

Constructing the Real Numbers from Cauchy Sequences

Chapter 4 ended at a hole. The Newton sequence $2, 3/2, 17/12, 577/408, \dots$ of Chapter 2 is Cauchy in $\mathbb{Q}$, and no rational is its limit (Theorem 4.6.1). Its terms crowd around a place where $\mathbb{Q}$ has no element. This chapter does not search for a number in the hole. It **takes the sequence itself as the number**. Many sequences crowd around the same hole: the decimal truncations $1, 7/5, 141/100, \dots$ approach it from below and never meet the Newton sequence. So two Cauchy sequences of rationals must count as one number when their difference dies out. That is an equivalence relation, and *Sets, Relations, and Cardinality*, Section 2.5 knows what to do with one: take its classes. *A real number is a Cauchy sequence of rationals, up to a null difference*. The chapter defines the sum, product and order of these classes on representatives and proves each well defined. The sum and product come from the quotient of a ring by an ideal of Chapter 3. The order needs a new idea, because comparing terms does not compare classes. The chapter then places $\mathbb{Q}$ inside the new field, proves it Archimedean with $\mathbb{Q}$ dense in it, and finds there a positive number whose square is 2. Until the field exists, every tolerance, margin and threshold is rational, so nothing about real numbers is assumed on the way to building them.

Then come the two forms of completeness. Every Cauchy sequence of real numbers converges. The proof builds the limit from rationals by a rule, so it uses no axiom of choice, and Section 5.5 shows where choice would have entered. Every non-empty set of real numbers that is bounded above has a least upper bound. This is proved by bisection in any ordered field that is Archimedean and Cauchy-complete, and one line of that proof needs Archimedes. So the constructed $\mathbb{R}$ is a complete ordered field, and this ends the construction of the number systems that began with the axioms of set theory in Chapter 1. Chapter 6 shows that any two complete ordered fields are isomorphic, and the optional Chapter 7 builds one from Dedekind cuts. *Analysis on the Real Line*, Chapter 1 takes the axioms of a complete ordered field as its starting point, and this chapter gives them a model. *Algebraic Preliminaries*, Chapter 1 builds the complex numbers on this $\mathbb{R}$. *Metric and Topological Spaces*, Chapter 2 repeats the construction for an arbitrary metric space.

What you need. Chapters 2 and 4 above all; Section 3.4 for one step of Section 5.2; a few facts about $\mathbb{N}$ from Chapter 1; *Sets, Relations, and Cardinality*, Chapters 1–3, chiefly Section 2.5; and *Logic and Proof*, Chapters 2–8, chiefly Chapters 5 and 6. From Chapter 4 of this book, nearly all of it, applied with $F = \mathbb{Q}$ and, once Section 5.3 has made $\mathbb{R}$ an ordered field, with $F = \mathbb{R}$. This covers the rules of an ordered field and the absolute value (Proposition 4.1.2, Lemma 4.1.1, Definition 4.1.1, Proposition 4.1.3), and powers and the copy of $\mathbb{Q}$ in an ordered field (Definition 4.2.1, Proposition 4.2.1, Theorem 4.2.2). It covers the Archimedean property, its reciprocal form, the integer part and density (Definition 4.2.2, Proposition 4.2.3, Proposition 4.2.4, Theorem 4.2.5). It covers sequences, limits and null sequences (Definition 4.3.1, Definition 4.3.2, Definition 4.3.3, Example (First Limits), Proposition 4.3.2, Theorem 4.3.4, Proposition 4.3.5, Proposition 4.4.1, Theorem 4.4.3, Proposition 4.4.4, Proposition 4.3.1, Proposition 4.3.3, Lemma 4.4.2). It covers Cauchy sequences, with the sign theorem and its companions (Definition 4.5.1, Proposition 4.5.1, Proposition 4.5.2, Theorem 4.5.3, Theorem 4.5.4, Proposition 4.5.5, Proposition 4.5.6). And it covers the Newton sequence with no rational limit (Theorem 4.6.1). The non-Archimedean field $\mathbb{Q}(t)$ (Example (A Non-Archimedean Ordered Field)) appears only in prose, a warning and an exercise; no proof cites it. From Chapter 2: commutative rings, fields and ordered fields (Definition 2.3.1, Definition 2.3.2, Proposition 2.3.2, Proposition 2.3.5, Proposition 2.5.2); the order from a positive set (Proposition 2.3.3), used here a third time; the lemma on rules on classes

(Theorem 2.2.1); least elements and discreteness in $\mathbb{Z}$ (Proposition 2.3.6, Theorem 2.3.4); the copies of $\mathbb{N}$ in $\mathbb{Z}$ and of $\mathbb{Z}$ in $\mathbb{Q}$ (Proposition 2.2.3, Proposition 2.4.4); $\mathbb{Q}$ as an Archimedean ordered field (Theorem 2.5.1, Theorem 2.5.3, Corollary 2.5.4, Proposition 2.5.5); and the Newton sequence and the gaps (Example (A Rational Sequence Aimed at the Gap), Theorem 2.6.1, Theorem 2.6.2). Review exercise R2 returns to the fractions of Sections 2.4 and 2.5 (Definition 2.4.1, Definition 2.5.1). From Section 3.4: ideals and the quotient ring (Definition 3.4.1, Theorem 3.4.1), which give Section 5.2 the ring laws of $\mathbb{R}$; exercise C2 of Section 5.2 mentions $\mathbb{Z}/4\mathbb{Z}$ (Definition 3.4.2). From Chapter 1: the citing rule (Corollary 1.7.3), by which induction and recursion apply to $\mathbb{N}$, and the arithmetic and order of $\mathbb{N}$ (Definition 1.5.1, Theorem 1.5.3, Theorem 1.6.1, Proposition 1.6.3), in Sections 5.1 and 5.6. From *Sets, Relations, and Cardinality*, Section 2.5: equivalence relations, classes and quotient sets (Sets, Relations, and Cardinality, Definition 2.5.1, Sets, Relations, and Cardinality, Definition 2.5.2, Sets, Relations, and Cardinality, Definition 2.5.4, Sets, Relations, and Cardinality, Theorem 2.5.1), with the two examples that showed what “well defined” means (Sets, Relations, and Cardinality, Proposition 2.5.5, Sets, Relations, and Cardinality, Example (A rule on classes that is not a function)). Further from that volume: functions and their equality (Sets, Relations, and Cardinality, Definition 1.4.1, Sets, Relations, and Cardinality, Proposition 1.4.1), from its Chapter 1; the recursion theorem (Sets, Relations, and Cardinality, Theorem 2.3.1), from its Section 2.3; and upper bounds and suprema (Sets, Relations, and Cardinality, Definition 2.6.2, Sets, Relations, and Cardinality, Definition 2.6.5), from its Section 2.6. From its Chapter 3, the axiom of choice (Sets, Relations, and Cardinality, Axiom 3.3.1) is named only to mark the step that Section 5.5 avoids. Review exercise R3 uses countability (Sets, Relations, and Cardinality, Theorem 3.2.2, Sets, Relations, and Cardinality, Theorem 3.4.1, Sets, Relations, and Cardinality, Theorem 1.4.4), and names Sets, Relations, and Cardinality, Proposition 3.5.2, from its Section 3.5, only as the route not taken, since its ($\Leftarrow$) direction uses choice. From *Logic and Proof*: the negation laws of its Chapter 6 (Logic and Proof, Proposition 6.2.1, Logic and Proof, Proposition 6.2.3, Logic and Proof, Proposition 6.2.2) and its rule $\exists E$ (Logic and Proof, Definition 5.6.1), which fixes one object at a time and is contrasted with choice. *Logic and Proof*, Chapter 2 gives the rules $\neg I$ and RAA (Logic and Proof, Definition 2.2.1, Logic and Proof, Definition 2.5.1), used in Section 5.3, and its Chapter 4 gives induction (Logic and Proof, Axiom 4.2.1). *Logic and Proof*, Chapter 8’s catalogue of what a proof may not do (Logic and Proof, Example (What is not allowed)) serves exercise C2 of Section 5.3. Nothing from *Analysis on the Real Line* is used, and no fact about $\mathbb{R}$ is assumed.

Time. About 25 hours on the core track (all of A, one B and one C per section, the exit checks); about 35 hours for everything.

5.-1.1 Roadmap

- **Sequences That Should Converge.** The Newton sequence and the decimal truncations, defined inside $\mathbb{Q}$ by powers of ten and a largest-integer argument in $\mathbb{Z}$, as two sequences aimed at one hole; the set $\mathcal{C}$ of Cauchy sequences of rationals; the relation $a \sim b$ when $a - b$ is null, proved an equivalence relation; the truncations equivalent to the Newton sequence; and finite changes and shifts, which never change the class.
- **The Real Numbers and Their Arithmetic.** $\mathbb{R}$ defined as the quotient set $\mathcal{C}/\sim$; sum, product and negative defined on representatives and proved well defined, the product through “null times bounded is null”; the null sequences as an ideal of $\mathcal{C}$, so that Chapter 3’s quotient theorem makes $\mathbb{R}$ a commutative ring; and inverses through the sign theorem of Chapter 4, after the failed termwise reciprocal.
- **Positive Classes and the Order.** Why termwise comparison defines no order on classes; positive sequences, eventually above a fixed rational margin, and why the margin is named before the index; positivity as a property of the class; the positive classes satisfying Chapter

2's recipe, which makes $\mathbb{R}$ an ordered field; termwise $\leq$ passing to classes, which strict $<$ does not; and the absolute value computed on representatives.

- **The Rationals Inside the Reals.** The map $\kappa(q) = [(q)_n]$, which preserves the operations and the order and is Chapter 4's embedding $\iota_{\mathbb{R}}: \mathbb{Q} \rightarrow \mathbb{R}$; $\mathbb{R}$ Archimedean, with a unique integer part; $\mathbb{Q}$ dense in $\mathbb{R}$, proved from the Cauchy property alone; rational tolerances sufficing for limits in $\mathbb{R}$; every Cauchy sequence of rationals converging in $\mathbb{R}$ to its own class; and the class of the Newton sequence, a positive real number whose square is 2.
- **Every Cauchy Sequence of Reals Converges.** Cauchy completeness of $\mathbb{R}$, proved through the rational approximants $m_k/(k+1)$, named by unique integer parts; where the axiom of choice would have entered (picking a representative of each term) and why the diagonal of arbitrary representatives does not work.
- **The Real Numbers Form a Complete Ordered Field.** Complete and Cauchy-complete ordered fields; the bound $2^k \geq k+1$ in any ordered field; the theorem that an Archimedean, Cauchy-complete ordered field is complete, proved by bisection with the Archimedean line marked; the constructed $\mathbb{R}$ a complete ordered field; a table of what $\mathbb{R}$ has and $\mathbb{Q}$ lacks; and review exercises that return to Chapter 4's sequences, Chapter 2's fractions, uncountability and recursion.

5.-1.2 Named moves

- **Representatives, then independence.** To define an operation or a property on classes, state it on representatives, then show that the class of the answer survives replacing each input by an equivalent one. It proves the sum, product and negative well defined in Section 5.2 and positivity a property of the class in Section 5.3. The termwise reciprocal of Section 5.2 and the termwise order of Section 5.3 fail it. Exercise B2 of Section 5.2 and Exercise C1 of Section 5.3 leave the last step, and then the last half, to you.
- **A term approximates its class.** For $a \in \mathcal{C}$, the rationals a_k converge in $\mathbb{R}$ to $[a]$. Estimate $|[a] - a_k|$ through the representative $(a_m - a_k)_m$, and aim the termwise bound below the target, since only $\leq$ passes to classes. It is proved in Section 5.4 and used in Section 5.5 to find the limit's candidate.
- **$\varepsilon/3$ through an approximant.** To compare two objects you cannot compare directly, here two rationals q_j, q_k known only through the reals x_j, x_k they approximate, pass through the approximated objects and give each of the three distances a third of the tolerance. It proves in Section 5.5 that the approximants form a Cauchy sequence of rationals. The triangle inequality is used in $\mathbb{R}$, and the result is carried back to $\mathbb{Q}$ through κ .
- **Bisect and keep the invariant.** Halve an interval by a rule that keeps its right end an upper bound and its left end not one; the ends converge to the supremum once the lengths tend to 0. Making the lengths tend to 0 is the one step that needs Archimedes. It proves in Section 5.6 that an Archimedean, Cauchy-complete ordered field is complete. The Quick check there and Exercise C2 of Section 5.6 run it by hand, and the latter runs it in $\mathbb{Q}$, where it finds no supremum.