

## 5.2 The Real Numbers and Their Arithmetic

Section 5.1 sorted the Cauchy sequences of rationals by a relation,  $a \sim b$  when  $a - b$  is null, and proved it an equivalence relation (Proposition 5.1.1). This section takes the classes as the new numbers. It defines their sum, product and negative on representatives, proves that the definitions do not depend on the representatives, and shows that the result is a field. The construction follows a pattern seen three times already: the integers modulo  $n$  in *Sets, Relations, and Cardinality*, Section 2.5, the integers and the rationals in Chapter 2, and the quotient of a ring by an ideal in Chapter 3. The order comes in Section 5.3.

### 5.2.1 The real numbers as classes

The Newton sequence and the decimal truncations are different sequences that aim at one number (Example (The Truncations and the Newton Sequence Are Equivalent)). Rather than search  $\mathbb{Q}$  for that number, we let the whole group of sequences aiming at it **be** the number.

*A real number is a Cauchy sequence of rationals, up to a null difference.*

#### Definition 5.2.1. The Real Numbers

The set of **real numbers** is the quotient set (Sets, Relations, and Cardinality, Definition 2.5.4)

$$\mathbb{R} := \mathcal{C} / \sim$$

of the set  $\mathcal{C}$  of Cauchy sequences of rationals (Definition 5.1.1) by the relation  $\sim$  of Definition 5.1.2. Its elements are the classes  $[a] = \{b \in \mathcal{C} : b \sim a\}$  for  $a \in \mathcal{C}$  (Sets, Relations, and Cardinality, Definition 2.5.2), and any  $b \in [a]$  is a **representative** of  $[a]$ .

In words: a real number is a set of Cauchy sequences of rationals, namely all those equivalent to one of them. By Sets, Relations, and Cardinality, Theorem 2.5.1 (b), for all  $a, b \in \mathcal{C}$ ,

$$[a] = [b] \iff a \sim b \iff a - b \text{ is null.}$$

This is the only tool we have, so far, for deciding when two real numbers are equal: compare representatives, and ask whether their difference is null.

#### Examples.

- *The Newton class.* By Example (The Truncations and the Newton Sequence Are Equivalent),  $[(x_n)_n] = [(d_k)_k]$ : one real number, with two quite different representatives. By Example (Finite Changes and Shifts) (b),  $(x_{n+1})_n$  is a third.
- *The class of a constant.* For  $q \in \mathbb{Q}$ , the class  $[(q)_n]$  contains  $(q)_n$ , and also  $(q + 1/(n+1))_n$ , which lies in  $\mathcal{C}$  by Theorem 4.5.3 (a), (b) and is equivalent to  $(q)_n$ , since the difference  $(1/(n+1))_n$  is null (Example (First Limits) (b)). Different constants give different classes, because constant sequences are equivalent only when they are equal (the degenerate example after Definition 5.1.2). Section 5.4 uses these classes to place a copy of  $\mathbb{Q}$  inside  $\mathbb{R}$ .
- *Zero.*  $[(0)_n]$  is the set of all null sequences in  $\mathcal{C}$ :  $b \sim (0)_n$  means that  $b - (0)_n = b$  is null. Every null sequence lies in  $\mathcal{C}$  (Theorem 4.5.3 (a)). So  $(1/(n+1))_n$  (Example (First Limits) (b)),  $((-1)^n/(n+1))_n$  (null times bounded, as  $|(-1)^n| = 1$ ; Proposition 4.4.1) and every sequence that is eventually 0 (null by Proposition 4.3.2 (a)) are representatives of it.

**What has been done, and what has not.** The symbol  $\mathbb{R}$  now names one definite set, built from  $\mathbb{Q}$  by the set-building tools of *Sets, Relations, and Cardinality*. In earlier volumes, the “familiar real numbers” were used informally; from here on in this volume  $\mathbb{R}$  means this set and nothing else, and every fact about it must be proved from the definition. At this moment  $\mathbb{R}$  has no addition, no multiplication, no order, and it does not contain  $\mathbb{Q}$ : the rational number 2 is not a class of sequences. All of these must be built. This section builds the arithmetic.

## 5.2.2 Arithmetic on representatives

To add two real numbers, add representatives term by term; likewise for products and negatives. The terms are rationals, where the arithmetic is known.

### Definition 5.2.2. Sum, Product and Negative of Real Numbers

For  $a, b \in \mathcal{C}$ , let

$$[a] + [b] := [a + b], \quad [a] [b] := [ab], \quad -[a] := [-a],$$

where  $a + b$ ,  $ab$  and  $-a$  are the termwise sum, product and negative. Let  $0_{\mathbb{R}} := [(0)_n]$  and  $1_{\mathbb{R}} := [(1)_n]$ .

The right-hand sides are classes of members of  $\mathcal{C}$ , because  $\mathcal{C}$  is closed under termwise sums, products and negatives and contains the constants (Theorem 4.5.3). That makes the right-hand sides **meaningful**. It does not yet make the definition **legitimate**. Each formula takes a class and reads off a representative; if two representatives of the same class gave different answers, the “operation” would assign two values to one input, and so would not be a function (Sets, Relations, and Cardinality, Definition 1.4.1).

This danger is real, and it has been met before. In *Sets, Relations, and Cardinality*, Section 2.5, the arithmetic of the integers modulo  $n$  was defined by the same kind of formula and survived the check (Sets, Relations, and Cardinality, Proposition 2.5.5), while the rule  $[a]_6 \mapsto [a]_4$  failed it (Sets, Relations, and Cardinality, Example (A rule on classes that is not a function)). Chapter 2 made the same check for the integers and for the rationals, through the lemma Theorem 2.2.1, which says that a rule on classes is a function as soon as equivalent inputs give equivalent outputs. Here the check is the same, and only the computation behind it is new.

**Representatives, then independence.** To define an operation on classes, define it on representatives; then show that the output class survives replacing each input by an equivalent one.

For sums this is a one-line computation. For products it is not, and the next theorem leans on Chapter 4’s result that Cauchy sequences are bounded (Proposition 4.5.2).

### Theorem 5.2.1. The Operations on Real Numbers Are Well Defined

Let  $a, a', b, b' \in \mathcal{C}$  with  $a \sim a'$  and  $b \sim b'$ . Then

$$a + b \sim a' + b', \quad ab \sim a'b', \quad -a \sim -a'.$$

Hence the rules of Definition 5.2.2 define functions  $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\cdot: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $-: \mathbb{R} \rightarrow \mathbb{R}$ .

*Idea.* For the product we must show that  $a_nb_n - a'_nb'_n$  dies out, knowing only that  $a_n - a'_n$  and  $b_n - b'_n$  do. Replace one factor at a time: subtract and add the mixed term  $a_nb'_n$ , so that

$$a_nb_n - a'_nb'_n = a_n(b_n - b'_n) + (a_n - a'_n)b'_n.$$

Each summand is a null sequence times another sequence. A null sequence times an **unbounded** one need not be null, so we need the multipliers  $(a_n)$  and  $(b'_n)$  bounded; they are, because they are Cauchy. Then each summand is null, and so is the sum.

*Proof.* Let  $a, a', b, b' \in \mathcal{C}$  with  $a \sim a'$  and  $b \sim b'$ ; so  $a - a'$  and  $b - b'$  are null (Definition 5.1.2).

*Sum.* For every  $n$ ,  $(a_n + b_n) - (a'_n + b'_n) = (a_n - a'_n) + (b_n - b'_n)$ , by the field axioms of  $\mathbb{Q}$ . So  $(a + b) - (a' + b')$  is the sum of two null sequences, which is null by Theorem 4.4.3 (a). Hence  $a + b \sim a' + b'$ .

*Negative.* For every  $n$ ,  $(-a_n) - (-a'_n) = -(a_n - a'_n)$  (Proposition 2.3.2 (c)). So  $(-a) - (-a') = -(a - a')$ , which is null by Theorem 4.4.3 (b). Hence  $-a \sim -a'$ .

*Product.* For every  $n$ , by the field axioms of  $\mathbb{Q}$ ,

$$a_nb_n - a'_nb'_n = a_n(b_n - b'_n) + (a_n - a'_n)b'_n. \quad (5.2.1)$$

The sequence  $a$  is in  $\mathcal{C}$ , so it is bounded (Proposition 4.5.2): there is  $M \in \mathbb{Q}_{>0}$  with  $|a_n| \leq M$  for every  $n$ . Since  $b - b'$  is null, the sequence  $((b_n - b'_n)a_n)_n$  is null by Proposition 4.4.1, and it is the first summand of Eq. 5.2.1, by commutativity in  $\mathbb{Q}$ . Likewise  $b' \in \mathcal{C}$  is bounded, by some  $M' \in \mathbb{Q}_{>0}$  (Proposition 4.5.2), and  $a - a'$  is null, so the second summand  $((a_n - a'_n)b'_n)_n$  is null (Proposition 4.4.1). By the sum rule (Theorem 4.4.3 (a)),  $ab - a'b'$  is null. Hence  $ab \sim a'b'$ .

*Functions.* The maps  $(a, b) \mapsto a + b$  and  $(a, b) \mapsto ab$  are functions  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and  $a \mapsto -a$  is a function  $\mathcal{C} \rightarrow \mathcal{C}$  (Theorem 4.5.3). Taking  $b' = b$  above ( $b \sim b$  by reflexivity) shows that each binary map respects  $\sim$  in its first argument; taking  $a' = a$  shows the same for the second. These are the hypotheses of Theorem 2.2.1 (b), and  $-a \sim -a'$  is the hypothesis of Theorem 2.2.1 (a). So the three rules define functions  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathbb{R} \rightarrow \mathbb{R}$ , as claimed. ■

The proof used two different bounds: one for  $a$ , which multiplies  $b_n - b'_n$ , and one for  $b'$ , which multiplies  $a_n - a'_n$ . A bound for  $b$  would be of no use, since  $b$  appears nowhere in Eq. 5.2.1 as a multiplier. Every Cauchy sequence is bounded, so both bounds exist; this is where the restriction of  $\sim$  to  $\mathcal{C}$  pays (compare the “Why only on  $\mathcal{C}$ ” paragraph of Section 5.1).

So  $\mathbb{R}$  has a sum, a product and a negative. For example,  $[(x_n)] + [(1)_n] = [(x_n + 1)_n]$ , and the answer does not change if  $(x_n)$  is replaced by  $(d_k)$ : the theorem says  $(x_n + 1) \sim (d_n + 1)$ .

### 5.2.3 The null sequences form an ideal

Chapter 3 made the check of well-definedness once and for all, for the quotient of a commutative ring by an ideal (Definition 3.4.1, Theorem 3.4.1). The construction of  $\mathbb{R}$  is an instance of it,

and seeing it that way gives the laws of arithmetic in  $\mathbb{R}$  without checking them one by one.

Let  $\mathcal{N} \subseteq \mathcal{C}$  be the set of null sequences of rationals; it lies in  $\mathcal{C}$  by Theorem 4.5.3 (a). Then

$$a \sim b \iff a - b \in \mathcal{N} \quad (a, b \in \mathcal{C}),$$

which is the relation  $\sim_{\mathcal{N}}$  of Theorem 3.4.1 (a). So  $\mathbb{R} = \mathcal{C}/\sim$  is  $\mathcal{C}/\mathcal{N}$  in the notation of Chapter 3, and the rules of Definition 5.2.2 are the rules of Theorem 3.4.1 (b). What must be checked is that  $\mathcal{C}$  is a commutative ring and  $\mathcal{N}$  an ideal of it. Once this is checked, Theorem 3.4.1 (b) gives the well-definedness in Theorem 5.2.1 a second time. Its proof changes one argument at a time, so the multipliers are  $b$  and  $a'$  rather than  $a$  and  $b'$ , and (I3) is supplied by “null times bounded is null”.

### Theorem 5.2.2. The Real Numbers Form a Commutative Ring

$(\mathbb{R}, +, \cdot)$ , with the operations of Definition 5.2.2, is a commutative ring (Definition 2.3.1) with zero  $0_{\mathbb{R}}$  and one  $1_{\mathbb{R}}$ : it satisfies (F1)–(F7) and (F9), the additive inverse of  $[a]$  is  $-[a]$ , and  $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ .

*Idea.* Two claims, then Chapter 3’s theorem. Termwise operations inherit every law of  $\mathbb{Q}$ , because a law about sequences is an equation between functions on  $\mathbb{N}$ , checked one index at a time. And the null sequences satisfy the three clauses of an ideal:  $(0)_n$  is null, sums of null sequences are null, and a null sequence times a Cauchy sequence is null. The quotient theorem then hands over all the laws at once. For  $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ , the constant 1 is not null.

*Proof.*

*Claim.*  $\mathcal{C}$ , with the termwise operations, is a commutative ring with zero  $(0)_n$ , one  $(1)_n$ , and  $-a$  the additive inverse of  $a$ .

*Proof.* By Theorem 4.5.3, the termwise sum and product of two members of  $\mathcal{C}$ , the termwise negative of a member, and the constants  $(0)_n$ ,  $(1)_n$  lie in  $\mathcal{C}$ ; so  $+$  and  $\cdot$  are functions  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ . Each of (F1)–(F7) and (F9) asserts an equality of sequences, that is, of functions  $\mathbb{N} \rightarrow \mathbb{Q}$ , so it holds if and only if it holds at every index  $n$  (Sets, Relations, and Cardinality, Proposition 1.4.1). At the index  $n$  it is the same clause in  $\mathbb{Q}$ , with  $a_n, b_n, c_n$  in place of  $a, b, c$ , and  $\mathbb{Q}$  is a field (Theorem 2.5.1). For example, (F9) at  $n$  reads  $a_n(b_n + c_n) = a_nb_n + a_nc_n$ , and (F4) reads  $a_n + (-a_n) = 0$ . ■

*Claim.*  $\mathcal{N}$  is an ideal of  $\mathcal{C}$  (Definition 3.4.1).

*Proof.* (I1) The zero sequence  $(0)_n$  is null (Example (First Limits) (a)). (I2) If  $a, b \in \mathcal{N}$ , then  $a + b \in \mathcal{N}$ , by the sum rule Theorem 4.4.3 (a). (I3) Let  $a \in \mathcal{N}$  and  $r \in \mathcal{C}$ . Then  $r$  is bounded (Proposition 4.5.2), so  $(a_nr_n)_n$  is null (Proposition 4.4.1), and  $ra = (r_na_n)_n = (a_nr_n)_n$  by commutativity in  $\mathbb{Q}$ . So  $ra \in \mathcal{N}$ . ■

As noted before the theorem,  $\sim$  is the relation  $\sim_{\mathcal{N}}$ , so  $\mathbb{R} = \mathcal{C}/\mathcal{N}$ , and the operations of Definition 5.2.2 are those of Theorem 3.4.1 (b). By Theorem 3.4.1 (c),  $\mathbb{R}$  is a commutative ring with zero  $[(0)_n] = 0_{\mathbb{R}}$  and one  $[(1)_n] = 1_{\mathbb{R}}$ . Its additive inverse of  $[a]$  is  $[-a] = -[a]$ , since  $[a] + [-a] = [a + (-a)] = [(0)_n]$  by the first claim.

Finally, by Theorem 3.4.1 (d),  $1_{\mathbb{R}} = 0_{\mathbb{R}}$  if and only if  $\mathcal{N} = \mathcal{C}$ . But  $(1)_n \in \mathcal{C}$  is not null: for  $\varepsilon = 1/2$ , every  $n$  has  $|1 - 0| = 1 \geq 1/2$ . So  $\mathcal{N} \neq \mathcal{C}$ , and  $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ . This proves the theorem. ■

To see what the quotient theorem did, here is distributivity unfolded. For  $a, b, c \in \mathcal{C}$ , using Definition 5.2.2 at every step but the middle one, and (F9) in  $\mathcal{C}$  there,

$$\begin{aligned} [a]([b] + [c]) &= [a][b + c] = [a(b + c)] \\ &= [ab + ac] = [ab] + [ac] = [a][b] + [a][c]. \end{aligned}$$

Every law goes the same way: move to representatives, use the law in  $\mathcal{C}$ , which is the law in  $\mathbb{Q}$  at each index, and move back. Only the legitimacy of “move to representatives” needed an argument, and Theorem 5.2.1 supplied it.

## 5.2.4 Inverses

A commutative ring is not yet a field. The missing clause is (F8): every  $x \neq 0_{\mathbb{R}}$  needs an inverse. The obvious idea is to invert term by term, and it fails; the example after the theorem shows how. What saves it is a fact from Chapter 4: a Cauchy sequence that is not null stays a **fixed distance** away from 0 from some index on (Theorem 4.5.4).

### Theorem 5.2.3. The Real Numbers Form a Field

Every  $x \in \mathbb{R}$  with  $x \neq 0_{\mathbb{R}}$  has a multiplicative inverse: there is  $y \in \mathbb{R}$  with  $xy = 1_{\mathbb{R}}$ . Hence  $\mathbb{R}$ , with the operations of Definition 5.2.2, is a field (Definition 2.3.1).

*Idea.* Write  $x = [a]$ . Since  $x \neq 0_{\mathbb{R}}$ ,  $a$  is not null, so from some index  $N$  on its terms are at least  $c > 0$  in absolute value. Invert  $a_n$  for  $n \geq N$ , and put anything harmless, say 1, before  $N$ ; Chapter 4 showed the result is again Cauchy (Proposition 4.5.6). The product with  $a$  is 1 from  $N$  on, and the first  $N$  terms do not matter. *Before reading on: the inverse is built from one representative  $a$ . Why does the class  $[b]$  not depend on which representative of  $x$  we started from?*

*Proof.* Let  $x \in \mathbb{R}$  with  $x \neq 0_{\mathbb{R}}$ . By Sets, Relations, and Cardinality, Definition 2.5.4,  $x = [a]$  for some  $a \in \mathcal{C}$ ; fix one such  $a$  ( $\exists E$ , Logic and Proof, Definition 5.6.1). The sequence  $a$  is not null: otherwise  $a \sim (0)_n$ , and  $x = [(0)_n] = 0_{\mathbb{R}}$  by Sets, Relations, and Cardinality, Theorem 2.5.1 (b).

By Theorem 4.5.4 (with  $F = \mathbb{Q}$ ) there are  $c \in \mathbb{Q}_{>0}$  and  $N \in \mathbb{N}$  such that  $a_n \geq c$  for every  $n \geq N$ , or  $a_n \leq -c$  for every  $n \geq N$ . Fix one such pair  $(c, N)$  ( $\exists E$  again). This fixes two objects, once; it is not a selection from infinitely many sets. In either case  $|a_n| \geq c$  for every  $n \geq N$ : if  $a_n \geq c > 0$ , then  $|a_n| = a_n$ ; if  $a_n \leq -c$ , then  $|a_n| = -a_n \geq c$  (Definition 4.1.1, Proposition 2.3.5 (a)).

By Proposition 4.5.6, the sequence  $b$  with  $b_n := 1/a_n$  for  $n \geq N$  and  $b_n := 1$  for  $n < N$  is in  $\mathcal{C}$ , and  $a_nb_n = 1$  for every  $n \geq N$ . So the sequence  $ab - (1)_n$  agrees with  $(0)_n$  from  $N$  on, and it is null by Proposition 4.3.2 (a). Hence  $ab \sim (1)_n$ , and

$$x[b] = [a][b] = [ab] = [(1)_n] = 1_{\mathbb{R}}.$$

So  $y := [b]$  is an inverse of  $x$ , and (F8) holds. Together with [Theorem 5.2.2](#), which gives (F1)–(F7), (F9) and  $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ , this shows that  $\mathbb{R}$  is a field. ■

The answer to the prompt: **inverses are unique**. By the theorem,  $\mathbb{R}$  is a field, so  $x$  has only one inverse (Lemma 4.1.1 (a) with  $F = \mathbb{R}$ ); the reason is the one-line computation  $y = y(xy') = (yx)y' = y'$  for two inverses  $y, y'$ . So starting from another representative  $a'$  of  $x$ , with its own  $c', N'$  and  $b'$ , gives  $[b'] = [b]$ ; we write  $x^{-1}$  for it.

### Example. Termwise Reciprocals Are Not a Definition

A shorter route to inverses would be: for  $x = [a] \neq 0_{\mathbb{R}}$ , let  $x^{-1} := [(1/a_n)_n]$ . Explain why this is not a definition, and why  $0_{\mathbb{R}}$  must be excluded in any case.

*Solution. It need not make sense.* The sequences  $a := (0, 1, 1, 1, \dots)$  and  $(1)_n$  agree from  $n = 1$  on;  $a$  is in  $\mathcal{C}$ , since  $|a_m - a_n| = 0$  for all  $m, n \geq 1$  (threshold  $N = 1$  for every tolerance), so the two sequences are equivalent (Example (Finite Changes and Shifts) (a)) and represent the same real number  $1_{\mathbb{R}} \neq 0_{\mathbb{R}}$ . With the representative  $a$ , the rule asks for  $1/a_0 = 1/0$ , which does not exist. A rule that cannot be applied to some representatives does not define a function on classes; it would need a statement of which representatives are allowed, and a proof that the answer does not depend on the allowed one. The proof of [Theorem 5.2.3](#) supplies both at once: it inverts only from an index  $N$  on, and the uniqueness of inverses makes the class independent of all the choices.

*Why  $0_{\mathbb{R}}$  is excluded.* The null sequence  $(1/(n+1))_n$  represents  $0_{\mathbb{R}}$  and has no zero term, so the termwise reciprocal  $(n+1)_n$  exists. But it is not bounded (Theorem 2.5.3), so it is not in  $\mathcal{C}$  (Proposition 4.5.2), and it represents no real number. No inverse of  $0_{\mathbb{R}}$  is possible in any case, since  $0_{\mathbb{R}} \cdot y = y \cdot 0_{\mathbb{R}} = 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$  for every  $y$  ((F6), Proposition 2.3.2 (b), [Theorem 5.2.2](#)).

**Warning. A termwise formula is not an operation until it is checked on every representative.** “ $[a]^{-1} := [(1/a_n)]$ ” fails on  $(0, 1, 1, 1, \dots)$ , a representative of  $1_{\mathbb{R}}$ , whose first term cannot be inverted. Every definition on  $\mathbb{R}$  must say what it does to **each** representative of a class, and then prove that the class of the answer is the same for all of them.

*Quick check.* Find a representative of  $[(1 + 1/(n+1))_n] \cdot [(2)_n] - [(2)_n]$ , and name its class.

*Solution.* By [Definition 5.2.2](#), with  $x - y := x + (-y)$  in the ring  $\mathbb{R}$ , the real number is  $[(1 + 1/(n+1)) \cdot 2] + [(-2)_n] = [(2 + 2/(n+1) - 2)_n]$ , so  $(2/(n+1))_n$  represents it. This sequence is null (Theorem 4.4.3 (b) with  $c = 2$ , applied to  $(1/(n+1))_n$ ), which is null by Example (First Limits) (b)), so the class is  $0_{\mathbb{R}}$ . In other words,  $[(1 + 1/(n+1))_n] \cdot [(2)_n] = [(2)_n]$ . The first factor is  $1_{\mathbb{R}}$ , since  $(1 + 1/(n+1))_n - (1)_n = (1/(n+1))_n$  is null, so  $(1 + 1/(n+1))_n \sim (1)_n$ ; it is  $1_{\mathbb{R}}$  represented by a sequence none of whose terms is 1.

$\mathbb{R}$  is now a field. It has no order yet: “positive” for a class of sequences needs care, and Section 5.3 defines it.

## 5.2.5 Exercises

### 5.2.5.1 A. Check your understanding

#### Exercise. A1

- Prove, directly from [Definition 5.2.2](#) and the field axioms of  $\mathbb{Q}$ , that  $([a][b])[c] = [a]([b][c])$  and  $1_{\mathbb{R}}[a] = [a]$  for all  $a, b, c \in \mathcal{C}$ .
- In the product part of the proof of [Theorem 5.2.1](#), which two of the four sequences  $a, a', b, b'$  must be bounded, and why is “ $b$  is bounded” not what [Eq. 5.2.1](#) needs?
- Run that proof on  $a = (2 + 1/(n+1))_n$ ,  $a' = (2)_n$ ,  $b = (x_n)_n$  (the Newton sequence) and  $b' = (d_n)_n$  (the truncations): check that  $a \sim a'$  and  $b \sim b'$ , write  $a_nb_n - a'_nb'_n$  as in [Eq. 5.2.1](#), and give a bound for each multiplier.
- State the idea of the proof of [Theorem 5.2.1](#) in two sentences.

*Solution.*

- By [Definition 5.2.2](#) twice,  $([a][b])[c] = [ab][c] = [(ab)c]$  and  $[a]([b][c]) = [a][bc] = [a(bc)]$ . For every  $n$ ,  $(a_nb_n)c_n = a_n(b_nc_n)$  by (F5) in  $\mathbb{Q}$ , so  $(ab)c = a(bc)$  as sequences (Sets, Relations, and Cardinality, Proposition 1.4.1), and the two classes are equal. Likewise  $1_{\mathbb{R}}[a] = [(1)_n][a] = [(1 \cdot a_n)_n] = [a]$ , by (F6) and (F7) in  $\mathbb{Q}$ .
- $a$  and  $b'$ . In [Eq. 5.2.1](#),  $a_n$  multiplies the null  $b_n - b'_n$ , and  $b'_n$  multiplies the null  $a_n - a'_n$ ; Proposition 4.4.1 needs the multiplier bounded. The sequence  $b$  appears in [Eq. 5.2.1](#) only inside the null factor  $b_n - b'_n$ , so its bound is never used.
- $a - a' = (1/(n+1))_n$  is null (Example (First Limits) (b)), so  $a \sim a'$  (both are in  $\mathcal{C}$  by Theorem 4.5.3 (a), (b));  $b \sim b'$  is Example (The Truncations and the Newton Sequence Are Equivalent). The split is

$$\left(2 + \frac{1}{n+1}\right)x_n - 2d_n = \left(2 + \frac{1}{n+1}\right)(x_n - d_n) + \frac{1}{n+1}d_n.$$

Multipliers:  $|2 + 1/(n+1)| \leq 3$ , since  $0 < 1/(n+1) \leq 1$ ; and  $|d_n| \leq 2$ , since  $1 \leq d_n < 2$  (Steps 1 and 3 of the solution of Example (The Truncations and the Newton Sequence Are Equivalent)). So the first summand is null (null times bounded by 3) and the second is null (null times bounded by 2).

- Changing both factors of a product at once is split into two changes, one factor at a time, by adding and subtracting a mixed term. Each change is a null sequence times a bounded one, because Cauchy sequences are bounded, so each is null, and so is their sum.

### 5.2.5.2 B. Practice

#### Exercise. B1: The negative, directly

Without using Theorem 3.4.1, prove that the rule  $-[a] := [-a]$  defines a function  $\mathbb{R} \rightarrow \mathbb{R}$ , and that  $[a] + (-[a]) = 0_{\mathbb{R}}$  for every  $a \in \mathcal{C}$ . Deduce that  $[a] - [b] = [a - b]$  for all  $a, b \in \mathcal{C}$ .

*Solution. Well defined.*  $-a \in \mathcal{C}$  for  $a \in \mathcal{C}$  (Theorem 4.5.3 (b)). Let  $a \sim a'$ . For every  $n$ ,  $(-a_n) - (-a'_n) = -(a_n - a'_n)$  (Proposition 2.3.2 (c)), and  $(-(a_n - a'_n))_n$  is null because  $a - a'$  is (Theorem 4.4.3 (b)). So  $-a \sim -a'$ , and Theorem 2.2.1 (a) gives a function  $\mathbb{R} \rightarrow \mathbb{R}$ .

*Inverse.*  $[a] + (-[a]) = [a] + [-a] = [a + (-a)] = [(a_n + (-a_n))_n] = [(0)_n] = 0_{\mathbb{R}}$ , by Definition 5.2.2 and (F4) in  $\mathbb{Q}$  at each index.

*Differences.*  $[a] - [b] = [a] + (-[b]) = [a] + [-b] = [a + (-b)] = [a - b]$ , by the definition  $x - y := x + (-y)$  in a ring and Definition 5.2.2.

### Exercise. B2: The larger of two reals, faded

For  $x, y \in \mathbb{Q}$  let  $\max(x, y) := x$  if  $x \geq y$ , and  $\max(x, y) := y$  otherwise. For  $a, b \in \mathcal{C}$  consider the rule

$$g([a], [b]) := [(\max(a_n, b_n))_n].$$

- (a) Prove that  $|\max(x, y) - \max(x', y')| \leq |x - x'| + |y - y'|$  for all  $x, y, x', y' \in \mathbb{Q}$ . *Hint:  $\max(x, y)$  is one of  $x, y$ , and is  $\geq$  both.*
- (b) Deduce from (a) that  $(\max(a_n, b_n))_n \in \mathcal{C}$  whenever  $a, b \in \mathcal{C}$ .
- (c) Here is the start of a proof that the rule  $g$  is well defined. *Name the move, then complete it.*  
 “Let  $a, a', b, b' \in \mathcal{C}$  with  $a \sim a'$  and  $b \sim b'$ . By (b), the sequences  $(\max(a_n, b_n))_n$  and  $(\max(a'_n, b'_n))_n$  are in  $\mathcal{C}$ . Let  $e_n := \max(a_n, b_n) - \max(a'_n, b'_n)$ . By (a), for every  $n$ ,

$$0 \leq |e_n| \leq |a_n - a'_n| + |b_n - b'_n|.$$

⟨ blank ⟩”

*Solution.*

- (a) Write  $M := \max(x, y)$  and  $M' := \max(x', y')$ . Note  $M \geq x$ ,  $M \geq y$ , and  $M$  is one of  $x, y$ ; likewise for  $M'$ . Suppose first  $M \geq M'$ . If  $M = x$ , then  $0 \leq M - M' \leq x - x' \leq |x - x'|$ , since  $M' \geq x'$  (Proposition 2.3.5 (b), Proposition 4.1.3 (c)). If  $M = y$ , then likewise  $0 \leq M - M' \leq y - y' \leq |y - y'|$ . Either way  $|M - M'| = M - M' \leq |x - x'| + |y - y'|$ . If  $M' > M$ , exchange the roles of  $(x, y)$  and  $(x', y')$ ; the right-hand side is unchanged, as  $|x' - x| = |x - x'|$ .
- (b) Let  $\varepsilon \in \mathbb{Q}_{>0}$ , and let  $N$  be a threshold with tolerance  $\varepsilon/2$  for both  $a$  and  $b$  (the larger of the two thresholds). For  $m, n \geq N$ , (a) with  $(x, y, x', y') = (a_m, b_m, a_n, b_n)$  gives  $|\max(a_m, b_m) - \max(a_n, b_n)| \leq |a_m - a_n| + |b_m - b_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ .
- (c) *The move:* representatives, then independence; here its last step, “a sequence squeezed between 0 and a null sequence is null, so the outputs are equivalent and the classes agree”. *Completion:* The sequences  $(|a_n - a'_n|)_n$  and  $(|b_n - b'_n|)_n$  are null, since  $||t| - 0| = |t - 0|$  for every  $t \in \mathbb{Q}$  (Definition 4.3.3). So their sum is null (Theorem 4.4.3 (a)). By the squeeze Proposition 4.4.4 (c), with lower sequence  $(0)_n$  and this upper sequence,  $(|e_n|)_n$  is null; hence  $(e_n)_n$  is null, by the same remark on  $|e_n - 0|$ . So  $(\max(a_n, b_n))_n \sim (\max(a'_n, b'_n))_n$ . Taking  $b' = b$ , and then  $a' = a$ , gives the two hypotheses of Theorem 2.2.1 (b), so  $g$  is a function  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ .

**Exercise. B3: A square root of two**

Let  $(x_n)$  be the Newton sequence. Show that  $[(x_n)] \cdot [(x_n)] = [(2)_n]$ .

*Solution.* By Definition 5.2.2,  $[(x_n)] \cdot [(x_n)] = [(x_n^2)_n]$ , so we must show that  $(x_n^2 - 2)_n$  is null. By the bound  $0 < x_n^2 - 2 \leq 1/(4n)$  for  $n \geq 1$  (Section 5.1, (NS1)–(NS2)), and  $1/(4n) \leq 1/(n+1)$  for  $n \geq 1$  (since  $4n - (n+1) = 3n - 1 > 0$ ; Proposition 4.1.2 (c)),

$$0 \leq x_n^2 - 2 \leq \frac{1}{n+1} \quad (n \geq 1).$$

The squeeze Proposition 4.4.4 (c), with  $N_0 = 1$ , lower sequence  $(0)_n$  and the null upper sequence  $(1/(n+1))_n$  (Example (First Limits) (b)), defined for every  $n \in \mathbb{N}$ , shows that  $(x_n^2 - 2)_n$  is null. So  $(x_n^2) \sim (2)_n$  and the classes are equal. (The inequality fails at  $n = 0$ , where  $x_0^2 - 2 = 2$ ; the squeeze does not need it there.) Section 5.4 returns to this class.

**5.2.5.3 C. Going deeper****Exercise. C1: Exit check: multiplication is well defined**

State precisely what must be proved for the rule  $[a][b] := [ab]$  to define a function  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , and prove it, including the fact that  $ab \in \mathcal{C}$  for  $a, b \in \mathcal{C}$ .

*Solution. What must be shown.* (i) For  $a, b \in \mathcal{C}$ , the termwise product  $ab$  is in  $\mathcal{C}$ , so that  $[ab]$  is a real number. (ii) If  $a \sim a'$  and  $b \sim b'$ , then  $ab \sim a'b'$ . By Theorem 2.2.1 (b), (i) and (ii) give exactly one function  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  with  $([a], [b]) \mapsto [ab]$ ; without (ii), one pair of classes could receive two values, and (Fn2) of Sets, Relations, and Cardinality, Definition 1.4.1 would fail.

(i) *Bound, then shrink.* Let  $a, b \in \mathcal{C}$ . Both are bounded (Proposition 4.5.2); let  $M \in \mathbb{Q}_{>0}$  be a bound for both (the larger of the two bounds). For all  $m, n$ ,

$$\begin{aligned} |a_m b_m - a_n b_n| &= |a_m(b_m - b_n) + (a_m - a_n)b_n| \\ &\leq M|b_m - b_n| + M|a_m - a_n|, \end{aligned}$$

by Proposition 4.1.3 (b), (e). Let  $\varepsilon \in \mathbb{Q}_{>0}$ , and let  $N$  be a threshold with tolerance  $\varepsilon/(2M)$  for both  $a$  and  $b$ . For  $m, n \geq N$ , the right side is less than  $M \cdot \varepsilon/(2M) + M \cdot \varepsilon/(2M) = \varepsilon$ . So  $ab \in \mathcal{C}$ . (This is Theorem 4.5.3 (c), which may be cited instead.)

(ii) *The add-and-subtract decomposition.* For every  $n$ ,

$$a_n b_n - a'_n b'_n = a_n(b_n - b'_n) + (a_n - a'_n)b'_n.$$

The multipliers  $a$  and  $b'$  are in  $\mathcal{C}$ , hence bounded (Proposition 4.5.2). The sequences  $b - b'$  and  $a - a'$  are null. By Proposition 4.4.1, both summands are null, and by the sum rule (Theorem 4.4.3 (a)) so is  $ab - a'b'$ . Hence  $ab \sim a'b'$ .

**Marking scheme.** (10 points)

- 2: states what must be shown (the product lies in  $\mathcal{C}$ ; equivalent inputs give equivalent outputs) and why this makes the rule a function.

- 2: proves  $ab \in \mathcal{C}$ , by “bound, then shrink” with the threshold’s dependence on  $\varepsilon$  and  $M$  clear, or by citing Theorem 4.5.3 (c).
- 2: the add-and-subtract decomposition, or any identity writing  $a_nb_n - a'_nb'_n$  as a sum of (null)  $\times$  (sequence) terms.
- 2: boundedness of the two multipliers, justified by “Cauchy implies bounded”.
- 2: concludes with null  $\times$  bounded is null and null  $+$  null is null.

A proof that changes one factor at a time (first  $ab \sim a'b$ , then  $a'b \sim a'b'$ ) earns the same points; the multipliers are then  $b$  and  $a'$ . A proof through the quotient ring is also valid: show that  $\mathcal{N}$  is an ideal of the commutative ring  $\mathcal{C}$ , and cite Theorem 3.4.1 (b). It earns the first two items as above; the decomposition item for identifying  $\sim$  with  $\sim_{\mathcal{N}}$  and proving (I1) and (I2); the multiplier item for “a Cauchy multiplier is bounded” (Proposition 4.5.2); and the last item for (I3), null  $\times$  Cauchy is null, together with a correct citation of the quotient theorem. The ideal clauses must be proved, not cited from Theorem 5.2.2. Fatal errors (0 for the step): “both differences are small, so the product difference is small” with no bound on the multipliers; a bound for  $b$  used where  $b'$  is the multiplier.

### Exercise. C2: No square is zero except zero

- Let  $a$  be a sequence of rationals such that  $(a_n^2)_n$  is null. Prove that  $a$  is null.
- Deduce, using only Definition 5.2.2 and (a), and not Theorem 5.2.3, that for  $x \in \mathbb{R}$ ,  $x^2 = 0_{\mathbb{R}}$  implies  $x = 0_{\mathbb{R}}$ .

*Solution.*

- Let  $\varepsilon \in \mathbb{Q}_{>0}$ . Then  $\varepsilon^2 \in \mathbb{Q}_{>0}$ , so there is  $N$  with  $|a_n^2| < \varepsilon^2$  for every  $n \geq N$ . For such  $n$ ,  $|a_n|^2 = a_n^2 = |a_n^2| < \varepsilon^2$  (Proposition 4.1.3 (b), and  $a_n^2 \geq 0$ ), and  $|a_n| \geq 0$ ,  $\varepsilon > 0$ ; so  $|a_n| < \varepsilon$  by Proposition 2.3.5 (g). The threshold for  $\varepsilon$  is the threshold of  $(a_n^2)_n$  for  $\varepsilon^2$ . So  $a$  is null.
- Let  $x = [a]$  with  $a \in \mathcal{C}$ . Then  $x^2 = [a][a] = [(a_n^2)_n]$  by Definition 5.2.2. If  $x^2 = 0_{\mathbb{R}} = [(0)_n]$ , then  $(a_n^2)_n \sim (0)_n$  (Sets, Relations, and Cardinality, Theorem 2.5.1 (b)), that is,  $(a_n^2)_n$  is null. By (a),  $a$  is null, so  $a \sim (0)_n$  and  $x = 0_{\mathbb{R}}$ .

In a field this also follows from inverses: if  $x \neq 0_{\mathbb{R}}$ , then  $x = x^{-1}x^2 = x^{-1}0_{\mathbb{R}} = 0_{\mathbb{R}}$ , a contradiction. The direct proof shows the fact is already visible in the representatives, before any inverse exists. It fails in some commutative rings: in  $\mathbb{Z}/4\mathbb{Z}$  of Chapter 3 (Definition 3.4.2),  $[2]^2 = [4] = [0]$  while  $[2] \neq [0]$ .