.6 (Sets, Relations, and Cardinality, Definition 2.6.1, Sets, Relations, and Cardinality, Definition 2.6.2, Sets, Relations, and Cardinality, Definition 2.6.5); and, for one review exercise, countable sets, Chapter 3 (Sets, Relations, and Cardinality, Theorem 3.2.2). From *Logic and Proof*: the quantifier rules and the negation of quantified statements, Chapters 5 and 6 (Logic and Proof, Definition 5.4.1, Logic and Proof, Definition 5.5.1, Logic and Proof, Definition 5.6.1, Logic and Proof, Proposition 6.2.3, Logic and Proof, Proposition 6.2.2), and the catalogue of illegal steps, Chapter 8, for the find-the-error exercises (Logic and Proof, Example (What is not allowed)).

Time. About 25 hours on the core track (all of A, one B and one C per section, the exit checks); about 30 hours for everything.

7.-1.1 Roadmap

- **Cuts.** The set below the gap at $\sqrt{2}$; the definition of a cut by three conditions, non-empty and not all of $\mathbb{Q}$, closed downward, no largest element, and the set $\mathbb{R}_D$ of all cuts; the rational cuts D_q and the cut σ of the gap; why each condition is there, with a set that has a largest element as the cautionary case; and the two lemmas the rest of the chapter rests on, separation of inside and outside and the walk up by ε .
- **Order by Inclusion, and Least Upper Bounds by Unions.** Cuts ordered by inclusion, which recovers the order of $\mathbb{Q}$ on rational cuts and is total because of separation; and the theorem that a non-empty set of cuts bounded above has a supremum, its union, with each hypothesis shown to be needed.
- **Addition and Negatives.** The sum of two cuts as the set of sums, and the zero cut D_0 ; the natural recipe for a negative, which fails on the rational cuts, and its repair by a positive margin; the key proof that a cut plus its negative is D_0 , by walking up; and the cuts as an ordered abelian group.
- **Multiplication.** Why the set of all products fails; the product of positive cuts, its laws and the walk up in multiplicative form; the extension to all cuts by absolute value and sign, through one lemma of sign rules; distributivity, the reciprocal, and the theorem that the cuts form an ordered field.
- **The Rationals Inside the Cuts, and a Complete Ordered Field.** The rational cuts as the copy of $\mathbb{Q}$, with its order and arithmetic; density of the rational cuts and the Archimedean property; the cut σ squares to D_2 and is not rational; and the cuts as a complete ordered field, by the union theorem of Section 7.2.
- **The Two Roads Give the Same Real Numbers.** The unique order-preserving field isomorphism from Chapter 5's field to the cuts, computed explicitly as “the rationals below”, and its inverse as a supremum; the two square roots of 2 matched; a table of what each construction pays for; and review exercises that return to separation by rationals (Chapter 6), limits and Cauchy sequences in an ordered field (Chapter 4), and countability (*Sets, Relations, and Cardinality*, Chapter 3).

7.-1.2 Named moves

- **Walk up by ε .** From a point inside a cut, add ε until you leave; the last step inside and the first step outside are ε apart (Lemma 7.1.3, Section 7.1). It proves the hard $\supseteq$ halves of the chapter: in full for the additive inverse in Section 7.3, and in a multiplicative form (Lemma 7.4.2) for the reciprocal in Section 7.4 and for $\sigma\sigma = D_2$ in Section 7.5. Exercise B2 of Section 7.4 leaves you its last step, and Exercise C2 of Section 7.5 (a cube root of 2) its last half.
- **A cut is a downset with an open top.** Inside is closed downward, outside is closed

upward, everything inside lies below everything outside, and the inside has no largest element (Lemma 7.1.1, Section 7.1). Most “is it a cut?” checks are this sentence read clause by clause; it also makes inclusion a total order in Section 7.2 and bounds sums and products from above in Sections 7.3 and 7.4.

- **Reduce to the positive case.** Define and prove for positive cuts, then extend to all cuts by signs once, through one lemma (Lemma 7.4.4, Section 7.4), never axiom by axiom. It gives the product and the reciprocal of Section 7.4, distributivity by moving the negative summand to the other side, and the products of rational cuts in Section 7.5.