

6.2 Building the Map from the Rationals Below

Section 6.1 gave every element x of an ordered field F a record, the set $\mathbb{Q}_F(x)$ of rationals below it, and showed that in an Archimedean field the record determines the element (Lemma 6.1.3). This section turns the record into a map. Given an Archimedean field F and a complete field L , we send $x \in F$ to the element of L with the same record. The result is the main theorem of the chapter in its one-sided form: there is exactly one order-preserving field map $F \rightarrow L$. Section 6.3 applies it twice, in both directions, to compare two complete ordered fields.

Two fields are in play throughout, so every proof here keeps ι_F and ι_L visible. Only the two lemmas, each about one field, write q for $\iota_F(q)$.

6.2.1 What the map must be

Suppose $\varphi: F \rightarrow L$ is an order-preserving field map (Definition 6.1.1). By Proposition 6.1.1 it fixes the rationals, $\varphi(\iota_F(q)) = \iota_L(q)$, and it preserves and reflects order. So for $q \in \mathbb{Q}$ and $x \in F$,

$$\iota_F(q) < x \iff \varphi(\iota_F(q)) < \varphi(x) \iff \iota_L(q) < \varphi(x).$$

In the notation of Section 6.1: $\mathbb{Q}_L(\varphi(x)) = \mathbb{Q}_F(x)$. *The image of x has exactly the rationals below it that x has.*

That already shows there can be at most one such map when L is Archimedean: two candidates for $\varphi(x)$ would have the same rationals below them, and rationals separate. For existence, we need an element of L with a prescribed set of rationals below it. When L is complete there is a natural candidate: the least element of L that lies above all the copies $\iota_L(q)$, $q \in \mathbb{Q}_F(x)$, that is, their supremum.

The check that this candidate respects sums and products goes through the record, not through suprema. For that we need to know how the rationals below $a + b$ and below ab are made from those below a and b .

6.2.2 Sums and products, measured by rationals

A rational q lies below $a + b$ exactly when it splits into a rational below a and a rational below b .

Lemma 6.2.1. Rationals below a Sum

Let F be an Archimedean ordered field, $a, b \in F$ and $q \in \mathbb{Q}$. Then $\iota_F(q) < a + b$ if and only if there is $r \in \mathbb{Q}$ with

$$\iota_F(r) < a \quad \text{and} \quad \iota_F(q - r) < b.$$

In other words, $q \in \mathbb{Q}_F(a + b) \iff \exists r \in \mathbb{Q} (r \in \mathbb{Q}_F(a) \wedge q - r \in \mathbb{Q}_F(b))$.

Idea. One direction is addition of inequalities. For the other, split q as $r + (q - r)$ with r a rational just below a : we need r in the gap between $q - b$ and a , which is non-empty exactly because $q < a + b$, and density supplies it.

Proof. We write q for $\iota_F(q)$; since ι_F preserves sums, it preserves differences, so $\iota_F(q - r) = q - r$ in F .

($\Leftarrow$) If $r < a$ and $q - r < b$, then $q = r + (q - r) < a + b$ by Proposition 2.3.5 (b).

($\Rightarrow$) Let $q < a + b$. Then $q - b < a$ (Proposition 2.3.5 (a)). By Theorem 4.2.5 there is $r \in \mathbb{Q}$ with $q - b < r < a$. Then $r < a$, and $q - r < b$ by Proposition 2.3.5 (a) again. ■

For products the same idea works for positive elements and positive rationals, with q/r in place of $q - r$.

Lemma 6.2.2. Rationals below a Product

Let F be an Archimedean ordered field, $a, b \in F$ with $a > 0$ and $b > 0$, and $q \in \mathbb{Q}$ with $q > 0$. Then $\iota_F(q) < ab$ if and only if there is $r \in \mathbb{Q}$ with

$$r > 0, \quad \iota_F(r) < a \quad \text{and} \quad \iota_F(q/r) < b.$$

For $q \leq 0$, $\iota_F(q) < ab$ holds always.

In words: below a product of positives, a positive rational splits as a product of a rational below each factor. Exercise B1 asks for the proof; the fold below has it.

🕒 **Second reading** · The proof of the product lemma (try Exercise B1 first)

Idea. As for sums, with division in place of subtraction: split q as $r \cdot (q/r)$ with r a rational in the gap between q/b and a , a gap of positive elements.

Proof. We write q for $\iota_F(q)$; since ι_F preserves products and $\iota_F(r) \neq 0$ for $r \neq 0$, it preserves quotients, so $\iota_F(q/r) = q/r$ in F .

If $q \leq 0$, then $q \leq 0 < ab$ by (O4). Let $q > 0$.

($\Leftarrow$) Let $r > 0$ with $r < a$ and $q/r < b$. Since $q/r > 0$ (Proposition 2.5.2 (a), (O4)), Proposition 2.3.5 (c) gives $q = r \cdot (q/r) < a \cdot (q/r)$; since $a > 0$, it gives $a \cdot (q/r) < ab$. So $q < ab$.

($\Rightarrow$) Let $q < ab$. Multiplying by $b^{-1} > 0$ (Proposition 2.5.2 (a), Proposition 2.3.5 (c)) gives $q/b < a$, and $q/b > 0$ by (O4). By Theorem 4.2.5 there is $r \in \mathbb{Q}$ with $q/b < r < a$. Then $r > q/b > 0$, so $r > 0$ in $\mathbb{Q}$, since ι_F reflects order. Multiplying $q/b < r$ by $b > 0$ gives $q < rb$, and multiplying by $r^{-1} > 0$ gives $q/r < b$. ■

Quick check. In $F = \mathbb{Q}$, with $a := 3/4$, $b := 1/2$ and $q := 1$, find a rational r as in Lemma 6.2.1. Which rationals r work?

Solution. We need $r < 3/4$ and $1 - r < 1/2$, that is, $1/2 < r < 3/4$. For example $r = 2/3$: then $1 - 2/3 = 1/3 < 1/2$. Exactly the rationals of the open gap between $q - b = 1/2$ and $a = 3/4$ work, as the proof predicts.

6.2.3 The map into a complete field

Here is the theorem. Its proof is the first key proof of the chapter.

Theorem 6.2.3. Embedding an Archimedean Field in a Complete One

Let F be an Archimedean ordered field and L a complete ordered field. Then there is **exactly one** order-preserving field map $\varphi: F \rightarrow L$. It is given by

$$\varphi(x) := \sup_L \{\iota_L(q) : q \in \mathbb{Q}_F(x)\} \quad (x \in F),$$

the supremum taken in L , and it satisfies

$$\mathbb{Q}_L(\varphi(x)) = \mathbb{Q}_F(x) \quad \text{for every } x \in F. \quad (\star)$$

Try it first (ten minutes): if φ exists and fixes $\mathbb{Q}$, which element of L must $\varphi(x)$ be?

Idea. The natural first attempt defines φ on the copy of $\mathbb{Q}$, by $\varphi(\iota_F(q)) := \iota_L(q)$, and then tries to “extend”. It stalls at once: an element $x \in F$ that is not a rational has no formula, and nothing in F names it. What we do know, from the previous subsection, is its record: $\varphi(x)$ must have exactly the rationals $\mathbb{Q}_F(x)$ below it. So we **construct a point as a supremum**: $\varphi(x)$ is the least upper bound in L of the copies of those rationals, and $(\star)$ says the construction has the right record. After that, sums, products and uniqueness are checked the same way: compute the record of both sides (for sums and products by the two lemmas) and let rationals separate in L . Monotonicity needs only one rational between x and y .

Step roadmap: ① $\varphi(x)$ exists; ② $(\star)$; ③ φ fixes the rationals; ④ φ is strictly increasing; ⑤ sums; ⑥ products and the unit; ⑦ uniqueness. *Before reading on: where is completeness of L used, and where the Archimedean property of F ?*

Proof. By Theorem 6.1.2, L is Archimedean. So Theorem 4.2.5, Lemma 6.1.3, Lemma 6.2.1 and Lemma 6.2.2 apply in L as well as in F . For $x \in F$ write $A_x := \{\iota_L(q) : q \in \mathbb{Q}_F(x)\} \subseteq L$.

Step 1: $\varphi(x)$ is defined. Let $x \in F$. Since F is Archimedean, there is $n \in \mathbb{N}$ with $-x < n \cdot 1_F$, so $\iota_F(-n) = -(n \cdot 1_F) < x$ (Proposition 2.3.5 (a)), and $-n \in \mathbb{Q}_F(x)$: the set A_x is non-empty. Applied to x , the Archimedean property gives $m \in \mathbb{N}$ with $x < m \cdot 1_F = \iota_F(m)$. For $q \in \mathbb{Q}_F(x)$, $\iota_F(q) < x < \iota_F(m)$, so $q < m$ because ι_F reflects order, and $\iota_L(q) < \iota_L(m)$. So $\iota_L(m)$ is an upper bound of A_x . Since L is complete, A_x has a supremum in L , and it is

unique; $\varphi(x)$ is that supremum. So $\varphi: F \rightarrow L$ is a function.

Step 2: $(\star)$. Let $x \in F$ and $q \in \mathbb{Q}$.

$(\subseteq)$ Let $q \in \mathbb{Q}_F(x)$. By density in F there is $r \in \mathbb{Q}$ with $\iota_F(q) < \iota_F(r) < x$. Then $r \in \mathbb{Q}_F(x)$, so $\iota_L(r) \in A_x$ and $\iota_L(r) \leq \varphi(x)$, since $\varphi(x)$ is an upper bound. Also $q < r$ (reflection in F), so $\iota_L(q) < \iota_L(r) \leq \varphi(x)$, and $q \in \mathbb{Q}_L(\varphi(x))$.

$(\supseteq)$ Let $\iota_L(q) < \varphi(x)$. Since $\varphi(x)$ is the **least** upper bound of A_x , $\iota_L(q)$ is not an upper bound, so there is $r \in \mathbb{Q}_F(x)$ with $\iota_L(q) < \iota_L(r)$ (totality). Then $q < r$ (reflection in L), so $\iota_F(q) < \iota_F(r) < x$, and $q \in \mathbb{Q}_F(x)$.

Step 3: φ fixes the rationals. Let $r \in \mathbb{Q}$. By $(\star)$ and the first example after Definition 6.1.2, used in F and in L ,

$$\mathbb{Q}_L(\varphi(\iota_F(r))) = \mathbb{Q}_F(\iota_F(r)) = \{q \in \mathbb{Q} : q < r\} = \mathbb{Q}_L(\iota_L(r)).$$

By Lemma 6.1.3 in L , $\varphi(\iota_F(r)) = \iota_L(r)$. In particular $\varphi(0_F) = 0_L$ and $\varphi(1_F) = 1_L$.

Step 4: φ is strictly increasing. Let $x < y$ in F . By density in F there is $r \in \mathbb{Q}$ with $x < \iota_F(r) < y$. Then $r \notin \mathbb{Q}_F(x)$ (by (O1)), so $r \notin \mathbb{Q}_L(\varphi(x))$ by $(\star)$, and $\varphi(x) \leq \iota_L(r)$ by totality. And $r \in \mathbb{Q}_F(y) = \mathbb{Q}_L(\varphi(y))$, so $\iota_L(r) < \varphi(y)$. Hence $\varphi(x) \leq \iota_L(r) < \varphi(y)$.

Step 5: sums. Let $x, y \in F$ and $q \in \mathbb{Q}$. Then

$$\begin{aligned} q \in \mathbb{Q}_L(\varphi(x+y)) &\iff q \in \mathbb{Q}_F(x+y) \\ &\iff \exists r \in \mathbb{Q} (r \in \mathbb{Q}_F(x) \wedge q-r \in \mathbb{Q}_F(y)) \\ &\iff \exists r \in \mathbb{Q} (r \in \mathbb{Q}_L(\varphi(x)) \wedge q-r \in \mathbb{Q}_L(\varphi(y))) \\ &\iff q \in \mathbb{Q}_L(\varphi(x) + \varphi(y)), \end{aligned}$$

by $(\star)$, Lemma 6.2.1 in F , $(\star)$ twice, and Lemma 6.2.1 in L . By Lemma 6.1.3 in L , $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Step 6: products and the unit. First let $x > 0$ and $y > 0$. By Steps 3 and 4, $\varphi(x) > \varphi(0_F) = 0_L$ and $\varphi(y) > 0_L$; and $xy > 0$, $\varphi(x)\varphi(y) > 0$ by (O4). Let $q \in \mathbb{Q}$. If $q \leq 0$, then $q \in \mathbb{Q}_F(xy) = \mathbb{Q}_L(\varphi(xy))$, and $q \in \mathbb{Q}_L(\varphi(x)\varphi(y))$, by the last clause of Lemma 6.2.2 in F and in L . If $q > 0$, then

$$\begin{aligned} q \in \mathbb{Q}_L(\varphi(xy)) &\iff q \in \mathbb{Q}_F(xy) \\ &\iff \exists r \in \mathbb{Q}_{>0} (r \in \mathbb{Q}_F(x) \wedge q/r \in \mathbb{Q}_F(y)) \\ &\iff \exists r \in \mathbb{Q}_{>0} (r \in \mathbb{Q}_L(\varphi(x)) \wedge q/r \in \mathbb{Q}_L(\varphi(y))) \\ &\iff q \in \mathbb{Q}_L(\varphi(x)\varphi(y)), \end{aligned}$$

by $(\star)$, Lemma 6.2.2 in F , $(\star)$ twice, and Lemma 6.2.2 in L (with $\varphi(x), \varphi(y) > 0$). By Lemma 6.1.3 in L , $\varphi(xy) = \varphi(x)\varphi(y)$ for $x, y > 0$.

Now the signs. By Steps 5 and 3, $\varphi(x) + \varphi(-x) = \varphi(0_F) = 0_L$, so $\varphi(-x) = -\varphi(x)$ (Proposition 2.3.2 (a)). If $x = 0$ or $y = 0$, both sides of $\varphi(xy) = \varphi(x)\varphi(y)$ are 0_L (Step 3 and Proposition 2.3.2 (b)). If $x < 0 < y$, then $-x > 0$ (Proposition 2.3.5 (a)) and $xy = -((-x)y)$ (Proposition 2.3.2 (c)), so

$$\varphi(xy) = -\varphi((-x)y) = -(\varphi(-x)\varphi(y)) = -(-\varphi(x))\varphi(y) = \varphi(x)\varphi(y).$$

The case $x > 0 > y$ follows by (F6), and if $x, y < 0$, then $xy = (-x)(-y)$ and $\varphi(xy) = (-\varphi(x))(-\varphi(y)) = \varphi(x)\varphi(y)$. Finally $\varphi(1_F) = 1_L$ by Step 3. With Steps 4 and 5, φ is an order-preserving field map.

Step 7: uniqueness. Let $\chi: F \rightarrow L$ be any order-preserving field map, and let $x \in F$, $q \in \mathbb{Q}$. By Proposition 6.1.1, $\chi(\iota_F(q)) = \iota_L(q)$. If $\iota_F(q) < x$, then $\iota_L(q) = \chi(\iota_F(q)) < \chi(x)$. If $\iota_L(q) < \chi(x)$, that is, $\chi(\iota_F(q)) < \chi(x)$, then $\iota_F(q) < x$, because χ reflects order (the remark after Definition 6.1.1). So

$$\mathbb{Q}_L(\chi(x)) = \mathbb{Q}_F(x) = \mathbb{Q}_L(\varphi(x)),$$

the second equality by $(\star)$. By Lemma 6.1.3 in L , $\chi(x) = \varphi(x)$. Since x was arbitrary, $\chi = \varphi$. This proves the theorem. ■

The answer to the prompt. Completeness of L is used in Step 1, where the supremum must exist, and in Step 2 ($\supseteq$), where its leastness turns “below $\varphi(x)$ ” into “below some $\iota_L(r)$ ”. The Archimedean property of F is used in Step 1 for the bounds $-n$ and m , in Steps 2 and 4 through density in F , and in Steps 5 and 6 through the two lemmas applied in F . Completeness of L is used a third time, indirectly: it makes L Archimedean (Theorem 6.1.2), which Steps 3, 5, 6 and 7 need for rationals to separate in L and for the lemmas in L .

Remark. Step 7 is the whole of “uniqueness”. Its idea, comparing the rationals below two candidate values, needs only that L is Archimedean, not that it is complete: Exercise C1 runs it for any two order-preserving field maps $F \rightarrow L$, with no appeal to $(\star)$. Section 6.3 leans on the uniqueness clause throughout: to show that the maps between two complete fields are inverse to each other, that the isomorphism they give is the only one, and that a complete ordered field has no order-preserving field automorphism other than the identity.

Remark. Neither hypothesis can be dropped from the existence half. *Without completeness of L :* take $F = \mathbb{R}$, which is Archimedean (Theorem 5.4.2), and $L = \mathbb{Q}$. An order-preserving field map $\varphi: \mathbb{R} \rightarrow \mathbb{Q}$ would fix 2 (Proposition 6.1.1), so for $\sqrt{2} \in \mathbb{R}$ (the element of Example (The Class of the Newton Sequence), as in Section 6.1), $\varphi(\sqrt{2})^2 = \varphi((\sqrt{2})^2) = \varphi(2) = 2$ in $\mathbb{Q}$, against Theorem 2.6.1. *Without the Archimedean property of F :* take $F = \mathbb{Q}(t)$ of Example (A Non-Archimedean Ordered Field) and any complete L . An order-preserving field map φ would fix each $n \cdot 1$ and be strictly increasing, so $n \cdot 1_L = \varphi(n \cdot 1) < \varphi(t)$ for every $n \in \mathbb{N}$, against the Archimedean property of L (Theorem 6.1.2).

Warning. The supremum in the definition of φ is taken **in** L , of the copies $\iota_L(q)$. It cannot be taken in $\mathbb{Q}$, or in F . In $\mathbb{Q}$: the set $\mathbb{Q}_{\mathbb{R}}(\sqrt{2})$ of Example (The Rationals below the Square Root of Two) (with $\sqrt{2}$ as in Section 6.1) has no supremum in $\mathbb{Q}$. It contains the set $A = \{p \in \mathbb{Q} : p^2 < 2\}$ of Theorem 2.6.2, and its other elements are negative, so below $0 \in A$; hence the two sets have the same upper bounds in $\mathbb{Q}$, and A has no least one. In F : the supremum of $\iota_F(\mathbb{Q}_F(x))$ always exists and is x itself, since x is an upper bound and, by density in F (Theorem 4.2.5), every $u < x$ lies below some $\iota_F(q) < x$. So it is an element of the wrong field.

Quick check. Take $L := F$, a complete ordered field. Which map $F \rightarrow F$ does [Theorem 6.2.3](#) produce, and why, without computing a single supremum?

Solution. The identity. The theorem says there is exactly one order-preserving field map $F \rightarrow F$ (F is Archimedean by [Theorem 6.1.2](#)), and id_F is one (the first example after [Definition 6.1.1](#)). So $\varphi = \text{id}_F$. This is the argument [Section 6.3](#) builds on.

6.2.4 Exercises

6.2.4.1 A. Check your understanding

Exercise. A1

These questions are about the proof of [Theorem 6.2.3](#).

- (a) Which steps use that L is complete, and which that F is Archimedean? Which steps use that L is Archimedean, and where does that come from?
- (b) Run the definition of φ with $F = \mathbb{Q}$ and $L = \mathbb{R}$, the constructed reals. What is φ ?
- (c) State the idea of the proof in two sentences.
- (d) Why may Step 1 not define $\varphi(x)$ as a supremum taken in F ?

Solution.

- (a) L complete: Step 1 (the supremum exists) and Step 2 ($\supseteq$) (its leastness). F Archimedean: Step 1 (the bounds $-n, m$), Steps 2 and 4 (density in F), and Steps 5 and 6 ([Lemma 6.2.1](#) and [Lemma 6.2.2](#) applied in F). L Archimedean: Steps 3, 5, 6 and 7, through [Lemma 6.1.3](#) in L and the two lemmas in L . It comes from the completeness of L , by [Theorem 6.1.2](#).
- (b) $\varphi = \iota_{\mathbb{R}} = \kappa$. By Step 3, $\varphi(\iota_{\mathbb{Q}}(r)) = \iota_{\mathbb{R}}(r)$ for every $r \in \mathbb{Q}$, and $\iota_{\mathbb{Q}}$ is the identity ([Chapter 4](#), after [Theorem 4.2.2](#)). So $\varphi(r) = \iota_{\mathbb{R}}(r)$, which is $\kappa(r)$ by [Theorem 5.4.1](#) (d). Concretely, $\varphi(r)$ is the supremum in $\mathbb{R}$ of $\{\kappa(q) : q < r\}$, and that supremum is $\kappa(r)$.
- (c) An order-preserving map that fixes the rationals must send x to an element with the same rationals below it, and in a complete L the supremum of their copies is such an element. Every field identity for φ is then checked by comparing the rationals below the two sides, which the sum and product lemmas compute, and rationals separate in L .
- (d) In F the supremum of $\iota_F(\mathbb{Q}_F(x))$ always exists, since F is Archimedean, and it is x itself ([Exercise B2](#) of [Section 6.1](#)): an element of F , not of L . So it does not give a map into L . The supremum must be of the copies $\iota_L(q)$, taken in L .

6.2.4.2 B. Practice

Exercise. B1

Prove [Lemma 6.2.2](#): let F be an Archimedean ordered field, $a, b \in F$ with $a, b > 0$, and

$q \in \mathbb{Q}$ with $q > 0$; show that $\iota_F(q) < ab$ if and only if some rational $r > 0$ has $\iota_F(r) < a$ and $\iota_F(q/r) < b$. Show also that $\iota_F(q) < ab$ for every rational $q \leq 0$.

Hint: imitate the proof of Lemma 6.2.1, with division in place of subtraction.

Solution. Write q for $\iota_F(q)$. Since ι_F preserves products, $\iota_F(r) \iota_F(r^{-1}) = \iota_F(1) = 1$ for $r \neq 0$, so $\iota_F(r^{-1}) = \iota_F(r)^{-1}$ (Lemma 4.1.1 (a)) and $\iota_F(q/r) = q/r$ in F .

If $q \leq 0$, then $q \leq 0 < ab$ by (O4).

Let $q > 0$. ($\Leftarrow$) Let $r > 0$, $r < a$, $q/r < b$. Then $q/r > 0$ (Proposition 2.5.2 (a), (O4)), so Proposition 2.3.5 (c) gives $q = r(q/r) < a(q/r)$, and, as $a > 0$, $a(q/r) < ab$. Hence $q < ab$.

($\Rightarrow$) Let $q < ab$. Since $b^{-1} > 0$, Proposition 2.3.5 (c) gives $q/b < a$, and $q/b > 0$. By Theorem 4.2.5 there is $r \in \mathbb{Q}$ with $q/b < \iota_F(r) < a$. Then $\iota_F(r) > 0 = \iota_F(0)$, so $r > 0$ since ι_F reflects order. From $q/b < r$ and $b > 0$, $q < rb$; multiplying by $r^{-1} > 0$, $q/r < b$. This r is as required.

Exercise. B2: Cube roots

Let F be a complete ordered field and $a \in F$ with $a > 0$. Let $S := \{y \in F : y \geq 0 \text{ and } y^3 < a\}$. The following steps prove that some $s \in F$ has $s^3 = a$.

- (1) S is non-empty ($0 \in S$) and bounded above by $1 + a$: if $y > 1 + a$, then $y > 1$, so $y^3 > y > a$. Let $s := \sup S$. Then $s > 0$, since $m := \min\{1, a/2\}$ has $0 < m \leq 1$, so $m^3 \leq m \leq a/2 < a$, and $m \in S$.
- (2) For $y, z \in F$, $z^3 - y^3 = (z - y)(z^2 + zy + y^2)$ and $z^2 + zy + y^2 = (z + y/2)^2 + (3/4)y^2$, which is > 0 unless $y = z = 0$ (if $y \neq 0$, then $(3/4)y^2 > 0$ by Proposition 2.3.5 (e), (c) and the square is ≥ 0 , so the sum is positive by (b); if $y = 0$ and $z \neq 0$, the expression is $z^2 > 0$ by (e)). Hence $y < z \iff y^3 < z^3$, for all $y, z \in F$.
- (3) The case $s^3 < a$ is impossible. Let $h := \min\{1, (a - s^3)/(2(3s^2 + 3s + 1))\} > 0$. Since $0 < h \leq 1$ and $s \geq 0$, $(s + h)^3 = s^3 + 3s^2h + 3sh^2 + h^3 \leq s^3 + h(3s^2 + 3s + 1) \leq s^3 + (a - s^3)/2 < a$, so $s + h \in S$, against s being an upper bound.
- (4) The case $s^3 > a$ is impossible. *Name the move, then complete it.*
- (5) Hence $s^3 = a$.

Then explain how to get a cube root of $a = 0$ and of $a < 0$, and why every $a \in F$ has **exactly one** cube root.

Solution. The move is **construct a point as a supremum**: s is built as $\sup S$, and each of the two wrong cases of trichotomy is refuted by exhibiting a point that contradicts one clause of the supremum. In step (3) it was an element of S above s , against (S1). In step (4) it is an upper bound below s , against (S2).

- (4) Suppose $s^3 > a$. Let $h := (s^3 - a)/(3s^2)$; it is positive, since $s > 0$ by (1). Also $h < s$: this says $s^3 - a < 3s^3$, which holds because $a > 0$ and $2s^3 > 0$. So $s - h > 0$, and

$$(s - h)^3 - a = (s^3 - a) - 3s^2h + 3sh^2 - h^3 = h^2(3s - h) > 0,$$

since $3s^2h = s^3 - a$, $h^2 > 0$ and $3s - h > 2s > 0$. Let $y > s - h$. By (2), $y^3 > (s - h)^3 > a$,

so $y \notin S$. Hence every element of S is $\leq s - h$, and $s - h$ is an upper bound of S smaller than s , against s being the least upper bound.

(5) Steps (3) and (4) leave only $s^3 = a$, by trichotomy (O1).

For $a = 0$, $0^3 = 0$. For $a < 0$, $-a > 0$ has a cube root c by the above, and $(-c)^3 = -c^3 = a$ (Proposition 2.3.2 (c)). Uniqueness: if $b^3 = c^3$, then neither $b < c$ nor $c < b$ can hold, since by (2) either would make the cubes differ; so $b = c$. So every element of a complete ordered field has exactly one cube root.

6.2.4.3 C. Going deeper

Exercise. C1: Exit check: at most one map into an Archimedean field

Let F and L be Archimedean ordered fields, and let $\chi_1, \chi_2: F \rightarrow L$ be order-preserving field maps. Prove that $\chi_1 = \chi_2$.

Solution (Hint 1). Compare the rationals below $\chi_1(x)$ and below $\chi_2(x)$.

Solution (Hint 2). By Proposition 6.1.1, $\chi_i(\iota_F(q)) = \iota_L(q)$ for $i = 1, 2$ and every $q \in \mathbb{Q}$.

Solution. Let $x \in F$ and $i \in \{1, 2\}$. By Proposition 6.1.1 (which needs no order hypothesis), $\chi_i(\iota_F(q)) = \iota_L(q)$ for every $q \in \mathbb{Q}$. Since χ_i is strictly increasing and reflects order (the remark after Definition 6.1.1),

$$\iota_F(q) < x \iff \chi_i(\iota_F(q)) < \chi_i(x) \iff \iota_L(q) < \chi_i(x).$$

So $\mathbb{Q}_L(\chi_1(x)) = \mathbb{Q}_F(x) = \mathbb{Q}_L(\chi_2(x))$. Since L is Archimedean, Lemma 6.1.3 gives $\chi_1(x) = \chi_2(x)$. As x was arbitrary, $\chi_1 = \chi_2$. (This is Step 7 of Theorem 6.2.3 with χ_2 in place of φ ; completeness of L is not needed, and neither is the Archimedean property of F .)

Why L must be Archimedean (remark, not proved here). The last step separates $\chi_1(x)$ from $\chi_2(x)$ by a rational strictly between them, which is density of $\iota_L(\mathbb{Q})$ in L , and density needs Archimedes (Theorem 4.2.5). In the non-Archimedean field $\mathbb{Q}(t)$ of Example (A Non-Archimedean Ordered Field), the elements $1/t$ and $2/t$ are different and have the same rationals below them, namely all $q \leq 0$: a positive rational q has $q > 1/n$ in $\mathbb{Q}$ for some $n \in \mathbb{Z}_{>0}$ (Theorem 2.5.3), so $\iota_{\mathbb{Q}(t)}(q) > \iota_{\mathbb{Q}(t)}(1/n) = 1/(n \cdot 1) > 2/t$, because $t > 2n \cdot 1$ (Example (A Non-Archimedean Ordered Field)). So the separating step fails in such a field. For the stronger statement proved above, with F arbitrary, the hypothesis on L cannot be dropped: on $\mathbb{Q}(t)$, the substitution $t \mapsto t + 1$ keeps degrees and leading coefficients, so it is an order-preserving field map $\mathbb{Q}(t) \rightarrow \mathbb{Q}(t)$ different from the identity (school polynomial algebra, as flagged in Example (A Non-Archimedean Ordered Field)). An example with F Archimedean needs more algebra than this volume has, and we give none.

Marking scheme. (10 points) - 2: each χ_i fixes $\mathbb{Q}$, with the reason (Proposition 6.1.1, or directly: induction for $\mathbb{N}$, then negatives and quotients). - 3: links the rationals below x to those below $\chi_i(x)$, using that χ_i fixes $\mathbb{Q}$ and reflects order. Either gives the full equivalence $\iota_F(q) < x \iff \iota_L(q) < \chi_i(x)$, or, for the squeeze route of the next item, applies reflection

by χ_1 and χ_2 to a rational strictly between $\chi_1(x)$ and $\chi_2(x)$. - 3: concludes $\chi_1(x) = \chi_2(x)$ by rationals separating, or by a rational strictly between the two, invoking the Archimedean property of L . - 2: states the conclusion for every $x \in F$. Fatal errors (0 for the step): using density in L without a hypothesis that gives it; assuming χ_i surjective.

Exercise. C2: Dense image, and when the map is onto

Let F be an Archimedean ordered field, L a complete ordered field, and $\varphi: F \rightarrow L$ the map of Theorem 6.2.3.

- (a) Prove that $\varphi(F)$ is dense in L : for all $u < v$ in L there is $x \in F$ with $u < \varphi(x) < v$.
- (b) Prove that if F is complete, then φ is surjective.
- (c) Prove that if φ is surjective, then F is complete.

Solution (Hint 1). For (b), given $y \in L$, look at the set $S := \mathbb{Q}_L(y)$ and its copy $\iota_F(S)$ in F .

Solution (Hint 2). For (b): $\iota_F(S)$ is non-empty and bounded above in F ; let $x := \sup_F \iota_F(S)$, and show $\mathbb{Q}_F(x) = S$.

Solution. By Theorem 6.1.2, L is Archimedean.

- (a) Let $u < v$ in L . By Theorem 4.2.5 in L there is $q \in \mathbb{Q}$ with $u < \iota_L(q) < v$. By Step 3 of the proof of Theorem 6.2.3, $\iota_L(q) = \varphi(\iota_F(q))$, so $x := \iota_F(q)$ works.
- (b) Let F be complete and $y \in L$. Let $S := \mathbb{Q}_L(y)$. Since L is Archimedean, there are $n, m \in \mathbb{N}$ with $-y < \iota_L(n)$ and $y < \iota_L(m)$. So $-n \in S$, and every $q \in S$ has $\iota_L(q) < y < \iota_L(m)$, hence $q < m$ and $\iota_F(q) < \iota_F(m)$. So $\iota_F(S)$ is non-empty and bounded above in F , and $x := \sup_F \iota_F(S)$ exists.
We show $\mathbb{Q}_F(x) = S$. ($\supseteq$) Let $q \in S$. By density in L there is $r \in \mathbb{Q}$ with $\iota_L(q) < \iota_L(r) < y$; then $r \in S$ and $q < r$, so $\iota_F(q) < \iota_F(r) \leq x$. ($\subseteq$) Let $\iota_F(q) < x$. Then $\iota_F(q)$ is not an upper bound of $\iota_F(S)$, so $\iota_F(q) < \iota_F(r)$ for some $r \in S$; hence $q < r$ and $\iota_L(q) < \iota_L(r) < y$, so $q \in S$.
By ($\star$), $\mathbb{Q}_L(\varphi(x)) = \mathbb{Q}_F(x) = S = \mathbb{Q}_L(y)$, so $\varphi(x) = y$ by Lemma 6.1.3 in L .
- (c) Let φ be surjective. It is injective (strictly increasing), so bijective, with an inverse $\varphi^{-1}: L \rightarrow F$ (Sets, Relations, and Cardinality, Theorem 1.5.4). Let $A \subseteq F$ be non-empty with an upper bound b . Then $\varphi(A)$ is non-empty and bounded above by $\varphi(b)$, since $a \leq b$ gives $\varphi(a) \leq \varphi(b)$. As L is complete, $s := \sup \varphi(A)$ exists; let $c := \varphi^{-1}(s)$.
Upper bound: for $a \in A$, $\varphi(a) \leq s = \varphi(c)$, so $a \leq c$, because φ reflects order (and is injective, for the case of equality). *Least:* if u is an upper bound of A , then $\varphi(u)$ is an upper bound of $\varphi(A)$, so $\varphi(c) = s \leq \varphi(u)$, and $c \leq u$ as before. So $c = \sup A$ (Sets, Relations, and Cardinality, Definition 2.6.5), and F is complete.

So φ is an order-preserving field isomorphism exactly when F is complete. Section 6.3 reaches the “if” direction by a different route, which computes no supremum at all.

Marking scheme. (10 points) - 2: (a) L is Archimedean (Theorem 6.1.2), so some $\iota_L(q)$ lies strictly between u and v , and $\iota_L(q) = \varphi(\iota_F(q))$ (Step 3). - 2: (b) $\iota_F(S)$ is non-empty and

bounded above in F (Archimedes in L , transferred through ι_F), so $x = \sup_F \iota_F(S)$ exists by completeness of F . - 2: (b) $\mathbb{Q}_F(x) = S$, both inclusions; ($\supseteq$) uses density in L . - 1: (b) $\mathbb{Q}_L(\varphi(x)) = \mathbb{Q}_F(x) = \mathbb{Q}_L(y)$ by $(\star)$, hence $\varphi(x) = y$ by Lemma 6.1.3 in L . - 3: (c) for non-empty $A \subseteq F$ bounded above by b , $\varphi(A)$ is bounded above by $\varphi(b)$, and $\varphi^{-1}(\sup \varphi(A))$ is an upper bound of A and the least one, both using that φ reflects order and is bijective. Fatal errors (0 for the step): using density of $\iota_L(\mathbb{Q})$ in L without the Archimedean property of L ; in (b), taking a supremum in F before checking that the set is non-empty and bounded above; in (c), assuming φ^{-1} preserves order without justification.