

The Complete Ordered Field Is Unique

Chapter 5 ended with one complete ordered field, whose elements are classes of Cauchy sequences of rationals. The optional Chapter 7 builds another, whose elements are sets of rationals, and *Analysis on the Real Line*, Chapter 1 builds none: it assumes that a complete ordered field exists and calls it $\mathbb{R}$. Are these the same $\mathbb{R}$? The axioms of an ordered field alone do not decide it. The field $\mathbb{R}(x)$ of rational functions, compared at large arguments, satisfies all of them and has an element above every natural number; the field $\mathbb{Q}(\sqrt{2})$ is even Archimedean, and still 3 has no square root in it. This chapter proves that completeness is the property that pins $\mathbb{R}$ down: any two complete ordered fields are linked by an order-preserving field isomorphism, and by only one. So “the” real numbers are well defined, and no theorem about them depends on how they were built.

The tool is one idea. In an Archimedean ordered field an element is determined by **the rationals below it**, and a map that respects sums, products, 1 and the order must send each element to one with the same rationals below it. In a complete field the supremum of the copies of those rationals is such an element. So from an Archimedean field into a complete one there is exactly one order-preserving field map, and each identity it satisfies is checked by comparing rationals below the two sides, with no arithmetic of suprema. When both fields are complete, maps run both ways, and the word “exactly” makes them inverse to each other without a computation. The same uniqueness leaves a complete ordered field no symmetry: a map of it to itself that respects only sums, products and 1 is the identity, because there the positive elements are the non-zero squares. The two non-examples show what each hypothesis buys: $\mathbb{Q}(\sqrt{2})$ has a symmetry exchanging $\sqrt{2}$ and $-\sqrt{2}$, which breaks the order, and $\mathbb{R}(x)$ has one sending x to $x+1$, which keeps it. The chapter, the last of the volume’s main line, ends by comparing the two ways to begin analysis, assuming the axioms or building a model, and by listing what *Analysis on the Real Line* takes from this volume.

What you need. Chapters 4 and 5 above all, and Chapter 2; *Sets, Relations, and Cardinality*, Chapters 1–3, chiefly Section 2.6; and *Logic and Proof*, Chapters 2, 4 and 6. From Chapter 4 of this book: the rules of an ordered field and the absolute value (Proposition 4.1.2, Lemma 4.1.1, Definition 4.1.1, Proposition 4.1.3); powers and the one copy ι_F of $\mathbb{Q}$ in an ordered field, with its uniqueness clause (Definition 4.2.1, Proposition 4.2.1, Theorem 4.2.2); the Archimedean property, density of $\mathbb{Q}$, and the non-Archimedean field $\mathbb{Q}(t)$ (Definition 4.2.2, Proposition 4.2.3, Theorem 4.2.5, Example (A Non-Archimedean Ordered Field)); and limits (Definition 4.3.2, Proposition 4.3.1), with Cauchy sequences in the exercises of Section 6.5 (Definition 4.5.1). From Chapter 5: complete ordered fields (Definition 5.6.1), the theorem that the Cauchy model $\mathbb{R}$ is one (Theorem 5.6.3), the map $\kappa = \iota_{\mathbb{R}}$ (Theorem 5.4.1), the class of the Newton sequence, a positive real number whose square is 2 (Example (The Class of the Newton Sequence)), and, for contrast, completeness from Archimedes and Cauchy-completeness (Theorem 5.6.2). From Chapter 2: the axioms (F1)–(F9) and (O1)–(O4) (Definition 2.3.1, Definition 2.3.2) and their rules (Proposition 2.3.2, Proposition 2.3.5, Proposition 2.5.2), $\mathbb{Q}$ Archimedean (Theorem 2.5.3), least elements (Proposition 2.3.6), and no rational square equal to 2 (Theorem 2.6.1, Theorem 2.6.2). Chapter 1 supplies one analogy (Theorem 1.7.1) and, in a review solution, the order arithmetic of ω (Proposition 1.6.3); Chapter 3 supplies even and odd integers (Corollary 3.1.3), used in Section 6.4, and one review exercise (Theorem 3.4.3, Theorem 3.5.2). From *Sets, Relations, and Cardinality*: injective and bijective maps and two-sided inverses, Sections 1.4 and 1.5 (Sets, Relations, and Cardinality, Definition 1.4.3, Sets, Relations, and Cardinality, Definition 1.4.5, Sets, Relations, and Cardinality, Theorem 1.5.4); finite sums, Section 2.3 (Sets,

Relations, and Cardinality, Definition 2.3.1); equivalence relations and quotient sets, Section 2.5 (Sets, Relations, and Cardinality, Definition 2.5.1, Sets, Relations, and Cardinality, Definition 2.5.2, Sets, Relations, and Cardinality, Definition 2.5.4); upper bounds and suprema, Section 2.6 (Sets, Relations, and Cardinality, Definition 2.6.2, Sets, Relations, and Cardinality, Definition 2.6.5); and, for exercises of Sections 6.4 and 6.5, countable and uncountable sets, Chapter 3 (Sets, Relations, and Cardinality, Theorem 3.2.2, Sets, Relations, and Cardinality, Theorem 3.4.1). From *Logic and Proof*: the contrapositive, Chapter 2 (Logic and Proof, Proposition 2.3.2); induction, Chapter 4 (Logic and Proof, Axiom 4.2.1); and the negation of quantifiers, unbounded and bounded, Chapter 6 (Logic and Proof, Proposition 6.2.1, Logic and Proof, Proposition 6.2.3).

Time. About 20 hours on the core track (all of A, one B and one C per section, the exit checks); about 30 hours for everything.

6.-1.1 Roadmap

- **What Completeness Gives at Once.** Order-preserving field maps, which reflect the order and fix the rationals; the theorem that a complete ordered field is Archimedean, by a supremum of the copy of $\mathbb{N}$ that cannot exist; the rationals below an element, $\mathbb{Q}_F(x)$, which determine x when F is Archimedean; and square roots in a complete ordered field, built as a supremum and unique.
- **Building the Map from the Rationals Below.** The rationals below a sum, and below a product of positive elements, computed from those below each term; and the main theorem in its one-sided form: from an Archimedean ordered field F into a complete one L there is exactly one order-preserving field map, which sends x to the supremum in L of the copies of the rationals below x and keeps the rationals below every element.
- **Two Complete Ordered Fields, One Isomorphism.** The uniqueness theorem, with the inverse map obtained from the uniqueness clause instead of computed; the relation $F \cong L$, every complete ordered field a copy of the Cauchy model, and the Dedekind cut of Chapter 7 read as a set of rationals below; and the automorphisms of a complete ordered field, of which there is one, the identity, even among the maps that respect only sums, products and 1.
- **Two Non-Examples.** Polynomial functions compared at large arguments, where the top term decides the sign; the ordered field $\mathbb{R}(x)$ of rational functions, which is not Archimedean and in which x is not a square; the field $\mathbb{Q}(\sqrt{2})$, Archimedean and not complete, with conjugation as a symmetry that breaks the order; the shift $x \mapsto x + 1$, a symmetry of $\mathbb{R}(x)$ that keeps it; and a table of which of $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{R}(x)$ and $\mathbb{R}$ are Archimedean, complete, have square roots of positive elements, and have no field automorphism except the identity.
- **Assume the Axioms or Build a Model.** The two ways to begin analysis and the two questions, existence and uniqueness, that assuming the axioms leaves open; what an isomorphism carries from one complete ordered field to another, with limits as the worked case, and what it does not carry, shown by a model whose zero is the empty set; a table of what *Analysis on the Real Line*, Chapter 1 assumes and where this series supplies it; and review exercises that return to Chapter 5's real numbers, Chapter 3's $\mathbb{Z}/5\mathbb{Z}$, equivalence relations and suprema.

6.-1.2 Named moves

- **Rationals separate.** To show $a = b$ in an Archimedean ordered field, show that the same rationals lie below each; otherwise a rational would lie strictly between them. It is proved in Section 6.1 and then proves every equation between elements of the complete field in

the main theorem of Section 6.2: the values of the map on the rationals, sums, products and uniqueness (Steps 3, 5, 6 and 7); the exit check C1 of Section 6.2 runs it alone. In $\mathbb{R}(x)$, where Archimedes fails, the elements $1/x$ and $2/x$ have the same rationals below them (Section 6.4).

- **Construct a point as a supremum.** To produce an element you cannot name, take the supremum of a well-chosen set and refute the two wrong cases of trichotomy, one with an element of the set above the supremum, the other with a smaller upper bound. It builds square roots in Section 6.1 and the map of Section 6.2. Exercise B2 of Section 6.2 (cube roots) leaves the last step to you, and Exercise C3 of Section 6.4 (a bounded increasing sequence converges to the supremum of its terms) the last half.
- **Uniqueness gives inverses.** If the only order-preserving field map $F \rightarrow F$ is the identity, then two maps $F \rightarrow L \rightarrow F$ compose to the identity without computing anything. It proves in Section 6.3 that the map between two complete ordered fields is bijective, and the same uniqueness clause shows there that a complete ordered field has no order-preserving automorphism except the identity; once square roots recover the order, also no field automorphism.