

4.2 The Rationals inside an Ordered Field; the Archimedean Property

In an ordered field F we can form $1, 1 + 1, 1 + 1 + 1$, and so on. They look like the natural numbers $1, 2, 3, \dots$, but they are elements of F , and two questions need answers before the next section can use them as thresholds. Do they behave like natural numbers, adding, multiplying and comparing as $\mathbb{N}$ does? And does every element of F lie below one of them? For $\mathbb{Q}$, Section 2.5 answered the second question with yes (Theorem 2.5.3). This section answers the first for every ordered field, and goes further: a whole copy of $\mathbb{Q}$ sits inside F , in exactly one way. The second question has the answer no in general. Fields where the answer is yes are called **Archimedean**, and in them every element has an integer part and the rationals are dense.

4.2.1 The natural numbers in an ordered field

Section 2.3 defined, in any commutative ring R and for any $a \in R$, the multiples $n \cdot a$ for $n \in \mathbb{N}$ (just before Proposition 2.3.5): $0 \cdot a = 0$ and $(n + 1) \cdot a = n \cdot a + a$. With $a = 1$ they are the elements $0, 1, 1 + 1, \dots$ of the opening paragraph. We give them a name, together with powers, which this chapter needs for estimates such as $2^k \geq k + 1$.

Definition 4.2.1. Natural Numbers and Powers in an Ordered Field

Let F be an ordered field, with unit element 1_F .

- (a) For $n \in \mathbb{N}$, $n \cdot 1_F \in F$ is the multiple of 1_F of Section 2.3: the map $\mathbb{N} \rightarrow F$, $n \mapsto n \cdot 1_F$, is the unique function with

$$0 \cdot 1_F = 0, \quad (n + 1) \cdot 1_F = n \cdot 1_F + 1_F \quad \text{for every } n \in \mathbb{N}.$$

- (b) For $x \in F$, the **powers** x^k ($k \in \mathbb{N}$) are given by Sets, Relations, and Cardinality, Theorem 2.3.1, read on $\mathbb{N} = \omega$ through Corollary 1.7.3 (b), with $X = F$, starting value 1_F and step $(k, y) \mapsto yx$:

$$x^0 := 1_F, \quad x^{k+1} := x^k x \quad \text{for every } k \in \mathbb{N}.$$

The subscript in 1_F is there because two fields will often be in play, F and $\mathbb{Q}$, each with its own 1; where only one field is in play the subscript is dropped. For natural numbers n, k , the power $n^k \in \mathbb{N}$ is the one of Section 1.6, defined (just before Proposition 1.6.4) by the same recursion with $X = \mathbb{N}$: $n^0 = 1$ and $n^{k+1} = n^k n$. So 2^k may mean a natural number or an element of F ; part (c) of the next proposition says the two readings match.

Examples. $1 \cdot 1_F = 0 \cdot 1_F + 1_F = 0 + 1_F = 1_F$, and $2 \cdot 1_F = 1_F + 1_F$, the element written 2 in Section 4.1. Also $x^1 = 1_F x = x$ and $x^2 = xx$, which is the square of Proposition 2.3.2 (d). The degenerate case: $0 \cdot 1_F = 0$, and $0^0 = 1_F$ by the first equation, a convention the recursion imposes and nothing below questions.

A non-example by minimal change. Keep the field axioms and drop the order. In the two-element field of Example (A field with two elements that cannot be ordered), the same recursion gives $2 \cdot 1 = 1 + 1 = 0 = 0 \cdot 1$. So the map $n \mapsto n \cdot 1$ is not injective there, and the elements $0 \cdot 1, 1 \cdot 1, 2 \cdot 1, \dots$ are not a copy of $\mathbb{N}$. The next proposition shows that the order axioms prevent this.

Proposition 4.2.1. The Natural Numbers inside an Ordered Field

Let F be an ordered field, and let $m, n, k \in \mathbb{N}$.

- (a) $(m + n) \cdot 1_F = m \cdot 1_F + n \cdot 1_F$ and $(mn) \cdot 1_F = (m \cdot 1_F)(n \cdot 1_F)$.
- (b) If $m < n$, then $m \cdot 1_F < n \cdot 1_F$. Hence $n \mapsto n \cdot 1_F$ is injective, and $n \cdot 1_F \geq 1_F > 0$ whenever $n \geq 1$.
- (c) $(n^k) \cdot 1_F = (n \cdot 1_F)^k$. In particular $(2^k) \cdot 1_F = (2 \cdot 1_F)^k$, the k -th power in F of $2 = 1 + 1$.
- (d) In $\mathbb{Q}$, $n \cdot 1_{\mathbb{Q}} = j(\iota(n))$, the image of n under the embeddings Proposition 2.2.3 and Proposition 2.4.4. With the identifications $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$, this says $n \cdot 1_{\mathbb{Q}} = n$.

Idea. The sum law in (a) and part (b) are rules for multiples (Proposition 2.3.5 (h)); the product law and (c) are inductions following the recursion defining the left side. The product law is the only step with content: multiplying by $n + 1$ is multiplying by n and adding one more copy, and the sum law and distributivity turn that into the same step on the right. Part (d) compares two functions on $\mathbb{N}$ that satisfy the same recursion, and the uniqueness half of the recursion theorem makes them equal.

Proof.

- (a) The sum law is Proposition 2.3.5 (h) with $R = F$ and $a = 1_F$. For the product law, fix m and use Logic and Proof, Axiom 4.2.1 (through Corollary 1.7.3 (b)) on n , for $P(n)$: “ $(mn) \cdot 1_F = (m \cdot 1_F)(n \cdot 1_F)$ ”. *Base:* $m \cdot 0 = 0$ by Definition 1.5.1 (b), so the left side is $0 \cdot 1_F = 0$, and the right side is $(m \cdot 1_F) \cdot 0 = 0$ by Proposition 2.3.2 (b). *Step:* assume $P(n)$. By Definition 1.5.1 (b) and (c), $m(n + 1) = mn + m$. Hence

$$\begin{aligned} (m(n + 1)) \cdot 1_F &= (mn) \cdot 1_F + m \cdot 1_F = (m \cdot 1_F)(n \cdot 1_F) + (m \cdot 1_F)1_F \\ &= (m \cdot 1_F)(n \cdot 1_F + 1_F) = (m \cdot 1_F)((n + 1) \cdot 1_F), \end{aligned}$$

by the sum law, $P(n)$ with (F7), (F9), and the recursion of (a) in Definition 4.2.1. So $P(n + 1)$ holds, and $P(n)$ holds for every n .

- (b) Since $1_F \neq 0$ in a field, Proposition 2.3.5 (h) gives $m \cdot 1_F < n \cdot 1_F$ whenever $m < n$. If $m \neq n$, then $m < n$ or $n < m$ by trichotomy in $\mathbb{N}$ (Theorem 1.6.1 (a)), so $m \cdot 1_F \neq n \cdot 1_F$ by (O1): the map is injective. Finally $1 \cdot 1_F = 1_F$ (the examples above), and for $n > 1$, $1_F = 1 \cdot 1_F < n \cdot 1_F$; so $n \cdot 1_F \geq 1_F$ for every $n \geq 1$, and $1_F > 0$ by Proposition 2.3.5 (e).
- (c) Fix n and induct on k . *Base:* $n^0 = 1$, so $(n^0) \cdot 1_F = 1 \cdot 1_F = 1_F = (n \cdot 1_F)^0$. *Step:* if $(n^k) \cdot 1_F = (n \cdot 1_F)^k$, then, since $n^{k+1} = n^k n$,

$$(n^{k+1}) \cdot 1_F = ((n^k) \cdot 1_F)(n \cdot 1_F) = (n \cdot 1_F)^k (n \cdot 1_F) = (n \cdot 1_F)^{k+1},$$

by the product law of (a), the hypothesis, and Definition 4.2.1 (b). The special case is $n = 2$, with $2 \cdot 1_F = 1_F + 1_F$.

- (d) Let $f(n) := j(\iota(n)) \in \mathbb{Q}$. The zero and one of $\mathbb{Z}$ are $\iota(0)$ and $\iota(1)$ (Theorem 2.3.1), and j sends them to the zero and one of $\mathbb{Q}$ (Proposition 2.4.4). So $f(0) = 0$, and, since ι and j preserve sums (Proposition 2.2.3 (a), Proposition 2.4.4),

$$f(n + 1) = j(\iota(n) + \iota(1)) = j(\iota(n)) + j(\iota(1)) = f(n) + 1_{\mathbb{Q}}.$$

So f and $n \mapsto n \cdot 1_{\mathbb{Q}}$ both satisfy the recursion of [Definition 4.2.1](#) (a) with $F = \mathbb{Q}$, and by the uniqueness in Sets, Relations, and Cardinality, Theorem 2.3.1 (through Corollary 1.7.3 (b)) they are equal. This proves the proposition. ■

So the order axioms do what the non-example above lacked: by (b), the elements $n \cdot 1_F$ are all different and increase with n , and by (a) they add and multiply as natural numbers do. In $\mathbb{Q}$, (d) says that nothing new has happened: $n \cdot 1_{\mathbb{Q}}$ is n . When only one field is in play, we will sometimes write n for $n \cdot 1_F$ in a general ordered field too; the statements of this section keep $n \cdot 1_F$, because there the difference between $n \in \mathbb{N}$ and its image in F is the point.

4.2.2 A copy of the rationals in every ordered field

The copy of $\mathbb{N}$ extends. Negatives exist in F , so the integers have images too, and inverses of non-zero elements exist, so fractions have images. The next theorem says that the result is a copy of $\mathbb{Q}$, compatible with $+$, $\cdot$ and $<$, and that there is no other way to place $\mathbb{Q}$ inside F . Later chapters of this volume use this map to compare an arbitrary ordered field with the rationals.

Every ordered field contains a copy of $\mathbb{Q}$, and only one.

Theorem 4.2.2. The Rationals Embed in Every Ordered Field

Let F be an ordered field. There is **exactly one** map $\iota_F: \mathbb{Q} \rightarrow F$ such that, for all $x, y \in \mathbb{Q}$,

$$\begin{aligned} \iota_F(1) &= 1_F, \\ \iota_F(x + y) &= \iota_F(x) + \iota_F(y), & \iota_F(xy) &= \iota_F(x) \iota_F(y). \end{aligned}$$

It satisfies $\iota_F(n) = n \cdot 1_F$ for every $n \in \mathbb{N}$. It is injective and **strictly increasing** ($x < y$ implies $\iota_F(x) < \iota_F(y)$), and it **reflects order**: $\iota_F(x) < \iota_F(y)$ implies $x < y$.

Idea. Uniqueness tells us how to build ι_F . A map respecting 1 and $+$ must send $n = 1 + \dots + 1$ to $n \cdot 1_F$; respecting negatives, it must send $-n$ to $-(n \cdot 1_F)$; and respecting products, it must send p/q to $(p \cdot 1_F)(q \cdot 1_F)^{-1}$. So we take that formula as the definition, and the work is to check it. **Step roadmap.** ① Extend $n \mapsto n \cdot 1_F$ to $\mathbb{Z}$ and check that it respects $+$, $\cdot$ and negation. ② Define ι_F on fractions and check that the value does not depend on the fraction chosen. ③ Check $+$ and $\cdot$ with the fraction rules. ④ Check the order. ⑤ Uniqueness: run the opening argument as a proof.

Proof. Step 1: the integers. By Proposition 2.2.3 (c), read through the identification $\mathbb{N} \subseteq \mathbb{Z}$, every $m \in \mathbb{Z}$ is n or $-n$ for some $n \in \mathbb{N}$. Define

$$m \cdot 1_F := \begin{cases} n \cdot 1_F & \text{if } m = n \text{ with } n \in \mathbb{N}, \\ -(n \cdot 1_F) & \text{if } m = -n \text{ with } n \in \mathbb{N}. \end{cases}$$

This is well defined. Within one case, n is determined by m : directly in the first case, and in the second because $-n = -n'$ gives $n = n'$ by Proposition 2.3.2 (a). The cases overlap

only at 0: if $n = -n'$ with $n, n' \in \mathbb{N}$, then $n + n' = 0$, and $n, n' \geq 0$ (Theorem 2.3.4 (b)); if $n' > 0$, then $n + n' > n + 0 \geq 0$ by Proposition 2.3.5 (b), against $n + n' = 0$, so $n' = 0$ and then $n = 0$. At $m = 0$ both cases give 0, since $-0 = 0$.

We check three rules, for $m, m' \in \mathbb{Z}$.

Negation: $(-m) \cdot 1_F = -(m \cdot 1_F)$. If $m = n \in \mathbb{N}$, this is the second case of the definition. If $m = -n$, then $-m = n$ by Proposition 2.3.2 (a), and $(-m) \cdot 1_F = n \cdot 1_F = -(-(n \cdot 1_F)) = -(m \cdot 1_F)$.

Sums: $(m + m') \cdot 1_F = m \cdot 1_F + m' \cdot 1_F$. If $m, m' \in \mathbb{N}$, this is Proposition 4.2.1 (a). We write out the mixed case $m = n$, $m' = -n'$ with $n, n' \in \mathbb{N}$. If $n' \leq n$, then $n = n' + k$ for some $k \in \mathbb{N}$ (Proposition 1.6.3 (a)), so $m + m' = (n' + k) - n' = k$ in $\mathbb{Z}$, and

$$\begin{aligned} m \cdot 1_F + m' \cdot 1_F &= (n' + k) \cdot 1_F - n' \cdot 1_F \\ &= n' \cdot 1_F + k \cdot 1_F - n' \cdot 1_F = k \cdot 1_F, \end{aligned}$$

by Proposition 4.2.1 (a). If $n < n'$, then $n' = n + k$, $m + m' = -k$, and the same computation gives $n \cdot 1_F - (n + k) \cdot 1_F = -(k \cdot 1_F) = (-k) \cdot 1_F$. The case $m = -n$, $m' = n'$ follows by (F1). If $m = -n$ and $m' = -n'$, then $m + m' = -(n + n')$, and $-((n + n') \cdot 1_F) = -(n \cdot 1_F) - (n' \cdot 1_F)$, by the sum law and uniqueness of negatives.

Products: $(mm') \cdot 1_F = (m \cdot 1_F)(m' \cdot 1_F)$. If $m, m' \in \mathbb{N}$, this is Proposition 4.2.1 (a). We write out $m = -n$, $m' = n'$: then $mm' = -(nn')$ by Proposition 2.3.2 (c), so

$$\begin{aligned} (mm') \cdot 1_F &= -((nn') \cdot 1_F) = -((n \cdot 1_F)(n' \cdot 1_F)) \\ &= -(n \cdot 1_F)(n' \cdot 1_F) = (m \cdot 1_F)(m' \cdot 1_F), \end{aligned}$$

by negation, Proposition 4.2.1 (a), and Proposition 2.3.2 (c) in F . The two remaining sign patterns are the same with signs: $(-n)(-n') = nn'$ in $\mathbb{Z}$ and $-(n \cdot 1_F)(-n' \cdot 1_F) = (n \cdot 1_F)(n' \cdot 1_F)$ in F , both by Proposition 2.3.2 (c), and $m = n$, $m' = -n'$ follows from the case written out by (F6).

Step 2: the definition on $\mathbb{Q}$. By Proposition 2.4.4 (b), every $x \in \mathbb{Q}$ is p/q for some $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$. Since $q \in \mathbb{Z}_{>0}$, that is, $q \in \mathbb{N}$ and $q \neq 0$, we have $q \geq 1$ (Theorem 1.6.1 (d)), and Proposition 4.2.1 (b) gives $q \cdot 1_F \geq 1_F > 0$, so $q \cdot 1_F \neq 0$. Define

$$\iota_F\left(\frac{p}{q}\right) := (p \cdot 1_F)(q \cdot 1_F)^{-1} = \frac{p \cdot 1_F}{q \cdot 1_F}.$$

The value does not depend on the fraction chosen: if $p/q = r/s$ with $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$, then $ps = rq$ (Proposition 2.4.4 (b)), so $(p \cdot 1_F)(s \cdot 1_F) = (r \cdot 1_F)(q \cdot 1_F)$ by the product rule of Step 1. Multiplying both sides by $(q \cdot 1_F)^{-1}(s \cdot 1_F)^{-1}$ and using (F5), (F6), (F8) gives $(p \cdot 1_F)(q \cdot 1_F)^{-1} = (r \cdot 1_F)(s \cdot 1_F)^{-1}$. So ι_F is a function $\mathbb{Q} \rightarrow F$. For $m \in \mathbb{Z}$, $m = m/1$, so $\iota_F(m) = (m \cdot 1_F)(1_F)^{-1} = m \cdot 1_F$; in particular $\iota_F(n) = n \cdot 1_F$ for $n \in \mathbb{N}$, and $\iota_F(1) = 1_F$.

Step 3: sums and products. Let $x = p/q$ and $y = r/s$ as in Step 2. In the field $\mathbb{Q}$, Lemma 4.1.1 (b) gives

$$x + y = \frac{ps + rq}{qs}, \quad xy = \frac{pr}{qs},$$

with $qs \in \mathbb{Z}_{>0}$ by (O4) in $\mathbb{Z}$. By the definition, Step 1, and Lemma 4.1.1 (b) in F ,

$$\begin{aligned}\iota_F(x + y) &= \frac{(p \cdot 1_F)(s \cdot 1_F) + (r \cdot 1_F)(q \cdot 1_F)}{(q \cdot 1_F)(s \cdot 1_F)} \\ &= \frac{p \cdot 1_F}{q \cdot 1_F} + \frac{r \cdot 1_F}{s \cdot 1_F} = \iota_F(x) + \iota_F(y),\end{aligned}$$

and in the same way $\iota_F(xy) = \frac{(p \cdot 1_F)(r \cdot 1_F)}{(q \cdot 1_F)(s \cdot 1_F)} = \iota_F(x) \iota_F(y)$.

Step 4: order. Let $x < y$ in $\mathbb{Q}$. Then $y - x > 0$, and writing $y - x = p/q$ with $q \in \mathbb{Z}_{>0}$, the comparison formula (Theorem 2.5.1 (a)) gives $p > 0$, so $p \in \mathbb{Z}_{>0}$ (Theorem 2.3.4 (b)). By Proposition 4.2.1 (b), $p \cdot 1_F > 0$ and $q \cdot 1_F > 0$; by Proposition 2.5.2 (a) and (O4), $\iota_F(y - x) > 0$. By Step 3, $\iota_F(y) = \iota_F(x) + \iota_F(y - x) > \iota_F(x)$, using (O3). So ι_F is strictly increasing. If $x \neq y$, then $x < y$ or $y < x$, and the images differ by (O1): ι_F is injective. If $\iota_F(x) < \iota_F(y)$, then neither $x = y$ nor $y < x$ can hold, since they would give $\iota_F(x) = \iota_F(y)$ or $\iota_F(y) < \iota_F(x)$, against (O1); so $x < y$ by trichotomy.

Step 5: uniqueness. Let $\psi: \mathbb{Q} \rightarrow F$ satisfy the three equations of the statement. Then $\psi(0) = \psi(0 + 0) = \psi(0) + \psi(0)$, and adding $-\psi(0)$ gives $\psi(0) = 0$. By Logic and Proof, Axiom 4.2.1 (through Corollary 1.7.3 (b)), $\psi(n) = n \cdot 1_F$ for every $n \in \mathbb{N}$: the base is $\psi(0) = 0 = 0 \cdot 1_F$, and $\psi(n + 1) = \psi(n) + \psi(1) = n \cdot 1_F + 1_F = (n + 1) \cdot 1_F$. For $x \in \mathbb{Q}$, $\psi(x) + \psi(-x) = \psi(0) = 0$, so $\psi(-x) = -\psi(x)$ by Proposition 2.3.2 (a); hence $\psi(m) = m \cdot 1_F$ for every $m \in \mathbb{Z}$. For $q \in \mathbb{Z}_{>0}$, $\psi(q) \psi(q^{-1}) = \psi(1) = 1_F$, so $\psi(q^{-1}) = (q \cdot 1_F)^{-1}$ by Lemma 4.1.1 (a). Finally $p/q = pq^{-1}$ (Proposition 2.4.4 (a)), so

$$\psi\left(\frac{p}{q}\right) = \psi(p) \psi(q^{-1}) = (p \cdot 1_F)(q \cdot 1_F)^{-1} = \iota_F\left(\frac{p}{q}\right).$$

So $\psi = \iota_F$. This proves the theorem. ■

For $F = \mathbb{Q}$, the identity map of $\mathbb{Q}$ satisfies the three equations, so by uniqueness $\iota_{\mathbb{Q}}$ is the identity. In general ι_F is the map of Step 2, and it carries $\mathbb{Q}$ onto a copy inside F that adds, multiplies and compares as $\mathbb{Q}$ does. So “ $\mathbb{Q} \subseteq F$ ” is an honest abuse of notation, as $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ was in Chapter 2, and where only one field is in play we write q for $\iota_F(q)$. The statements below keep ι_F visible.

Remark. Uniqueness has a consequence that later chapters of this volume use when comparing two ordered fields. Let L be another ordered field and $\chi: F \rightarrow L$ a map with $\chi(1_F) = 1_L$ that respects $+$ and $\cdot$. Then $\chi \circ \iota_F: \mathbb{Q} \rightarrow L$ satisfies the three equations of the theorem, so $\chi \circ \iota_F = \iota_L$: such a map fixes the rationals.

4.2.3 The Archimedean property

Section 2.5 proved that every rational lies below a natural number (Theorem 2.5.3). By Proposition 4.2.1 (d), that theorem says: for every $x \in \mathbb{Q}$ there is $n \in \mathbb{N}$ with $x < n \cdot 1_{\mathbb{Q}}$. This statement makes sense in any ordered field, and it deserves a name, because it can fail.

No element of the field lies above all the natural numbers.

Definition 4.2.2. Archimedean Ordered Field

An ordered field F is **Archimedean** if **for every** $x \in F$ **there is** $n \in \mathbb{N}$ with $x < n \cdot 1_F$. In symbols,

$$\forall x \in F \quad \exists n \in \mathbb{N} \quad x < n \cdot 1_F.$$

In words: whatever element of F we pick, some natural number, read in F , lies strictly above it. The n may depend on x ; it is chosen after x .

Examples. $\mathbb{Q}$ is Archimedean: this is Theorem 2.5.3, by Proposition 4.2.1 (d). The constructed real numbers of the next chapter are Archimedean too (proved there). A non-example, $\mathbb{Q}(t)$, comes at the end of this section: there the element t lies above every $n \cdot 1$.

Why the name. Archimedes used, as an axiom about lengths, that a smaller length laid off often enough exceeds any larger one. That is Exercise B1: in an Archimedean field, for $y > 0$ and any x , some multiple $(n \cdot 1_F)y$ exceeds x . The definition is the case $y = 1_F$.

The form in which the next section uses the property is about reciprocals, as in Corollary 2.5.4, now in an arbitrary Archimedean ordered field.

Proposition 4.2.3. Small Reciprocals in an Archimedean Field

In an Archimedean ordered field F , **for every** $\varepsilon \in F_{>0}$ **there is** $N \in \mathbb{Z}_{>0}$ such that $1/(n \cdot 1_F) < \varepsilon$ **for every** $n \in \mathbb{N}$ with $n \geq N$. In symbols,

$$\forall \varepsilon \in F_{>0} \quad \exists N \in \mathbb{Z}_{>0} \quad \forall n \in \mathbb{N} \quad \left(n \geq N \Rightarrow \frac{1}{n \cdot 1_F} < \varepsilon \right).$$

Proof. Let $\varepsilon \in F_{>0}$. Then $1/\varepsilon > 0$ by Proposition 2.5.2 (a). By Definition 4.2.2 there is $N \in \mathbb{N}$ with $1/\varepsilon < N \cdot 1_F$; it depends on ε . Then $N \neq 0$, since $0 \cdot 1_F = 0 < 1/\varepsilon$, so $N \in \mathbb{Z}_{>0}$. Let $n \geq N$. Then $n \cdot 1_F \geq N \cdot 1_F$ (Proposition 4.2.1 (b), or equality), so $0 < 1/\varepsilon < n \cdot 1_F$, and Proposition 2.5.2 (b) gives

$$\frac{1}{n \cdot 1_F} < \frac{1}{1/\varepsilon} = \varepsilon,$$

where $1/(1/\varepsilon) = (\varepsilon^{-1})^{-1} = \varepsilon$ by Lemma 4.1.1 (a). ■

The order of the quantifiers is as in Section 2.5: ε first, then N , depending on ε , and then **every** $n \geq N$ at once. That last clause is what Section 4.3 will call “eventually”.

Quick check. Find $n \in \mathbb{N}$ with $n > 355/113$ in $\mathbb{Q}$. Then find the least $n \in \mathbb{Z}_{>0}$ with $1/n < 7/500$.

Solution. By the comparison formula (Theorem 2.5.1 (a)), $355/113 < n/1$ if and only if $355 < 113n$. Since $113 \cdot 3 = 339 < 355$ and $113 \cdot 4 = 452 > 355$, $n = 4$ works (and is the least such n).

For $n \in \mathbb{Z}_{>0}$, $1/n < 7/500$ if and only if $500 < 7n$. Since $7 \cdot 71 = 497 < 500$ and $7 \cdot 72 = 504 > 500$, the least such n is 72; every $n \geq 72$ works too, as [Proposition 4.2.3](#) promises.

4.2.4 Integer part and density

In $\mathbb{Q}$, the Archimedean property gave every rational an integer part (Proposition 2.5.5). About $\mathbb{Q}$ the proof used only Archimedes, applied to x and to $-x$; the rest (least elements and discreteness of $\mathbb{Z}$, totality of the order) holds in any ordered field. So it runs in any Archimedean field, with ι_F carrying the integers into F .

Proposition 4.2.4. Integer Part in an Archimedean Field

Suppose that the ordered field F is Archimedean. Then for every $x \in F$ there is **exactly one** $m \in \mathbb{Z}$ with

$$\iota_F(m) \leq x < \iota_F(m+1).$$

Idea. The integer just above x should be the least integer k with $x < \iota_F(k)$. Archimedes applied to x shows that such k exist; applied to $-x$, it bounds them below, so Proposition 2.3.6 supplies a least one. The comparisons happen in F , and ι_F reflecting order ([Theorem 4.2.2](#)) carries them back to $\mathbb{Z}$.

Proof. Let $x \in F$ and $S := \{k \in \mathbb{Z} : x < \iota_F(k)\}$. By [Definition 4.2.2](#) there is $n \in \mathbb{N}$ with $x < n \cdot 1_F = \iota_F(n)$ ([Theorem 4.2.2](#)), so $n \in S$ and $S \neq \emptyset$. Applied to $-x$, it gives $n' \in \mathbb{N}$ with $-x < \iota_F(n')$, that is, $\iota_F(-n') = -\iota_F(n') < x$ (Proposition 2.3.5 (a); ι_F respects negatives, as in Step 5 of the proof of [Theorem 4.2.2](#)). For $k \in S$, $\iota_F(-n') < x < \iota_F(k)$, so $-n' < k$ because ι_F reflects order. Thus $-n'$ is a lower bound of S , and by Proposition 2.3.6 S has a least element l . Let $m := l - 1$, so $l = m + 1$.

Existence. $x < \iota_F(m+1)$, since $l \in S$. And $m \notin S$, since $m < l$ and l is least. So $x < \iota_F(m)$ fails, and $\iota_F(m) \leq x$ by totality (Proposition 4.1.2 (a)).

Uniqueness. Let $m, m' \in \mathbb{Z}$ both satisfy the condition, and suppose $m < m'$. By discreteness ([Theorem 2.3.4 \(c\)](#)), $m+1 \leq m'$, so $\iota_F(m+1) \leq \iota_F(m')$ ([Theorem 4.2.2](#)). Then

$$x < \iota_F(m+1) \leq \iota_F(m') \leq x,$$

so $x < x$, against (O1). By symmetry $m' < m$ is impossible, so $m = m'$. ■

For $F = \mathbb{Q}$, $\iota_{\mathbb{Q}}$ is the identity and this is Proposition 2.5.5. The constructed real numbers of the next chapter will use it with F the new field, where m is an integer approximant of a number not yet known to be anything else.

Between two rationals there is always a third, their midpoint (Section 2.5). In an Archimedean field more is true: between any two elements, however close, lies a **rational**.

Theorem 4.2.5. The Rationals Are Dense in an Archimedean Field

In an Archimedean ordered field F , for all $x, y \in F$ with $x < y$ there is $q \in \mathbb{Q}$ with $x < \iota_F(q) < y$.

Idea. Look for q with denominator n : we need an integer k with $nx < k < ny$, after scaling by n . Integers are spaced 1 apart, so such a k exists once the scaled gap $n(y - x)$ exceeds 1; that is where Archimedes enters, through [Proposition 4.2.3](#). The integer to take is the one just above nx , which the integer part supplies.

Proof. Since $y - x > 0$, [Proposition 4.2.3](#) gives $n \in \mathbb{Z}_{>0}$ with $1/(n \cdot 1_F) < y - x$. Let $c := n \cdot 1_F$, which is positive ([Proposition 4.2.1](#) (b)). Multiplying by c ([Proposition 2.3.5](#) (c)) gives $1_F < c(y - x) = cy - cx$, by (F9) and [Proposition 2.3.2](#) (c); adding cx by (O3), $cx + 1_F < cy$. Let m be the integer part of cx ([Proposition 4.2.4](#)), so that $\iota_F(m) \leq cx < \iota_F(m + 1)$. Then

$$cx < \iota_F(m + 1) = \iota_F(m) + 1_F \leq cx + 1_F < cy,$$

by [Theorem 4.2.2](#) and [Proposition 2.3.5](#) (b). Multiplying by $c^{-1} > 0$ ([Proposition 2.5.2](#) (a)), [Proposition 2.3.5](#) (c)) gives $x < \iota_F(m + 1)c^{-1} < y$. Let $q := (m + 1)/n \in \mathbb{Q}$. Since $c = \iota_F(n)$ and ι_F respects products, $\iota_F(q)c = \iota_F(m + 1)$, so $\iota_F(q) = \iota_F(m + 1)c^{-1}$, and $x < \iota_F(q) < y$. ■

Applied again, the theorem shows that an interval of an Archimedean field contains infinitely many rationals ([Exercise C1](#)). Chapter 6 uses the theorem with F a field of real numbers; Chapter 5 uses it only in an exercise.

4.2.5 A field that is not Archimedean

Section 2.5 proved the Archimedean property of $\mathbb{Q}$ from the construction of $\mathbb{Q}$: every rational is a quotient of integers. Nothing in (F1)–(F9) and (O1)–(O4) forces it. The standard witness uses polynomials, which this volume has not built; *Algebraic Preliminaries*, Chapter 4 builds them as formal polynomials. We use the school algebra of polynomials in this one example, and say so. No proof of a numbered result in this chapter cites it; it serves only as a counterexample, in prose, warnings and exercises.

Example. A Non-Archimedean Ordered Field

Let $\mathbb{Q}(t)$ be the set of quotients p/q of polynomials p, q in an indeterminate t with rational coefficients, $q \neq 0$, where $p/q = r/s$ means $ps = rq$, with the usual sum and product of fractions. Call p/q with $p \neq 0$ **positive** if the leading coefficients of p and q have the same sign. With this positive set, $\mathbb{Q}(t)$ is an ordered field; we state this and do not verify it. Show that $t > n \cdot 1$ for every $n \in \mathbb{N}$, so that $\mathbb{Q}(t)$ is **not** Archimedean, and that $0 < 1/t < 1/(n \cdot 1)$ for every $n \geq 1$.

Solution. In $\mathbb{Q}(t)$ the element $n \cdot 1$ is the constant polynomial n , by the recursion of [Definition 4.2.1](#) (a) and the addition of constants. The difference $t - n \cdot 1 = (t - n)/1$ has numerator

$t - n$, with leading coefficient 1, and denominator 1, with leading coefficient 1. The signs agree, so $t - n \cdot 1$ is positive, that is, $n \cdot 1 < t$ (Proposition 2.3.3). This holds for every $n \in \mathbb{N}$, so $x = t$ has no n with $x < n \cdot 1$: $\mathbb{Q}(t)$ is not Archimedean.

Next, $1/t$ has numerator 1 and denominator t , both with leading coefficient 1, so $1/t > 0$. For $n \geq 1$, Lemma 4.1.1 (c) gives

$$\frac{1}{n \cdot 1} - \frac{1}{t} = \frac{t - n}{nt},$$

whose numerator has leading coefficient 1 and whose denominator nt has leading coefficient $n > 0$. So the difference is positive, and $1/t < 1/(n \cdot 1)$.

Warning. The Archimedean property does not follow from the ordered-field axioms: $\mathbb{Q}(t)$ satisfies (F1)–(F9) and (O1)–(O4) and fails it. So in a general ordered field the move “take $\varepsilon = 1/n$ ” does not reach every ε : in $\mathbb{Q}(t)$, the tolerance $\varepsilon = 1/t$ is below every $1/n$. Each later use of “ $1/n$ eventually drops below ε ” must name the Archimedean property.

The density theorem fails in $\mathbb{Q}(t)$ too, so its hypothesis is needed. There $0 < 1/t$, and no rational lies strictly between them. Suppose $q \in \mathbb{Q}$ had $0 < \iota_{\mathbb{Q}(t)}(q) < 1/t$, with $\iota_{\mathbb{Q}(t)}$ the embedding of Theorem 4.2.2. Then $q > 0$, because $\iota_{\mathbb{Q}(t)}$ reflects order. By Proposition 2.5.2 (b) and Lemma 4.1.1 (a), $t = 1/(1/t) < 1/\iota_{\mathbb{Q}(t)}(q) = \iota_{\mathbb{Q}(t)}(1/q)$. By Theorem 2.5.3, $1/q < n$ for some $n \in \mathbb{N}$, so $\iota_{\mathbb{Q}(t)}(1/q) < \iota_{\mathbb{Q}(t)}(n) = n \cdot 1$, and $t < n \cdot 1$, against the example.

4.2.6 Exercises

4.2.6.1 A. Check your understanding

Exercise. A1

- (a) Write “ F is **not** Archimedean” in symbols, moving the negation inside by Logic and Proof, Proposition 6.2.1.
- (b) Give two different elements of $\mathbb{Q}(t)$ (Example (A Non-Archimedean Ordered Field)) that lie above every $n \cdot 1$.
- (c) In an ordered field, can $n \cdot 1_F = 0$ for some $n \geq 1$? Give a reason.
- (d) What is $\iota_{\mathbb{Q}}$? Which part of Theorem 4.2.2 tells you?

Solution.

- (a) The definition is $\forall x \in F \exists n \in \mathbb{N} \ x < n \cdot 1_F$. By Logic and Proof, Proposition 6.2.1, in its bounded form Logic and Proof, Proposition 6.2.3, applied twice, its negation is $\exists x \in F \forall n \in \mathbb{N} \ \neg(x < n \cdot 1_F)$, and by totality (Proposition 4.1.2 (a)) $\neg(x < n \cdot 1_F)$ is $n \cdot 1_F \leq x$:

$$\exists x \in F \ \forall n \in \mathbb{N} \ n \cdot 1_F \leq x.$$
- (b) t , as in Example (A Non-Archimedean Ordered Field), and $t + 1$, since $t + 1 > t > n \cdot 1$. Also t^2 : $t^2 - n \cdot 1 = (t^2 - n)/1$ has leading coefficients 1 and 1.
- (c) No. By Proposition 4.2.1 (b), $n \cdot 1_F \geq 1_F > 0$ for $n \geq 1$. (The two-element field of

Example (A field with two elements that cannot be ordered) has $2 \cdot 1 = 0$, and it cannot be ordered.)

- (d) The identity map of $\mathbb{Q}$: it satisfies $\text{id}(1) = 1$ and respects $+$ and $\cdot$, and the theorem says **exactly one** map does.

4.2.6.2 B. Practice

Exercise. B1: Archimedes' form

Suppose the ordered field F is Archimedean, and take $x \in F$ and $y \in F_{>0}$. Prove that there is $n \in \mathbb{N}$ with $(n \cdot 1_F)y > x$.

Solution. Since $y > 0$, $y^{-1} > 0$ (Proposition 2.5.2 (a)). By Definition 4.2.2 applied to x/y , there is $n \in \mathbb{N}$ with $x/y < n \cdot 1_F$. Multiplying by $y > 0$ (Proposition 2.3.5 (c)) gives $(xy^{-1})y < (n \cdot 1_F)y$, and $(xy^{-1})y = x(y^{-1}y) = x$ by (F5), (F6), (F8). Hence $(n \cdot 1_F)y > x$.

Exercise. B2: Smaller than every reciprocal

- (a) Let F be Archimedean, and let $x \in F$ satisfy $0 \leq x < 1/(n \cdot 1_F)$ for every $n \in \mathbb{N}$ with $n \geq 1$. Prove that $x = 0$.
- (b) Give an element $x \neq 0$ of $\mathbb{Q}(t)$ (Example (A Non-Archimedean Ordered Field)) with this property.

Solution.

- (a) Suppose, for RAA (Logic and Proof, Definition 2.5.1), that $x \neq 0$. Then $x > 0$, since $x \geq 0$. By Proposition 4.2.3 with $\varepsilon := x$, there is $N \in \mathbb{Z}_{>0}$ with $1/(N \cdot 1_F) < x$. The hypothesis with $n = N \geq 1$ gives $x < 1/(N \cdot 1_F)$. By (O2), $x < x$, against (O1). So $x = 0$.
- (b) $x = 1/t$: by Example (A Non-Archimedean Ordered Field), $0 < 1/t < 1/(n \cdot 1)$ for every $n \geq 1$, and $1/t \neq 0$. So part (a) fails without the Archimedean hypothesis.

Exercise. B3: Rationals with a power of two as denominator

- (a) Prove that $2^k \geq k + 1$ in $\mathbb{Q}$ for every $k \in \mathbb{N}$.
- (b) Let $a, b \in \mathbb{Q}$ with $a < b$. Prove that there are $m \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $a < (m+1)/2^k < b$.
- (c) Carry out the construction of (b) for $a = 1/3$, $b = 1/2$, taking k least with $1/(k+1) < b - a$.

Hint: follow the proof of Theorem 4.2.5, with 2^k in place of n .

Solution.

- (a) Read 2^k as $(2^k) \cdot 1_{\mathbb{Q}} = (2 \cdot 1_{\mathbb{Q}})^k$ (Proposition 4.2.1 (c), (d)). We use Logic and Proof, Axiom 4.2.1 (through Corollary 1.7.3 (b)) on k for $P(k)$: “ $(2^k) \cdot 1_{\mathbb{Q}} \geq (k+1) \cdot 1_{\mathbb{Q}}$ ”.

Base: $2^0 = 1 = 0 + 1$, so both sides are $1 \cdot 1_{\mathbb{Q}}$. *Step:* assume $P(k)$. Then

$$\begin{aligned} (2^{k+1}) \cdot 1_{\mathbb{Q}} &= ((2^k) \cdot 1_{\mathbb{Q}})(2 \cdot 1_{\mathbb{Q}}) \geq ((k+1) \cdot 1_{\mathbb{Q}})(2 \cdot 1_{\mathbb{Q}}) \\ &= (2k+2) \cdot 1_{\mathbb{Q}} \geq (k+2) \cdot 1_{\mathbb{Q}}, \end{aligned}$$

by [Proposition 4.2.1](#) (a) and $2^{k+1} = 2^k \cdot 2$; then Proposition 4.1.2 (d), all factors being non-negative by [Proposition 4.2.1](#) (b); then (a) again with $(k+1)2 = 2k+2$ (Theorem 1.5.3); and finally [Proposition 4.2.1](#) (b), since $k+2 \leq 2k+2$. This is $P(k+1)$.

- (b) By [Proposition 4.2.3](#) with $F = \mathbb{Q}$ and $\varepsilon := b-a > 0$, there is $k \in \mathbb{N}$ with $1/(k+1) < b-a$ (take $k+1 \geq N$). By (a) and Proposition 4.1.2 (c), $1/2^k \leq 1/(k+1) < b-a$. Let $m := \lfloor 2^k a \rfloor$ (Proposition 2.5.5), so that $m \leq 2^k a < m+1$. Then $2^k a < m+1 \leq 2^k a + 1$, so

$$a < \frac{m+1}{2^k} \leq a + \frac{1}{2^k} < b,$$

dividing by $2^k > 0$ (Proposition 2.3.5 (c), Proposition 2.5.2 (a)) and using $1/2^k < b-a$ with (O3).

- (c) $b-a = 1/6$. For $k \in \mathbb{N}$, $1/(k+1) < 1/6$ if and only if $6 < k+1$, so the least k is 6, with $2^6 = 64$. Then $2^6 a = 64/3$, and $\lfloor 64/3 \rfloor = 21$, since $21 \cdot 3 = 63 \leq 64 < 66 = 22 \cdot 3$. So $(m+1)/2^k = 22/64 = 11/32$. Check: $1/3 < 11/32$ since $32 < 33$, and $11/32 < 1/2$ since $22 < 32$. (The bound $1/2^k \leq 1/(k+1)$ is far from sharp: $k=3$ already gives $3/8$.)

4.2.6.3 C. Going deeper

Exercise. C1: Many rationals, and rational approximation

Throughout, F is an Archimedean ordered field.

- (a) Let $x < y$ in F . Prove that the set $S := \{q \in \mathbb{Q} : x < \iota_F(q) < y\}$ is infinite (Sets, Relations, and Cardinality, Definition 3.1.3).
 (b) Let $x \in F$ and $n \in \mathbb{Z}_{>0}$. Prove that there is $q \in \mathbb{Q}$ with $\iota_F(q) \leq x < \iota_F(q + 1/n)$.

Solution (Hint 1). For (a), apply [Theorem 4.2.5](#) twice; for (b), apply [Proposition 4.2.4](#) to a multiple of x .

Solution (Hint 2). For (a): let $r, s \in \mathbb{Q}$ with $x < \iota_F(r) < \iota_F(s) < y$; it is enough to find infinitely many rationals strictly between r and s . For (b): let m be the integer part of $(n \cdot 1_F)x$.

Solution.

- (a) By [Theorem 4.2.5](#) there is $s \in \mathbb{Q}$ with $x < \iota_F(s) < y$, and, applied to $x < \iota_F(s)$, there is $r \in \mathbb{Q}$ with $x < \iota_F(r) < \iota_F(s)$. Then $r < s$, since ι_F reflects order ([Theorem 4.2.2](#)). For $k \in \mathbb{Z}_{>0}$, let

$$s_k := r + \frac{s-r}{k+1} \in \mathbb{Q}.$$

Since $k+1 > 1 > 0$, $0 < 1/(k+1) < 1$ (Proposition 2.5.2), so $0 < (s-r)/(k+1) < s-r$ (Proposition 2.3.5 (c)), and $r < s_k < s$ by (O3). As ι_F is strictly increasing, $x < \iota_F(r) < \iota_F(s_k) < \iota_F(s) < y$, so $s_k \in S$. If $k < l$, then $1/(l+1) < 1/(k+1)$ (Proposition 2.5.2 (b), Proposition 4.2.1 (b)), so $s_l < s_k$: the map $k \mapsto s_k$ is injective. Suppose S were finite, with m elements for some $m \in \mathbb{N}$ (Sets, Relations, and Cardinality, Definition 3.1.2). The map $\{1, \dots, m+1\} \rightarrow S$, $k \mapsto s_k$, would be injective, against Sets, Relations, and Cardinality, Corollary 3.1.3 (a). So S is infinite.

- (b) Let $c := n \cdot 1_F > 0$ (Proposition 4.2.1 (b)) and let m be the integer part of cx (Proposition 4.2.4): $\iota_F(m) \leq cx < \iota_F(m+1)$. Multiplying by $c^{-1} > 0$ (Proposition 2.5.2 (a), Proposition 2.3.5 (c), with equality allowed on the left) gives $\iota_F(m)c^{-1} \leq x < \iota_F(m+1)c^{-1}$. Let $q := m/n$. Since $c = \iota_F(n)$ and ι_F respects products, $\iota_F(m)c^{-1} = \iota_F(m/n) = \iota_F(q)$ and $\iota_F(m+1)c^{-1} = \iota_F((m+1)/n) = \iota_F(q + 1/n)$, using $(m+1)/n = m/n + 1/n$ (Lemma 4.1.1 (b) in $\mathbb{Q}$). So $\iota_F(q) \leq x < \iota_F(q + 1/n)$.

Marking scheme. (10 points) - 2: (a) obtains two rationals $r < s$ inside the interval, by density twice, with $r < s$ justified (order reflected). - 2: (a) exhibits infinitely many distinct rationals between r and s (any explicit injective family, or an induction producing one more each time), each checked to lie strictly between. - 2: (a) concludes that S is infinite from an injection, citing a finiteness result (pigeonhole or subsets of finite sets). - Second route for (a), same 6 points: 2 for a largest element of a finite non-empty S (Sets, Relations, and Cardinality, Proposition 2.3.8), 2 for density above it, 2 for the contradiction. - 2: (b) applies the integer part to $(n \cdot 1_F)x$. - 2: (b) divides by $n \cdot 1_F > 0$ and identifies the bounds as $\iota_F(q)$, $\iota_F(q + 1/n)$. Fatal errors (0 for the step): using density in an arbitrary ordered field (the Archimedean hypothesis must be named); “infinitely many because the construction never stops” with no finiteness argument; dividing by $n \cdot 1_F$ without its sign.

Exercise. C2: Where the proof for the rationals breaks

In $\mathbb{Q}(t)$ (Example (A Non-Archimedean Ordered Field)):

- Show that the set $\{n \cdot 1 : n \in \mathbb{N}\}$ is bounded above.
- The proof of Theorem 2.5.3 shows that every rational lies below a natural number. Which step of that proof has no analogue for the element t of $\mathbb{Q}(t)$? Prove that it has none.

Solution.

- By Example (A Non-Archimedean Ordered Field), $n \cdot 1 < t$ for every $n \in \mathbb{N}$, so t is an upper bound.
- The first step: “write $x = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ ”. The rest of the proof (the case split on the sign of p , discreteness of $\mathbb{Z}$, the comparison formula) works for any element of the form $\iota_F(p)/\iota_F(q)$ of an ordered field F , in particular of $\mathbb{Q}(t)$. But t is not of this form. Indeed, $\iota_{\mathbb{Q}(t)}(p)/\iota_{\mathbb{Q}(t)}(q) = \iota_{\mathbb{Q}(t)}(p/q)$ by Theorem 4.2.2; if $t = \iota_{\mathbb{Q}(t)}(r)$ with $r \in \mathbb{Q}$, then Theorem 2.5.3 gives $n \in \mathbb{N}$ with $r < n$, so $t = \iota_{\mathbb{Q}(t)}(r) < \iota_{\mathbb{Q}(t)}(n) = n \cdot 1$, against (a). So t is not the image of any rational, and the proof has nothing to start from.

Marking scheme. (10 points) - 3: (a) exhibits an upper bound and justifies it from the positive set of $\mathbb{Q}(t)$. - 3: (b) names the representation $x = p/q$ with integers p, q as the step

that fails. - 4: (b) proves that t is not $\iota_{\mathbb{Q}(t)}(r)$ for any rational r , using the Archimedean property of $\mathbb{Q}$ and that $\iota_{\mathbb{Q}(t)}$ is strictly increasing. Fatal errors (0 for the step): naming the comparison formula or discreteness as the failing step (both hold for images of rationals); asserting that t “is not a fraction” without proof.

Exercise. C3: The order of the rationals is the only one

Prove that the usual order of $\mathbb{Q}$ (Definition 2.5.1) is the only relation $<$ on $\mathbb{Q}$ that makes $\mathbb{Q}$, with its usual operations, an ordered field.

Solution (Hint 1). Which elements must be positive in any such order? Start with 1, then $n \cdot 1$, then p/q .

Solution (Hint 2). By Proposition 2.3.3, an order on a commutative ring is determined by its set of positive elements. Let P be the positive set of an order $\prec$ making $\mathbb{Q}$ an ordered field; show $P = \mathbb{Q}_{>0}$.

Solution. Let $\prec$ be a relation making $\mathbb{Q}$ an ordered field, and $P := \{x \in \mathbb{Q} : 0 \prec x\}$. By Proposition 2.3.3, P satisfies (P1) and (P2), and $x \prec y$ holds if and only if $y - x \in P$. The usual order has positive set $\mathbb{Q}_{>0}$ (Theorem 2.5.1) and is described by $\mathbb{Q}_{>0}$ in the same way. So it suffices to prove $P = \mathbb{Q}_{>0}$.

The multiples $n \cdot 1_{\mathbb{Q}}$ are defined by the recursion of Definition 4.2.1 (a), which uses only addition, so they do not depend on the order; by Proposition 4.2.1 (d), $n \cdot 1_{\mathbb{Q}} = n$. Applying Proposition 2.3.5 (h) to the ordered field $(\mathbb{Q}, \prec)$, in which $1 \neq 0$, gives $0 \prec n$ for every $n \in \mathbb{Z}_{>0}$: so $\mathbb{Z}_{>0} \subseteq P$.

Let $x \in \mathbb{Q}_{>0}$. Then $x = p/q$ with $p, q \in \mathbb{Z}_{>0}$ (Proposition 2.4.4 (b) and the comparison formula, Theorem 2.5.1 (a)), so $x = p \cdot q^{-1}$. Here $p, q \in P$, so $q^{-1} \in P$ by Proposition 2.5.2 (a) applied in $(\mathbb{Q}, \prec)$, and $x \in P$ by (P1). Hence $\mathbb{Q}_{>0} \subseteq P$.

Conversely, let $x \in P$. Then $x \neq 0$ by (P2). If $x \notin \mathbb{Q}_{>0}$, then $x < 0$ in the usual order, so $-x \in \mathbb{Q}_{>0} \subseteq P$; but then both x and $-x$ lie in P , against (P2). So $x \in \mathbb{Q}_{>0}$, and $P = \mathbb{Q}_{>0}$. Hence $\prec$ is the usual order.

Marking scheme. (10 points) - 2: reduces the claim to the positive sets, citing Proposition 2.3.3. - 2: shows every positive integer is positive for the new order, noting that $n \cdot 1$ does not depend on the order. - 3: shows every usual-positive p/q is in P , via inverses and products. - 3: shows $P \subseteq \mathbb{Q}_{>0}$ by (P2). - Route through Theorem 4.2.2 for $F = (\mathbb{Q}, \prec)$, same 10 points: 4 for $\iota_F = \text{id}$ by uniqueness (the equations ignore the order), 6 for reading off the order. Fatal errors (0 for the step): assuming $1 \succ 0$ without $1 \neq 0$ or a rule; using the usual order’s comparison formula to compare elements in the new order.