

7.1 Cuts

This section opens Chapter 7, which is optional: nothing outside the chapter depends on it, and the volume's main line ended with Chapter 6. Chapter 5 built a real number as a class of Cauchy sequences of rationals, and Chapter 6 proved that any two complete ordered fields are linked by exactly one order-preserving field isomorphism. Here we begin a second construction, Dedekind's, in which a real number is a single set of rationals: the set of all rationals **below** it. This section defines these sets, called cuts, tests the definition on the rationals and on the gap that Section 2.6 found, and proves two lemmas on which the rest of the chapter rests. The inside and the outside of a cut are separated, and one can walk up through a cut in steps of any fixed size until one steps out.

Throughout the chapter, lower-case Latin letters such as p, q, r, s stand for rationals and m, n, k for natural numbers; Section 7.6, which brings back the field of Chapter 5, says where that changes. Nothing about real numbers is assumed: they are what is being built.

7.1.1 The set below the gap

Section 2.6 proved that the set $A := \{x \in \mathbb{Q} : x^2 < 2\}$ is non-empty and bounded above in $\mathbb{Q}$, and has no supremum in $\mathbb{Q}$ (Theorem 2.6.2 (a)). Its rational upper bounds crowd down toward a place where $\mathbb{Q}$ has no element. Chapter 5 filled that place with a new object, a class of sequences. Dedekind noticed that something already present in $\mathbb{Q}$ describes the place exactly: the set of rationals that lie below it.

That set is not quite A . The set A contains -1 , since $(-1)^2 = 1 < 2$, but not -2 , since $(-2)^2 = 4 > 2$; it is a window of rationals around 0, not “everything below” anything. Adding the non-positive rationals repairs this. Let

$$\sigma := \{p \in \mathbb{Q} : p \leq 0 \text{ or } p^2 < 2\}.$$

This set has the same upper bounds in $\mathbb{Q}$ as A . It contains A , so every upper bound of σ bounds A . Conversely, let u be an upper bound of A ; since $1 \in A$, $u \geq 1 > 0$, and every element p of σ outside A has $p \leq 0 < u$. The conditions (S1) and (S2) of a supremum mention only the set of upper bounds (Sets, Relations, and Cardinality, Definition 2.6.5), so σ , like A , has no supremum in $\mathbb{Q}$; this is the argument of Theorem 2.6.2 (b).

So no rational sits at the top of σ , yet σ itself is a perfectly good set of rationals. Dedekind's move is to **take the set as the number**. The rest of the chapter makes this precise: it says which sets of rationals qualify, orders them, adds and multiplies them, and proves that they form a complete ordered field.

7.1.2 Dedekind cuts

Which subsets of $\mathbb{Q}$ should count as “the set of rationals below a number”? Three features of σ suggest the answer. It is neither empty nor all of $\mathbb{Q}$: a number should have rationals below it and rationals above it. With any rational it contains every smaller one. And it has no largest element, as [Example \(The cut of the gap\)](#) below shows. The third feature is the least obvious, and the paragraph “Why this definition” after the examples explains it.

A real number is the set of rationals below it.

Definition 7.1.1. Dedekind Cut

A subset $\alpha \subseteq \mathbb{Q}$ is a **cut** (or **Dedekind cut**) if

- (C1) $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$;
- (C2) **for every** $p \in \alpha$ and **every** $q \in \mathbb{Q}$ with $q < p$, $q \in \alpha$;
- (C3) α has **no largest element**: **for every** $p \in \alpha$ **there exists** $p' \in \alpha$ with $p' > p$.

The set of all cuts is

$$\mathbb{R}_D := \{\alpha \in \mathcal{P}(\mathbb{Q}) : \alpha \text{ satisfies (C1), (C2) and (C3)}\},$$

a subset of the power set $\mathcal{P}(\mathbb{Q})$ (Sets, Relations, and Cardinality, Definition 1.2.2).

In symbols, with bounded quantifiers,

- (C1) $\exists p \in \mathbb{Q} (p \in \alpha) \wedge \exists q \in \mathbb{Q} (q \notin \alpha)$,
- (C2) $\forall p \in \alpha \forall q \in \mathbb{Q} (q < p \Rightarrow q \in \alpha)$,
- (C3) $\forall p \in \alpha \exists p' \in \alpha (p' > p)$.

In words, clause by clause. (C1) says that the cut is a proper, non-empty part of $\mathbb{Q}$: some rational is in it and some rational is not. (C2) says that the cut is **closed downward**: whatever rational it contains, it contains every smaller one too. (C3) says that it has **no top**: above each of its elements it has another. The letters $\alpha, \beta, \gamma, \delta$ are reserved for cuts in this chapter.

Well-definedness. The collection $\mathbb{R}_D$ is a set: it is carved out of the set $\mathcal{P}(\mathbb{Q})$ by a condition, which (Z6) of Axiom 1.1.1 allows, and $\mathcal{P}(\mathbb{Q})$ exists by (Z5). A real number in the sense of this chapter is therefore literally an element of $\mathcal{P}(\mathbb{Q})$, a subset of $\mathbb{Q}$.

Negations. We shall often show that a set is **not** a cut, so the three negations are worth writing out once. By Logic and Proof, Proposition 6.2.3 and Logic and Proof, Proposition 6.2.2, a subset $\alpha \subseteq \mathbb{Q}$ fails

- (C1) if $\alpha = \emptyset$ or $\alpha = \mathbb{Q}$;
- (C2) if $\exists p \in \alpha \exists q \in \mathbb{Q} (q < p \wedge q \notin \alpha)$: some element has a smaller rational missing;
- (C3) if $\exists p \in \alpha \forall p' \in \alpha (p' \leq p)$: α has a **largest element**.

In the last line, “not $p' > p$ ” became $p' \leq p$ by trichotomy in $\mathbb{Q}$ (Proposition 2.3.5 (i)).

The first examples are the rationals themselves. For a rational q , the rationals below q form a cut, and this is how the old numbers will sit among the new ones.

Definition 7.1.2. Rational Cut

For $q \in \mathbb{Q}$, let

$$D_q := \{r \in \mathbb{Q} : r < q\}.$$

It is a cut (checked below), the **rational cut** of q . A cut α is **rational** if $\alpha = D_q$ for some $q \in \mathbb{Q}$.

D_q is a cut. (C1): $q - 1 \in D_q$, since $q - (q - 1) = 1 > 0$ (Proposition 2.3.5 (e), as $1 \neq 0$ in the field $\mathbb{Q}$, Theorem 2.5.1), so $q - 1 < q$ by Proposition 2.3.5 (a); and $q \notin D_q$, since $q < q$ fails by (O1). (C2): if $p \in D_q$ and $r < p$, then $r < p < q$, so $r < q$ by (O2), and $r \in D_q$. (C3): if $r \in D_q$,

the midpoint $(r + q)/2$ satisfies $r < (r + q)/2 < q$ (Section 2.5, “Between two rationals”), so it lies in D_q and is larger than r .

The letter D is for Dedekind.

Example. The cut of the gap

Show that $\sigma = \{p \in \mathbb{Q} : p \leq 0 \text{ or } p^2 < 2\}$ is a cut.

Solution. (C1): $0 \in \sigma$, since $0 \leq 0$. And $2 \notin \sigma$: $2 > 0$, and $2^2 = 4 > 2$.

(C2): let $p \in \sigma$ and $q \in \mathbb{Q}$ with $q < p$. If $q \leq 0$, then $q \in \sigma$. Otherwise $0 < q < p$. Then $p > 0$, so $p \leq 0$ fails and $p^2 < 2$. Since $0 \leq q < p$, Proposition 2.3.5 (g) gives $q^2 < p^2$, so $q^2 < 2$ by (O2), and $q \in \sigma$.

(C3): let $p \in \sigma$. If $p \leq 0$, then $1 \in \sigma$ (as $1^2 = 1 < 2$) and $1 > 0 \geq p$. If $p > 0$, then $p^2 < 2$. Put

$$d := 2 - p^2, \quad h := \frac{d}{2p + 2}.$$

Here $d > 0$, and $d < 2$ because $p^2 > 0$ (Proposition 2.3.5 (e)); also $2p + 2 > 2$, since $p > 0$. So $0 < h < 1$ (Proposition 2.5.2 (a) for $h > 0$; $d < 2p + 2$ for $h < 1$). From $0 < h < 1$, $h^2 < h$ by Proposition 2.3.5 (c). Hence

$$\begin{aligned} (p + h)^2 &= p^2 + 2ph + h^2 < p^2 + 2ph + h \\ &< p^2 + h(2p + 2) = p^2 + d = 2, \end{aligned}$$

where the second inequality adds $h > 0$ (O3). So $p + h \in \sigma$, and $p + h > p$.

The step size h depends on p : the closer p^2 is to 2, the smaller d and so h . That is unavoidable, since a fixed step would jump out of σ from points near the top. Section 7.5 proves that σ is not a rational cut, $\sigma \neq D_q$ for every $q \in \mathbb{Q}$, and Exercise C1 below asks for a direct proof. So σ is a genuinely new number: the cut that will square to 2.

Quick check. Which of the following subsets of $\mathbb{Q}$ are cuts? For each that is not, name a condition that fails.

- (a) $\{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 4\}$;
- (b) $\{p \in \mathbb{Q} : p^2 < 2\}$;
- (c) $\{p \in \mathbb{Q} : p^3 < 5\}$;
- (d) $\emptyset$.

Solution.

- (a) It is a cut, because it equals D_2 . If $p < 0$, then $p < 2$. If $p \geq 0$ and $p^2 < 4 = 2^2$, then $p < 2$ by Proposition 2.3.5 (g). Conversely, if $p < 2$, either $p < 0$, or $0 \leq p < 2$ and then $p^2 < 4$ by (g).
- (b) Not a cut: (C2) fails. The set contains 1, and $-2 < 1$, but $(-2)^2 = 4 > 2$, so -2 is missing.

- (c) A cut. Call the set E . *Cubes are monotone on non-negatives*: for $0 \leq q < p$, $p^3 - q^3 = (p - q)(p^2 + pq + q^2)$, where $p - q > 0$ and $p^2 + pq + q^2 \geq p^2 > 0$ (Proposition 2.3.5 (e), with $pq, q^2 \geq 0$); so $p^3 - q^3 > 0$ by (O4). (C1): $0 \in E$, and $2 \notin E$, as $2^3 = 8 > 5$. (C2): let $q < p \in E$. If $q \leq 0$, then $q^3 \leq 0 < 5$ ($q^3 = q \cdot q^2$, a non-positive times a non-negative). If $q > 0$, then $0 < q < p$ and $q^3 < p^3 < 5$. (C3): let $p \in E$. If $p \leq 0$, then $1 \in E$ is larger. If $p > 0$, put $d := 5 - p^3$ and $h := d/(3p^2 + 3p + 5)$. Then $0 < d < 5 < 3p^2 + 3p + 5$, so $0 < h < 1$, hence $h^3 < h^2 < h$ by Proposition 2.3.5 (c), and

$$\begin{aligned} (p + h)^3 &= p^3 + 3p^2h + 3ph^2 + h^3 \\ &< p^3 + h(3p^2 + 3p + 1) < p^3 + d = 5. \end{aligned}$$

So $p + h \in E$ and $p + h > p$.

- (d) Not a cut: (C1) fails, since $\emptyset$ has no element.

Why this definition. Each clause excludes something that would spoil the arithmetic to come.

- Without (C1), $\emptyset$ and $\mathbb{Q}$ would be numbers, below and above every rational. Section 7.3 adds cuts as sets, and $\mathbb{Q} + D_0 = \mathbb{Q} = \mathbb{Q} + D_1$ there would make cancellation fail.
- Without (C2), $\{p : p^2 < 2\}$ of the Quick check would be a number. It contains 1 but not the smaller rational -2 , so it is not the set of **all** rationals below anything.
- Without (C3), each rational q would be represented twice, by D_q and by $\{r \in \mathbb{Q} : r \leq q\}$. The next example shows that this is more than untidy: the sum of sets cannot tell the two apart.

Example. Two candidates for one rational

Optional: nothing later depends on this.

Let $E_1 := \{r \in \mathbb{Q} : r \leq 1\}$. Show that E_1 satisfies (C1) and (C2) but not (C3), and that, with the sum of sets $X + Y := \{x + y : x \in X, y \in Y\}$ of *Sets, Relations, and Cardinality*, Section 1.6 (Sets, Relations, and Cardinality, Example (Reflections, scalings and sums as images)),

$$E_1 + D_1 = D_2 = D_1 + D_1, \quad \text{although } E_1 \neq D_1.$$

Solution. The conditions. (C1): $0 \in E_1$ and $2 \notin E_1$. (C2): $q < p \leq 1$ gives $q < 1$. (C3) fails: $1 \in E_1$, and every $p' \in E_1$ has $p' \leq 1$, so 1 is a largest element. And $E_1 \neq D_1$, since $1 \in E_1 \setminus D_1$.

$E_1 + D_1 = D_2$. ($\subseteq$): if $a \leq 1$ and $b < 1$, then $a + b < 2$ by Proposition 2.3.5 (b). ($\supseteq$): if $s < 2$, then $s = 1 + (s - 1)$ with $1 \in E_1$ and $s - 1 < 1$, so $s - 1 \in D_1$.

$D_1 + D_1 = D_2$. ($\subseteq$): $a < 1$ and $b < 1$ give $a + b < 2$, by (b) again. ($\supseteq$): if $s < 2$, then $s = s/2 + s/2$, and multiplying $s < 2$ by $1/2 > 0$ (Proposition 2.3.5 (c)) gives $s/2 < 1$, so $s/2 \in D_1$.

So if E_1 were admitted as a number, the equation $\xi + D_1 = D_2$ would have two solutions, E_1 and D_1 , and addition could not be canceled, so the numbers could not form a group under addition.

Condition (C3) chooses between the two sets $\{r \in \mathbb{Q} : r < 1\}$ and $\{r \in \mathbb{Q} : r \leq 1\}$ that describe

“the rationals up to 1”: it keeps the one that leaves 1 out.

Warning. A cut is a set of rationals, not a rational. “ $\sigma < 2$ ” compares a set with a number and means nothing yet. Section 7.2 orders cuts, so that $\sigma < D_2$ acquires a meaning, and Section 7.5 shows that D_2 plays the part of the rational 2 among the cuts. Until then, every comparison in this chapter is between rationals, or between sets by inclusion.

7.1.3 Inside and outside

The defining conditions speak about the inside of a cut. The first lemma says what they imply about the outside: everything outside lies above everything inside.

Lemma 7.1.1. Separation

Let α be a cut, $p \in \alpha$ and $q \in \mathbb{Q} \setminus \alpha$. Then $p < q$. Consequently the complement $\mathbb{Q} \setminus \alpha$ is **closed upward**: if $q \notin \alpha$ and $q' \in \mathbb{Q}$ with $q' > q$, then $q' \notin \alpha$.

Proof. Suppose $p < q$ fails. By trichotomy (Proposition 2.3.5 (i)), $q \leq p$. If $q = p$, then $q \in \alpha$; if $q < p$, then $q \in \alpha$ by (C2). Either way $q \in \alpha$, against $q \notin \alpha$. So $p < q$.

For the consequence, let $q \notin \alpha$ and $q' > q$, and suppose $q' \in \alpha$. The first part, with q' in the role of p , gives $q' < q$, which together with $q < q'$ contradicts (O1). So $q' \notin \alpha$. ■

So a cut and its complement split $\mathbb{Q}$ into a lower piece and an upper piece, every element of the lower below every element of the upper. In particular every rational outside a cut is an upper bound of it. Together with (C3), this gives the picture to keep in mind, and one of the chapter’s named moves:

A cut is a downset with an open top.

Inside is closed downward, outside is closed upward, everything inside lies below everything outside, and the inside has no largest element. The outside may have a least element: the complement of D_q is $\{r \in \mathbb{Q} : r \geq q\}$, with least element q . Most “is it a cut?” checks in this chapter are this sentence, read clause by clause.

The second result locates a rational relative to a cut by comparing two sets. It is the bridge between a rational q and its cut D_q .

Proposition 7.1.2. Where a Rational Lies

Let α be a cut and $q \in \mathbb{Q}$. Then

- (a) $q \in \alpha$ if and only if $D_q \subsetneq \alpha$;
- (b) $q \notin \alpha$ if and only if $\alpha \subseteq D_q$.

Idea. If q is in α , closure downward brings everything below q along, and q itself is the extra element. If q is outside, separation puts all of α below q . The two converses come from the same two facts.

Proof.

- (a) $(\Rightarrow)$ Let $q \in \alpha$. For $r \in D_q$, $r < q$, so $r \in \alpha$ by (C2); thus $D_q \subseteq \alpha$. The inclusion is proper, because $q \in \alpha$ and $q \notin D_q$ (as $q < q$ fails, (O1)). $(\Leftarrow)$ Let $D_q \subsetneq \alpha$. Then some $r \in \alpha$ has $r \notin D_q$, that is, $r < q$ fails, so $q \leq r$ (Proposition 2.3.5 (i)). If $q = r$, then $q \in \alpha$; if $q < r$, then $q \in \alpha$ by (C2).
- (b) $(\Rightarrow)$ Let $q \notin \alpha$. For $r \in \alpha$, Lemma 7.1.1 gives $r < q$, so $r \in D_q$; thus $\alpha \subseteq D_q$. $(\Leftarrow)$ Let $\alpha \subseteq D_q$. Since $q \notin D_q$, also $q \notin \alpha$. ■

Since every rational is either in α or not, exactly one of $D_q \subsetneq \alpha$ and $\alpha \subseteq D_q$ holds. Section 7.2 rewrites (a) as $q \in \alpha \iff D_q < \alpha$, once inclusion has become the order on cuts.

7.1.4 Walking up through a cut

Many proofs in this chapter need two rationals, one inside a cut and one outside, that are **close together**: within a prescribed rational distance $\varepsilon > 0$. Separation says that the inside lies below the outside, but not that the two come close. They do, and the reason is a walk: start inside, climb in steps of ε , and stop at the first step that leaves the cut.

Lemma 7.1.3. The Step Lemma

Let α be a cut, $a \in \alpha$, and $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$. Then there is $n \in \mathbb{N}$ with

$$a + n\varepsilon \in \alpha \quad \text{and} \quad a + (n+1)\varepsilon \notin \alpha.$$

Idea. Picture the rungs $a, a+\varepsilon, a+2\varepsilon, \dots$. The bottom rung a is inside. Some rung is outside: pick any rational b outside α ; the rationals are Archimedean, so finitely many steps of size ε climb past b , and everything above b is outside by separation. Among the rungs outside there is a lowest one, because a non-empty set of natural numbers has a least element. It is not the bottom rung, so the rung just below it exists and is inside. The n found depends on α , a and ε .

Proof. By (C1), there is $b \in \mathbb{Q}$ with $b \notin \alpha$. By Theorem 2.5.3, applied to the rational $(b-a)/\varepsilon$, there is $m \in \mathbb{N}$ with $(b-a)/\varepsilon < m$. Multiplying by $\varepsilon > 0$ (Proposition 2.3.5 (c)) gives $b-a < m\varepsilon$, so $b < a+m\varepsilon$ by (O3). Since $b \notin \alpha$, Lemma 7.1.1 gives $a+m\varepsilon \notin \alpha$.

Let

$$K := \{k \in \mathbb{N} : a + k\varepsilon \notin \alpha\}.$$

Then $m \in K$, so K is a non-empty set of integers, and 0 is a lower bound of it. By Proposition 2.3.6, K has a least element k_0 . Since $a + 0 \cdot \varepsilon = a \in \alpha$, $0 \notin K$, so $k_0 \neq 0$ and

$k_0 > 0$. By discreteness (Theorem 2.3.4 (c)), $k_0 \geq 1$. Put $n := k_0 - 1$. Then $n \geq 0$ by (O3), so $n \in \mathbb{N}$ (Theorem 2.3.4 (b)). Since $n < k_0$ and k_0 is least in K , $n \notin K$, that is, $a + n\varepsilon \in \alpha$. And $a + (n + 1)\varepsilon = a + k_0\varepsilon \notin \alpha$, since $k_0 \in K$. This proves the lemma. ■

Since $n \geq 0$ and $\varepsilon > 0$, the point $p := a + n\varepsilon$ found by the lemma satisfies $p \geq a$. So the lemma hands over a pair $p \in \alpha$, $p + \varepsilon \notin \alpha$: a point inside and a point outside, exactly ε apart, with the inside point as high as we like above a given element. This is another of the chapter's named moves.

Walk up by ε : from a point inside a cut, add ε until you leave; the last step inside and the first step outside are ε apart.

It drives the hardest “ $\supseteq$ ” halves to come: the additive inverse in Section 7.3 and, in its multiplicative form of Section 7.4, the multiplicative inverse in Section 7.4 and the square of σ in Section 7.5. Note what it needs from $\mathbb{Q}$: the Archimedean property, to know that the walk leaves the cut at all, and least elements and discreteness of the integers, to find the first step out and the step before it.

Here is one walk, through σ from $a = 1$ in steps of $\varepsilon = 1/10$. The rungs $1, 11/10, 12/10, \dots$ are positive, so a rung r lies in σ exactly when $r^2 < 2$. Now $(14/10)^2 = 196/100 < 2$, because $196 < 200$, and the lower positive rungs have smaller squares (Proposition 2.3.5 (g)), so the rungs for $k = 0, \dots, 4$ are inside. But $(15/10)^2 = 225/100 > 2$. So the least k with $1 + k/10 \notin \sigma$ is $k_0 = 5$, the n of the lemma is 4, and the pair is $7/5 \in \sigma$, $3/2 \notin \sigma$.

Quick check. Continue the walk with a finer step: start at $a = 7/5 \in \sigma$ and take $\varepsilon = 1/100$. Which pair $p \in \sigma$, $p + \varepsilon \notin \sigma$ does the lemma give?

Solution. The rungs are $140/100, 141/100, 142/100, \dots$. Since $141^2 = 19881 < 20000$, $(141/100)^2 < 2$, so $141/100 \in \sigma$. Since $142^2 = 20164 > 20000$, $142/100 \notin \sigma$. So $k_0 = 2$, $n = 1$, and the pair is $p = 141/100$, $p + \varepsilon = 142/100 = 71/50$.

Each walk traps the top of σ between two rationals ε apart, and a smaller ε traps it more tightly. No walk ever lands on the top, since σ has none.

7.1.5 Exercises

7.1.5.1 A. Check your understanding

Exercise. A1

- (a) State (C1)–(C3) for a subset $\alpha \subseteq \mathbb{Q}$, and write the negation of (C3) in symbols.
- (b) For each set, name every condition among (C1)–(C3) that fails: (i) $\{p \in \mathbb{Q} : p \leq 3\}$; (ii) $\{p \in \mathbb{Q} : p > 0\}$; (iii) $\{p \in \mathbb{Q} : p^2 \leq 0\}$; (iv) $\{p \in \mathbb{Q} : p < p + 1\}$.
- (c) Let $q, r \in \mathbb{Q}$. Is $D_q \cup D_r$ a cut? Is $D_q \cap D_r$? Identify each.

Solution.

- (a) (C1) $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$; (C2) for every $p \in \alpha$ and every $q \in \mathbb{Q}$ with $q < p$, $q \in \alpha$; (C3) for every $p \in \alpha$ there is $p' \in \alpha$ with $p' > p$. The negation of (C3) is $\exists p \in \alpha \forall p' \in \alpha (p' \leq p)$: α has a largest element.
- (b) (i) Only (C3) fails: 3 is a largest element. (C1) holds (0 in, 4 out) and (C2) holds ($q < p \leq 3$ gives $q < 3$). (ii) Only (C2) fails: 1 is in, and $0 < 1$ is not. (C1) holds (1 in, 0 out), and (C3) holds ($p+1 > p > 0$). (iii) The set is $\{0\}$, since $p \neq 0$ gives $p^2 > 0$ (Proposition 2.3.5 (e)). (C2) fails ($-1 < 0$, -1 missing) and (C3) fails (0 is largest); (C1) holds. (iv) The set is $\mathbb{Q}$, since $0 < 1$ gives $p < p+1$ for every p (O3). (C1) fails; (C2) and (C3) hold.
- (c) Both are cuts. If $q \leq r$, then $D_q \subseteq D_r$ ($s < q \leq r$ gives $s < r$), so $D_q \cup D_r = D_r$ and $D_q \cap D_r = D_q$; if $r \leq q$, the roles swap. So $D_q \cup D_r = D_{\max(q,r)}$ and $D_q \cap D_r = D_{\min(q,r)}$, rational cuts by Definition 7.1.2.

7.1.5.2 B. Practice

Exercise. B1

Prove that $\{p \in \mathbb{Q} : 3p < 2\}$ is a cut, and that it equals $D_{2/3}$.

Solution. For $p \in \mathbb{Q}$, $3p < 2$ implies $p < 2/3$, by multiplying with $1/3 > 0$ (Proposition 2.5.2 (a), Proposition 2.3.5 (c)); and $p < 2/3$ implies $3p < 2$, by multiplying with $3 > 0$. So the set is $\{p \in \mathbb{Q} : p < 2/3\} = D_{2/3}$, and it is a cut because every rational cut is one (Definition 7.1.2, with the check after it).

Exercise. B2

Prove that $\{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 3\}$ is a cut. For (C3), give an explicit larger element.

Solution. Call the set β . (C1): $0 \in \beta$, since $0^2 = 0 < 3$; and $2 \notin \beta$, since $2 \geq 0$ and $2^2 = 4 > 3$.

(C2): let $q < p \in \beta$. If $q < 0$, then $q \in \beta$. Otherwise $0 \leq q < p$, so $p > 0$, $p < 0$ fails, and $p^2 < 3$. By Proposition 2.3.5 (g), $q^2 < p^2 < 3$, so $q \in \beta$.

(C3): let $p \in \beta$. If $p < 0$, then $0 \in \beta$ and $0 > p$. If $p \geq 0$, then $p^2 < 3$. Put

$$d := 3 - p^2, \quad h := \frac{d}{2p + 4}.$$

Then $0 < d \leq 3 < 4 \leq 2p + 4$, so $0 < h < 1$ and $h^2 < h$ (Proposition 2.3.5 (c)). Hence

$$\begin{aligned} (p + h)^2 &= p^2 + 2ph + h^2 < p^2 + h(2p + 1) \\ &< p^2 + h(2p + 4) = p^2 + d = 3, \end{aligned}$$

so $p + h \in \beta$ and $p + h > p$. (The denominator is chosen to exceed d , so that $h < 1$, and to be at least $2p + 1$; any such denominator works.)

7.1.5.3 C. Going deeper

Exercise. C1: The cut of the gap is not rational

Prove directly from the definitions that $\sigma \neq D_q$ for every $q \in \mathbb{Q}$.

Solution (Hint 1). If $\sigma = D_q$, where would q itself sit relative to σ ? What does that say about q and q^2 ?

Solution (Hint 2). You will reach $q > 0$ and $q^2 \geq 2$. If $q^2 > 2$, look for a smaller positive rational whose square still exceeds 2; Section 2.6 has one.

Solution. Suppose $\sigma = D_q$ for some $q \in \mathbb{Q}$. Since $q \notin D_q$, $q \notin \sigma$: so $q \leq 0$ fails, and $q^2 < 2$ fails. That is, $q > 0$ and $q^2 \geq 2$.

Case $q^2 = 2$. Impossible, by Theorem 2.6.1.

Case $q^2 > 2$. Let $q' := (q^2 + 2)/(2q)$, the rational $t(q)$ of Section 2.6. It is positive, as a quotient of positives (Proposition 2.5.2 (a), (O4)). By the identities Eq. 2.6.1,

$$q - q' = \frac{q^2 - 2}{2q} > 0, \quad q'^2 - 2 = \frac{(q^2 - 2)^2}{4q^2} > 0,$$

the first a quotient of positives, the second because $q^2 - 2 \neq 0$ (Proposition 2.3.5 (e)). So $q' < q$, that is, $q' \in D_q = \sigma$. But $q' > 0$ and $q'^2 > 2$, so $q' \notin \sigma$. This contradiction shows that $\sigma \neq D_q$.

Marking scheme. (10 points) - 3: from $q \notin D_q = \sigma$, derives both $q > 0$ and $q^2 \geq 2$. - 2: excludes $q^2 = 2$, citing that no rational squares to 2. - 3: for $q^2 > 2$, exhibits a rational q' with $0 < q' < q$ and $q'^2 > 2$, with both inequalities verified (any such q' earns the points). - 2: concludes: q' lies in D_q but not in σ . Fatal errors (0 for the step): “ $q = \sqrt{2}$ ” or any use of a square root of 2 as a number; a candidate q' whose square is not checked to exceed 2.