

6.1 What Completeness Gives at Once

Chapter 5 ended with one complete ordered field, the classes of Cauchy sequences of rationals (Theorem 5.6.3). The optional Chapter 7 builds another from Dedekind cuts, and *Analysis on the Real Line* simply assumes that one exists. This chapter asks whether these are the same field. Before that question can be asked we need a precise meaning for “the same”: a map between ordered fields that respects everything. This section fixes that meaning, and then collects what completeness gives at once: the Archimedean property, a way to tell elements apart by the rationals below them, and square roots.

Standing setting. Throughout the chapter, F and L are ordered fields (Definition 2.3.2), so $\leq$ is a total order on each (Proposition 4.1.2 (a)) and upper bounds and suprema in the sense of *Sets, Relations, and Cardinality*, Section 2.6 apply to their subsets (Sets, Relations, and Cardinality, Definition 2.6.2, Sets, Relations, and Cardinality, Definition 2.6.5). “Complete” is always the property of Definition 5.6.1: every non-empty subset that is bounded above has a supremum. We do not redefine it.

Each ordered field F carries its own copy of $\mathbb{Q}$: the map $\iota_F: \mathbb{Q} \rightarrow F$ of Theorem 4.2.2, the only map with $\iota_F(1) = 1_F$ that preserves sums and products. It is injective, strictly increasing and reflects order, and $\iota_F(n) = n \cdot 1_F$ for $n \in \mathbb{N}$. For the constructed reals, $\iota_{\mathbb{R}}$ is the map κ of Chapter 5 (Theorem 5.4.1 (d)). When only one field is in play, a proof may write q for $\iota_F(q)$, as Chapter 4 did; statements keep ι_F visible, and so does every argument that moves between two fields.

6.1.1 Maps that respect everything

To say that two ordered fields are “the same”, we want a dictionary between them: a map that turns sums into sums, products into products, 1 into 1, and smaller into smaller. An isomorphism of Peano systems in Chapter 1 (Theorem 1.7.1) was such a dictionary for 0 and the successor; here the structure is larger.

A map that respects addition, multiplication, the unit and the order.

Definition 6.1.1. Order-Preserving Field Map

Let F and L be ordered fields. A map $\chi: F \rightarrow L$ is an **order-preserving field map** if, for all $a, b \in F$,

$$\chi(a + b) = \chi(a) + \chi(b), \quad \chi(ab) = \chi(a)\chi(b), \quad \chi(1_F) = 1_L,$$

and $a < b$ implies $\chi(a) < \chi(b)$. An **order-preserving field isomorphism** is an order-preserving field map that is **bijective** (Sets, Relations, and Cardinality, Definition 1.4.5). An **order-preserving field automorphism** of F is an order-preserving field isomorphism $F \rightarrow F$.

In words: the first two equations say that computing in F and then translating gives the same answer as translating and then computing in L ; the third pins down the unit; the last says that χ is **strictly** increasing.

Two consequences come free. A strictly increasing map is injective: if $a \neq b$, then $a < b$ or $b < a$ by (O1), and the images differ in the same way. And it **reflects order**: if $\chi(a) < \chi(b)$, then $a < b$. For $a = b$ would give $\chi(a) = \chi(b)$, and $b < a$ would give $\chi(b) < \chi(a)$; both contradict

$\chi(a) < \chi(b)$ by (O1), so only $a < b$ is left. (This is the contrapositive of “ $b \leq a$ implies $\chi(b) \leq \chi(a)$ ”, Logic and Proof, Proposition 2.3.2, read with trichotomy.) So an order-preserving field map satisfies $a < b \iff \chi(a) < \chi(b)$.

Examples.

The identity. id_F satisfies all four clauses trivially; it is an order-preserving field automorphism of F .

The copy of the rationals. For every ordered field F , $\iota_F: \mathbb{Q} \rightarrow F$ is an order-preserving field map, by Theorem 4.2.2. It is injective and usually not surjective: $\iota_{\mathbb{R}} = \kappa$ misses the class r of the Newton sequence, which is not $\kappa(q)$ for any rational q (Example (The Class of the Newton Sequence)).

Doubling. On $\mathbb{Q}$, $\chi(x) := 2x$ preserves sums and is strictly increasing, but $\chi(1) = 2 \neq 1$, and $\chi(1 \cdot 1) = 2 \neq 4 = \chi(1)\chi(1)$. It is not a field map: the clauses about multiplication and the unit fail, the additive ones survive.

◉ Second reading · A degenerate case

The zero map $\chi(a) := 0_L$ preserves sums and products, since $0 + 0 = 0$ and $0 \cdot 0 = 0$. It fails the unit clause, because $1_L \neq 0_L$ in a field, and it fails strict increase. In Definition 6.1.1 the unit clause in fact follows from the other three: additivity gives $\chi(0) = \chi(0) + \chi(0)$, so $\chi(0) = 0$; strict increase then gives $\chi(1) > \chi(0) = 0$; and $\chi(1) = \chi(1)^2$ with $\chi(1) \neq 0$ forces $\chi(1) = 1$. It is stated anyway because Proposition 6.1.1, Theorem 6.3.4 and the remark after its proof assume sums, products and the unit but no order. Without the unit hypothesis, the zero map would be a counterexample to each of them.

Why this definition. Preserving sums, products and the unit is what makes χ a dictionary for the field operations; strict increase makes it a dictionary for the order. Section 6.4 shows a map of $\mathbb{Q}(\sqrt{2})$ to itself that respects sums, products and the unit but sends a positive element to a negative one, so the order clause does not follow from the others in general. Section 6.3 shows that for complete ordered fields it does.

Quick check. Which clauses of Definition 6.1.1 does the map $\chi: \mathbb{Q} \rightarrow \mathbb{Q}$, $\chi(x) := -x$, satisfy?

Solution. Only the sum clause: $-(a+b) = (-a)+(-b)$. The unit clause fails, $\chi(1) = -1 \neq 1$, and so does the product clause, since $\chi(1 \cdot 1) = -1$ while $\chi(1)\chi(1) = (-1)(-1) = 1$; and the order is reversed, $0 < 1$ but $\chi(1) = -1 < 0 = \chi(0)$.

The first result about these maps needs no order hypothesis at all. A map that preserves sums, products and the unit cannot move a rational number.

Proposition 6.1.1. Field Maps Fix the Rationals

Let F and L be ordered fields, and let $\chi: F \rightarrow L$ satisfy $\chi(a+b) = \chi(a) + \chi(b)$, $\chi(ab) =$

$\chi(a)\chi(b)$ for all $a, b \in F$, and $\chi(1_F) = 1_L$. Then

$$\chi \circ \iota_F = \iota_L, \quad \text{that is,} \quad \chi(\iota_F(q)) = \iota_L(q) \quad \text{for every } q \in \mathbb{Q}.$$

Proof. The composite $\chi \circ \iota_F: \mathbb{Q} \rightarrow L$ sends 1 to $\chi(1_F) = 1_L$, and for $x, y \in \mathbb{Q}$,

$$\chi(\iota_F(x + y)) = \chi(\iota_F(x) + \iota_F(y)) = \chi(\iota_F(x)) + \chi(\iota_F(y)),$$

by Theorem 4.2.2 for F and the hypothesis on χ ; products are the same with $\cdot$ in place of $+$. So $\chi \circ \iota_F$ has the three properties that, by the uniqueness clause of Theorem 4.2.2 for L , only ι_L has. Hence $\chi \circ \iota_F = \iota_L$. ■

This is the remark after Theorem 4.2.2, now with a label. Read with the identifications $q = \iota_F(q)$ and $q = \iota_L(q)$, it says $\chi(q) = q$: such a map **fixes the rationals**. If χ is also order-preserving, more follows. For $q \in \mathbb{Q}$ and $x \in F$, $\iota_F(q) < x$ holds if and only if $\chi(\iota_F(q)) < \chi(x)$, that is, if and only if $\iota_L(q) < \chi(x)$. The rationals below x in F are exactly the rationals below $\chi(x)$ in L . That observation drives the whole chapter.

6.1.2 A complete ordered field is Archimedean

Chapter 4 called F Archimedean if every element lies below some $n \cdot 1_F$ (Definition 4.2.2), and showed that this can fail: in $\mathbb{Q}(t)$ the element t lies above every $n \cdot 1$ (Example (A Non-Archimedean Ordered Field)). Chapter 5 assumed it, together with Cauchy-completeness, to prove completeness (Theorem 5.6.2). For the supremum form of completeness, no assumption is needed: it implies the Archimedean property. We write $\iota_F(\mathbb{N}) := \{\iota_F(n) : n \in \mathbb{N}\} = \{n \cdot 1_F : n \in \mathbb{N}\}$.

Theorem 6.1.2. A Complete Ordered Field Is Archimedean

Let F be a complete ordered field. Then F is Archimedean (Definition 4.2.2).

Idea. If some element bounded $\iota_F(\mathbb{N})$ above, completeness would give a **least** upper bound s . Step down by 1: $s - 1$ is not an upper bound, so some $n \cdot 1_F$ is above it, and one more step, $(n + 1) \cdot 1_F$, is above s . The set $\iota_F(\mathbb{N})$ is closed under adding 1, so it cannot have a least upper bound, and hence no upper bound at all.

Proof. Suppose, for $\neg I$, that $\iota_F(\mathbb{N})$ is bounded above, and fix an upper bound $b \in F$ $[\exists E]$, so that $n \cdot 1_F \leq b$ for every $n \in \mathbb{N}$. The set $\iota_F(\mathbb{N})$ is non-empty, since $0 \cdot 1_F \in \iota_F(\mathbb{N})$. Since F is complete, $\iota_F(\mathbb{N})$ has a supremum; fix $s := \sup \iota_F(\mathbb{N})$ $[\exists E]$.

Since $s - 1_F < s$ and s is the least upper bound, $s - 1_F$ is not an upper bound of $\iota_F(\mathbb{N})$. So some $n \in \mathbb{N}$ has $n \cdot 1_F \not\leq s - 1_F$ (Logic and Proof, Proposition 6.2.3 (a)), that is, $n \cdot 1_F > s - 1_F$ by totality (Proposition 4.1.2 (a)). Then, by Definition 4.2.1 (a) and (O3),

$$(n + 1) \cdot 1_F = n \cdot 1_F + 1_F > (s - 1_F) + 1_F = s,$$

and $(n+1) \cdot 1_F \in \iota_F(\mathbb{N})$. This contradicts that s is an upper bound of $\iota_F(\mathbb{N})$. Hence $\iota_F(\mathbb{N})$ is not bounded above $[\neg I]$.

So no $x \in F$ is an upper bound of $\iota_F(\mathbb{N})$ (Logic and Proof, Proposition 6.2.1). For $x \in F$, as above, some $n \in \mathbb{N}$ has $n \cdot 1_F \not\leq x$ (Logic and Proof, Proposition 6.2.3 (a)), that is, $x < n \cdot 1_F$ by totality. Hence F is Archimedean (Definition 4.2.2). ■

Both hypotheses of completeness were used: “non-empty” through $0 \cdot 1_F$, “bounded above” through b . The converse fails: $\mathbb{Q}$ is Archimedean (Theorem 2.5.3) and not complete (Theorem 2.6.2). And dropping completeness can lose the conclusion: $\mathbb{Q}(t)$ is an ordered field that is not Archimedean.

The proof never claimed that s is an element of $\iota_F(\mathbb{N})$; a supremum need not belong to its set. Exercise C2 shows how tempting that claim is.

Warning. Completeness here is about **subsets**, not sequences. Chapter 5 proved that the constructed $\mathbb{R}$ is complete by going through Cauchy sequences, and Theorem 5.6.2 needed the Archimedean property as a separate hypothesis to get from “every Cauchy sequence converges” to “every bounded non-empty set has a supremum”. In this chapter “complete” is the supremum property of Definition 5.6.1, and the theorem above shows that it brings the Archimedean property with it. Do not replace it by Cauchy-completeness in the proofs below: Chapter 5 records, without proof, that a Cauchy-complete ordered field need not be Archimedean.

Every complete ordered field therefore has the consequences of the Archimedean property that Chapter 4 proved. The one we use most is density (Theorem 4.2.5): in an Archimedean ordered field F , for all $x < y$ there is $q \in \mathbb{Q}$ with $x < \iota_F(q) < y$. We cite it, not reprove it; Section 6.2 uses it in two fields at once.

Here is the Archimedean property at work on a specific supremum. The pattern has two halves: check the candidate is an upper bound, then show that anything smaller is not one.

Example. A Supremum Computed by Archimedes

Let F be a complete ordered field. Show that $\sup\{\iota_F(1 - 1/n) : n \in \mathbb{Z}_{>0}\} = 1_F$.

Solution. Let $A := \{\iota_F(1 - 1/n) : n \in \mathbb{Z}_{>0}\}$.

Upper bound. For $n \in \mathbb{Z}_{>0}$, $1/n > 0$ in $\mathbb{Q}$, so $1 - 1/n < 1$, and $\iota_F(1 - 1/n) < \iota_F(1) = 1_F$ because ι_F is strictly increasing (Theorem 4.2.2). So 1_F is an upper bound of A .

Least. Let b be an upper bound of A , and suppose, for contradiction, that $b < 1_F$, so $1_F - b > 0$. By Theorem 6.1.2, F is Archimedean, so Proposition 4.2.3 (with $\varepsilon := 1_F - b$, applied at $n = N$) gives $N \in \mathbb{Z}_{>0}$ with $1/(N \cdot 1_F) < 1_F - b$. Since ι_F preserves products and $\iota_F(N) = N \cdot 1_F$, we have $\iota_F(1/N) \cdot (N \cdot 1_F) = \iota_F(1) = 1_F$, so $\iota_F(1/N) = 1/(N \cdot 1_F)$ by uniqueness of inverses (Lemma 4.1.1 (a)). Since ι_F preserves sums, it preserves negatives,

and adding $b - 1/(N \cdot 1_F)$ to both sides of $1/(N \cdot 1_F) < 1_F - b$ (O3) gives

$$\iota_F(1 - 1/N) = 1_F - \frac{1}{N \cdot 1_F} > b.$$

This element of A lies above b , against b being an upper bound. So $1_F \leq b$ by totality, for every upper bound b , and $1_F = \sup A$ (Sets, Relations, and Cardinality, Definition 2.6.5).

The supremum 1_F is not an element of A : every element is strictly below it. Completeness entered only through the Archimedean property; in an ordered field that is Archimedean but not complete, such as $\mathbb{Q}$ (Theorem 2.5.3), the same two halves give the same answer. Exercises B1 and B2 follow this pattern.

6.1.3 The rationals below an element

How can we tell two elements of a field apart? In an Archimedean field, by a rational between them. That suggests recording, for each element, which rationals lie below it.

An element is measured by the rationals below it.

Definition 6.1.2. The Rationals below an Element

Let F be an ordered field and $x \in F$. The **rationals below** x form the set

$$\mathbb{Q}_F(x) := \{q \in \mathbb{Q} : \iota_F(q) < x\} \subseteq \mathbb{Q}.$$

In words: $\mathbb{Q}_F(x)$ is a set of **rational numbers**, not of elements of F ; a rational q belongs to it when its copy $\iota_F(q)$ lies **strictly** below x .

Two properties hold in every ordered field and are used without comment. The set is **closed downward**: if $q \in \mathbb{Q}_F(x)$ and $p < q$, then $\iota_F(p) < \iota_F(q) < x$, so $p \in \mathbb{Q}_F(x)$. And if $x \leq y$, then $\mathbb{Q}_F(x) \subseteq \mathbb{Q}_F(y)$, by transitivity.

Examples.

A rational element. For $r \in \mathbb{Q}$, $\mathbb{Q}_F(\iota_F(r)) = \{q \in \mathbb{Q} : q < r\}$. Indeed $\iota_F(q) < \iota_F(r)$ holds if and only if $q < r$, because ι_F is strictly increasing and reflects order. In particular $r \notin \mathbb{Q}_F(\iota_F(r))$: the inequality is strict.

Example. The Rationals below the Square Root of Two

Let $r \in \mathbb{R}$ be the class of the Newton sequence, so that $r > 0$ and $r^2 = 2$ (Example (The Class of the Newton Sequence)). Show that

$$\mathbb{Q}_{\mathbb{R}}(r) = \{q \in \mathbb{Q} : q < 0\} \cup \{q \in \mathbb{Q} : q \geq 0 \text{ and } q^2 < 2\}.$$

Solution. Let $q \in \mathbb{Q}$, and recall $\iota_{\mathbb{R}} = \kappa$ (Theorem 5.4.1 (d)). If $q < 0$, then $\kappa(q) < \kappa(0) = 0 < r$, so $q \in \mathbb{Q}_{\mathbb{R}}(r)$. Let $q \geq 0$, so $\kappa(q) \geq 0$. Since also $r > 0$, Proposition 2.3.5 (g) in $\mathbb{R}$ gives

$$\kappa(q) < r \iff \kappa(q)^2 < r^2 = 2.$$

Now $\kappa(q)^2 = \kappa(q^2)$ and $2 = \kappa(2)$ (Theorem 5.4.1 (a)), and $\kappa(q^2) < \kappa(2)$ if and only if $q^2 < 2$

(Theorem 5.4.1 (b)). So for $q \geq 0$, $q \in \mathbb{Q}_{\mathbb{R}}(r)$ if and only if $q^2 < 2$. Together the two cases give the displayed set.

The set of Section 2.6, $\{x \in \mathbb{Q} : x^2 < 2\}$ (Theorem 2.6.2), is **not** $\mathbb{Q}_{\mathbb{R}}(r)$: it omits -2 , whose square is 4. That set is not closed downward, and a rationals-below set always is.

Quick check. Let F be an ordered field and $x := \iota_F(3/2)$. Describe $\mathbb{Q}_F(x)$, and say whether $3/2 \in \mathbb{Q}_F(x)$.

Solution. By the first example, $\mathbb{Q}_F(x) = \{q \in \mathbb{Q} : q < 3/2\}$. It does not contain $3/2$, since $\iota_F(3/2) < \iota_F(3/2)$ is false by (O1).

The point of the definition is that in an Archimedean field, these sets lose no information.

Lemma 6.1.3. Rationals Separate

Let F be an Archimedean ordered field, and let $x, y \in F$. Then

$$x \leq y \iff \mathbb{Q}_F(x) \subseteq \mathbb{Q}_F(y).$$

Consequently $x = y \iff \mathbb{Q}_F(x) = \mathbb{Q}_F(y)$.

Proof. ($\Rightarrow$) This is the second property after Definition 6.1.2: if $\iota_F(q) < x$ and $x \leq y$, then $\iota_F(q) < y$.

($\Leftarrow$) We prove the contrapositive (Logic and Proof, Proposition 2.3.2). Suppose that $x \leq y$ fails. Then $y < x$ by totality (Proposition 4.1.2 (a)). By Theorem 4.2.5 there is $q \in \mathbb{Q}$ with $y < \iota_F(q) < x$. Then $q \in \mathbb{Q}_F(x)$, and $q \notin \mathbb{Q}_F(y)$, since $\iota_F(q) < y$ and $y < \iota_F(q)$ cannot both hold (O1). So $\mathbb{Q}_F(x) \not\subseteq \mathbb{Q}_F(y)$.

For the consequence: $x = y$ holds if and only if $x \leq y$ and $y \leq x$, by antisymmetry of the total order $\leq$; by the equivalence just proved, that is $\mathbb{Q}_F(x) \subseteq \mathbb{Q}_F(y)$ and $\mathbb{Q}_F(y) \subseteq \mathbb{Q}_F(x)$, that is, $\mathbb{Q}_F(x) = \mathbb{Q}_F(y)$. ■

This is the first named move of the chapter, **rationals separate**: to prove two elements of an Archimedean field equal, show that the same rationals lie below each. A rational strictly between them would otherwise exist. Section 6.2 uses it in four of the seven steps of its main proof (Steps 3, 5, 6 and 7), each time to prove that two elements of the complete field are equal.

🕒 **Second reading** • Without Archimedes the sets cannot tell elements apart

The direction ($\Rightarrow$) and the remark that equal elements have equal sets hold in every ordered field. The direction ($\Leftarrow$) needs density, and density needs Archimedes. In the non-Archimedean field

$\mathbb{Q}(t)$ of Example (A Non-Archimedean Ordered Field) (school polynomial algebra, as flagged there; nothing below depends on it), every rational q satisfies $q < n$ for some $n \in \mathbb{N}$ (Theorem 2.5.3), and $n \cdot 1 < t$, so $\iota_{\mathbb{Q}(t)}(q) < t < t + 1$. Hence $\mathbb{Q}_{\mathbb{Q}(t)}(t) = \mathbb{Q} = \mathbb{Q}_{\mathbb{Q}(t)}(t + 1)$, although $t \neq t + 1$. Both elements lie above all of the rationals, and the rationals cannot see the difference.

6.1.4 Square roots

Completeness lets us build elements we cannot name in advance, as the supremum of a well-chosen set. We show the method once, in full, on square roots; it is the second named move, **construct a point as a supremum**. Exercise B2 of Section 6.2 runs it again on cube roots.

Proposition 6.1.4. Square Roots in a Complete Ordered Field

Let F be a complete ordered field and $a \in F$ with $a \geq 0$. Then there is **exactly one** $b \in F$ with $b \geq 0$ and $b^2 = a$.

Idea. The square root of a should be the top of the set S of non-negative y with $y^2 < a$. Let $s := \sup S$. Two of the three cases of trichotomy must be ruled out.

If $s^2 < a$, there is room above s : we look for $h > 0$ with $(s + h)^2 < a$, which puts $s + h$ in S above its supremum. Working backwards, for $0 < h \leq 1$ we have $h^2 \leq h$, so

$$(s + h)^2 = s^2 + 2sh + h^2 \leq s^2 + h(2s + 1),$$

and it is enough that $h(2s + 1)$ be at most half the room $a - s^2$. So $h := \min\{1, (a - s^2)/(2(2s + 1))\}$.

If $s^2 > a$, then s overshoots: we look for $h > 0$ with $(s - h)^2 \geq a$ still, which makes $s - h$ a smaller upper bound. Since $(s - h)^2 \geq s^2 - 2sh$, it is enough that $2sh = s^2 - a$, so $h := (s^2 - a)/(2s)$.

Proof. Existence. If $a = 0$, then $b := 0$ works. Let $a > 0$, and let

$$S := \{y \in F : y \geq 0 \text{ and } y^2 < a\}.$$

Then $0 \in S$, since $0^2 = 0 < a$. And $a + 1$ is an upper bound of S : if $y > a + 1$, then $y > 1 > 0$, so $y^2 = y \cdot y > y \cdot 1 = y > a$ (Proposition 2.3.5 (c)), and $y \notin S$; so every $y \in S$ satisfies $y \leq a + 1$ by totality. Since F is complete, fix $s := \sup S$ $[\exists E]$. Since $0 \in S$, $s \geq 0$.

Case $s^2 < a$. Let $h := \min\{1, (a - s^2)/(2(2s + 1))\}$. It is defined and positive, since $2s + 1 \geq 1 > 0$ and $a - s^2 > 0$ (Proposition 2.5.2 (a), (O4)). Since $0 < h \leq 1$, $h^2 \leq h$ (Proposition 2.3.5 (c)), so

$$\begin{aligned} (s + h)^2 &= s^2 + 2sh + h^2 \leq s^2 + h(2s + 1) \\ &\leq s^2 + \frac{a - s^2}{2} < a, \end{aligned}$$

where the second inequality uses $h \leq (a - s^2)/(2(2s + 1))$ and $2s + 1 > 0$, and the last uses

$(a - s^2)/2 < a - s^2$ (Proposition 4.1.2 (b)). So $s + h \geq 0$ lies in S , and $s + h > s$, against s being an upper bound of S .

Case $s^2 > a$. Then $s \neq 0$, since $0^2 = 0 < a$; so $s > 0$. Let $h := (s^2 - a)/(2s)$. It is positive, since $s^2 - a > 0$ and $2s > 0$ (Proposition 2.5.2 (a), (O4)). Then $s - h = (s^2 + a)/(2s) > 0$ by the same reasons, since $s^2 + a > 0$; and

$$(s - h)^2 = s^2 - 2sh + h^2 = s^2 - (s^2 - a) + h^2 = a + h^2 > a.$$

We claim that $s - h$ is an upper bound of S . Let $y > s - h$. Then $y > 0$, and Proposition 2.3.5 (g) gives $y^2 > (s - h)^2 > a$, so $y \notin S$. Hence every element of S is $\leq s - h$. But $s - h < s$, against s being the **least** upper bound.

Both cases are impossible, so $s^2 = a$ by trichotomy (O1), and $b := s \geq 0$ works.

Uniqueness. Let $b, c \geq 0$ with $b^2 = c^2 = a$. Then $(b - c)(b + c) = b^2 - c^2 = 0$, so $b = c$ or $b + c = 0$ (Proposition 2.3.5 (f)). If $b + c = 0$, then $b = -c \leq 0 \leq b$, so $b = 0$ and $c = -b = 0$. In either case $b = c$. ■

For $a \geq 0$ in a complete ordered field we write $\sqrt{a}$ for this b . In the constructed reals, the class r of Example (The Class of the Newton Sequence) has $r > 0$ and $r^2 = 2$, so $r = \sqrt{2}$ by the uniqueness clause, and [Example \(The Rationals below the Square Root of Two\)](#) describes $\mathbb{Q}_{\mathbb{R}}(\sqrt{2})$.

The constants follow the Idea: each h depends on s and a only, and each was found backwards and then placed first. Completeness was used once, to produce s ; the rest is arithmetic in an ordered field. Without completeness the statement fails: in $\mathbb{Q}$, the element $2 \geq 0$ has no square root (Theorem 2.6.1). Section 6.4 meets a field that is Archimedean and still lacks some square roots.

6.1.5 Exercises

6.1.5.1 A. Check your understanding

Exercise. A1

- (a) True or false, with a reason: in **every** ordered field F , if $\mathbb{Q}_F(x) \neq \mathbb{Q}_F(y)$ then $x \neq y$. Is the converse true in every ordered field?
- (b) Write “ F is Archimedean” in symbols, and state its negation with the quantifiers moved to the front (Logic and Proof, Proposition 6.2.1).
- (c) The eight lines (i)–(viii) below are about a complete ordered field F ; six of them, in a suitable order, prove [Theorem 6.1.2](#). Write an admissible order, and say why each of the other two lines is wrong.
 - (i) Then $\iota_F(n + 1) = \iota_F(n) + 1 > s$, and $n + 1 \in \mathbb{N}$.
 - (ii) For every $n \in \mathbb{N}$, $\iota_F(n) > s - 1$.
 - (iii) The set $\iota_F(\mathbb{N})$ is non-empty ($0 \in \mathbb{N}$), so $s := \sup \iota_F(\mathbb{N})$ exists.
 - (iv) This contradicts that s is an upper bound of $\iota_F(\mathbb{N})$.
 - (v) Suppose that $\iota_F(\mathbb{N})$ is bounded above.

- (vi) So $\iota_F(n) > s - 1$ for some $n \in \mathbb{N}$.
- (vii) So $s \in \iota_F(\mathbb{N})$, say $s = \iota_F(m)$.
- (viii) Since $s - 1 < s$, the element $s - 1$ is not an upper bound of $\iota_F(\mathbb{N})$.

Solution.

- (a) True. It is the contrapositive (Logic and Proof, Proposition 2.3.2) of “if $x = y$ then $\mathbb{Q}_F(x) = \mathbb{Q}_F(y)$ ”, which holds because equal elements have the same rationals below them. The converse, “if $x \neq y$ then $\mathbb{Q}_F(x) \neq \mathbb{Q}_F(y)$ ”, is the direction ($\Leftarrow$) of [Lemma 6.1.3](#), and it needs the Archimedean property: in $\mathbb{Q}(t)$, $t \neq t + 1$ and both have every rational below them (the second-reading note after [Lemma 6.1.3](#)).
- (b) $\forall x \in F \exists n \in \mathbb{N} \ x < n \cdot 1_F$. Negation: $\exists x \in F \forall n \in \mathbb{N} \ n \cdot 1_F \leq x$, using $\neg(x < y) \iff y \leq x$ (totality and (O1)). In words: some element lies above or at every $n \cdot 1_F$.
- (c) (v), (iii), (viii), (vi), (i), (iv). No other order is admissible: each line uses the one before it ((iii) needs the bound of (v) for the supremum to exist, (viii) needs s , (vi) unpacks (viii), (i) uses the n of (vi), and (iv) is the contradiction with (i)). The contradiction in (iv) refutes (v), so $\iota_F(\mathbb{N})$ is not bounded above; unpacked with the negation laws, as at the end of the proof of [Theorem 6.1.2](#), that is the statement of (b).

Line (vii) is wrong: a supremum need not belong to the set (in every ordered field, the set of negative elements has supremum 0, which is not negative: the first example after Definition 5.6.1). Line (ii) has the wrong quantifier: “ $s - 1$ is not an upper bound” gives **some** n with $\iota_F(n) > s - 1$, by Logic and Proof, Proposition 6.2.3, not every n (and for $n = 0$ it may be false).

6.1.5.2 B. Practice

Exercise. B1

Let F be a complete ordered field. Prove that $\inf\{\iota_F(1/n) : n \in \mathbb{Z}_{>0}\} = 0$.

Solution. Let $T := \{\iota_F(1/n) : n \in \mathbb{Z}_{>0}\}$. For $n \in \mathbb{Z}_{>0}$, $1/n > 0$ in $\mathbb{Q}$, so $\iota_F(1/n) > 0$: 0 is a lower bound of T .

Let c be a lower bound of T , and suppose, for contradiction, that $c > 0$. By [Theorem 6.1.2](#), F is Archimedean, so Proposition 4.2.3 (with $\varepsilon := c$, applied at $n = N$) gives $N \in \mathbb{Z}_{>0}$ with $1/(N \cdot 1_F) < c$. Since ι_F preserves products and $\iota_F(N) = N \cdot 1_F$, we have $\iota_F(1/N) \cdot (N \cdot 1_F) = \iota_F(1) = 1_F$, so $\iota_F(1/N) = 1/(N \cdot 1_F)$ by uniqueness of inverses (Lemma 4.1.1 (a)). So the element $\iota_F(1/N)$ of T lies below c , against c being a lower bound. Hence every lower bound c satisfies $c \leq 0$, and 0 is the greatest lower bound (Sets, Relations, and Cardinality, Definition 2.6.5).

Exercise. B2

Let F be an Archimedean ordered field and $x \in F$. Prove that the set $\iota_F(\mathbb{Q}_F(x)) = \{\iota_F(q) : q \in \mathbb{Q}_F(x)\}$ has supremum x in F .

Solution. Upper bound. Every element $\iota_F(q)$ of the set satisfies $\iota_F(q) < x$ by the definition of $\mathbb{Q}_F(x)$.

Least. Let u be an upper bound, and suppose $u < x$. By Theorem 4.2.5 there is $q \in \mathbb{Q}$ with $u < \iota_F(q) < x$. Then $q \in \mathbb{Q}_F(x)$, so $\iota_F(q)$ is in the set and exceeds u , against u being an upper bound. So $x \leq u$ by totality, for every upper bound u . Hence $x = \sup \iota_F(\mathbb{Q}_F(x))$.

No completeness was used: the supremum exists because we exhibited it.

6.1.5.3 C. Going deeper

Exercise. C1: An Archimedean field is no larger than the subsets of the rationals

Let F be an Archimedean ordered field. Prove that the map $F \rightarrow \mathcal{P}(\mathbb{Q})$, $x \mapsto \mathbb{Q}_F(x)$, is injective. What does this say about the size of F ?

Solution. Let $x, y \in F$ with $\mathbb{Q}_F(x) = \mathbb{Q}_F(y)$. By Lemma 6.1.3, $x = y$. So the map is injective (Sets, Relations, and Cardinality, Definition 1.4.3).

An injection $F \rightarrow \mathcal{P}(\mathbb{Q})$ says that F has at most as many elements as $\mathcal{P}(\mathbb{Q})$, in the sense of *Sets, Relations, and Cardinality*, Chapter 3. In particular every Archimedean ordered field, and so (by Theorem 6.1.2) every complete ordered field, is no larger than the set of all sets of rationals. The non-Archimedean field $\mathbb{Q}(t)$ shows that the hypothesis is needed for injectivity: t and $t + 1$ have the same image.

Exercise. C2: Find the error

Here is a “proof” that a complete ordered field F is Archimedean.

Suppose that $\iota_F(\mathbb{N})$ is bounded above, and let s be its supremum. Since s is the least upper bound of $\iota_F(\mathbb{N})$, it is the largest element of $\iota_F(\mathbb{N})$, so $s = \iota_F(m)$ for some $m \in \mathbb{N}$. But $\iota_F(m + 1) > s$, a contradiction.

Is the conclusion true? Find the line that fails, give a counterexample to that line, and repair the proof.

Solution. The conclusion is true: it is Theorem 6.1.2.

The illegal step is “the least upper bound of a set is its largest element”. A supremum is an upper bound that lies below every other upper bound (Sets, Relations, and Cardinality, Definition 2.6.5); nothing says it belongs to the set. Counterexample to the step: in the constructed $\mathbb{R}$, $A := \{1 - 1/n : n \in \mathbb{Z}_{>0}\}$ has supremum 1, and $1 \notin A$. Indeed every element is < 1 ; and if $b < 1$, Archimedes in $\mathbb{R}$ (Theorem 5.4.2, with Proposition 4.2.3) gives $n \in \mathbb{Z}_{>0}$ with $1/n < 1 - b$, so $1 - 1/n > b$ and b is not an upper bound. So the step fails for this set, though the theorem’s conclusion is not affected.

Repair: replace the second and third sentences by the argument of the proof of Theorem 6.1.2. Since $s - 1 < s$, $s - 1$ is not an upper bound, so $\iota_F(n) > s - 1$ for some n ; then $\iota_F(n + 1) = \iota_F(n) + 1 > s$, against s being an upper bound. The repair uses that s is **least** only through

“ $s - 1$ is not an upper bound”, which hands us an element of the set just below s , not s itself.

Marking scheme. (10 points) - 3: says the conclusion is true, with the reason (the theorem).
- 3: names the line “the supremum is the largest element” and says which property of suprema it wrongly assumes. - 2: a counterexample **to that step**: a set whose supremum is not an element of it, with the supremum verified. - 2: a correct repair that uses that s is **least**, without assuming s lies in the set: for example, $s - 1$ is not an upper bound, so some $\iota_F(n) > s - 1$ and then $\iota_F(n + 1) > s$; or, equivalently, if $\iota_F(n + 1) \leq s$ for all n then $s - 1$ is a smaller upper bound (the case $n = 0$ uses $1 \leq s$). Fatal errors (0 for the step): declaring the conclusion false; a counterexample to the theorem rather than to the step.