

4.4 The Algebra and Order of Limits

So far every limit has been found by hand: we computed the distance from a_n to the candidate limit and bounded it, as in Example (First Limits) and Exercise B1 of Section 4.3. This section proves the rules that make most such work unnecessary. Sums, multiples, products and quotients of convergent sequences converge to the sum, multiple, product and quotient of the limits; limits respect $\leq$; and a sequence trapped between two sequences with a common limit has that limit too. Everything holds in an arbitrary ordered field F . No result of this section uses the Archimedean property; only the examples and exercises set in $\mathbb{Q}$ do, each time through Example (First Limits) (b) or the non-example after Definition 4.3.4.

Throughout, F is an ordered field and $F_{>0}$ its set of positive elements, as in Section 4.3.

4.4.1 Two tools

The first tool says that a small factor wins against a factor that merely stays bounded.

Proposition 4.4.1. Null Times Bounded Is Null

Let (a_n) be a null sequence in F , and let (b_n) be a sequence with $|b_n| \leq M$ for every n , where $M \in F_{>0}$. Then $(a_n b_n)$ is null.

Proof. Let $\varepsilon \in F_{>0}$. Then $\varepsilon/M \in F_{>0}$ (Proposition 2.5.2 (a) and (O4)). Since (a_n) is null, there is N (depending on ε and M) with $|a_n| < \varepsilon/M$ for every $n \geq N$. For such n , by Proposition 4.1.3 (b),

$$|a_n b_n - 0| = |a_n| |b_n| \leq |a_n| M < \frac{\varepsilon}{M} M = \varepsilon,$$

where the first inequality multiplies $|b_n| \leq M$ by $|a_n| \geq 0$ (Proposition 4.1.2 (d)), and the second multiplies $|a_n| < \varepsilon/M$ by $M > 0$ (Proposition 2.3.5 (c)).

■

For instance, $((-1)^n/(n+1))$ is null in $\mathbb{Q}$: it is $1/(n+1)$, null by Example (First Limits) (b), times $(-1)^n$, bounded by 1. The bound M is fixed before ε and enters the tolerance ε/M . That is the pattern of the product law below.

The second tool is the fact behind every division by b_n : if $b_n \rightarrow b \neq 0$, the terms eventually stay away from 0, by a margin we can name.

Lemma 4.4.2. Terms Near a Non-zero Limit

Let $b_n \rightarrow b$ in F with $b \neq 0$. Then there is $N_0 \in \mathbb{N}$ with $|b_n| > |b|/2$ for every $n \geq N_0$. If $b > 0$, then moreover $b_n > b/2$ for every $n \geq N_0$.

Proof. Since $b \neq 0$, $|b| > 0$ (Proposition 4.1.3 (a)), so $\varepsilon := |b|/2 \in F_{>0}$ (Proposition 4.1.2 (b)). Choose N_0 (depending on (b_n) and b) with $|b_n - b| < |b|/2$ for every $n \geq N_0$. For such n , the reverse triangle inequality (Proposition 4.1.3 (f)), together with $x \leq |x|$ and $|-x| = |x|$

(parts (c), (b)), gives

$$|b| - |b_n| \leq ||b| - |b_n|| \leq |b - b_n| = |b_n - b| < \frac{|b|}{2},$$

so $|b_n| > |b| - |b|/2 = |b|/2$, adding $|b_n| - |b|/2$ to both sides (Proposition 2.3.5 (b)). If $b > 0$, then $|b| = b$, and Proposition 4.1.3 (d) turns $|b_n - b| < b/2$ into $b - b/2 < b_n$, that is, $b_n > b/2$. ■

In particular $b_n \neq 0$ for every $n \geq N_0$, since $|b_n| > |b|/2 > 0$. The lemma needs $b \neq 0$. In $\mathbb{Q}$, $b_n := (1 + (-1)^n)/(n+1)$ tends to 0 (the warning after Example (First Limits) in Section 4.3: $0 \leq b_n \leq 2/(n+1)$, and $1/(n+1) \rightarrow 0$ by Example (First Limits) (b), which uses Archimedes), yet $b_n = 0$ at every odd n , so for $b = 0$ no N_0 has $|b_n| > |b|/2 = 0$ for every $n \geq N_0$. Even $1/(n+1)$, which tends to 0 (Example (First Limits) (b)) and is never 0, is bounded away from 0 by no positive margin.

4.4.2 The limit laws

Theorem 4.4.3. Limit Laws

Let $a_n \rightarrow a$ and $b_n \rightarrow b$ in F , and let $c \in F$. Then:

- (a) $a_n + b_n \rightarrow a + b$;
- (b) $ca_n \rightarrow ca$; in particular $-a_n \rightarrow -a$ and $a_n - b_n \rightarrow a - b$;
- (c) $a_n b_n \rightarrow ab$;
- (d) if $b \neq 0$ and $b_n \neq 0$ for every n , then $a_n/b_n \rightarrow a/b$.

Idea. Work backwards from each target distance. For (a), $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$: a sum of two terms, each of which we can make small, so we give each half the tolerance. For (c), write

$$a_n b_n - ab = a_n(b_n - b) + b(a_n - a).$$

The factors $b_n - b$ and $a_n - a$ can be made small, but a_n cannot. It is, however, bounded (Proposition 4.3.5), so we replace $|a_n|$ by a bound M and ask $|b_n - b|$ to be smaller than $\varepsilon/(2M)$. For (d), it is enough to treat $1/b_n$ and then use (c). *Before reading on: why must M be fixed before ε ?*

Proof.

- (a) Let $\varepsilon \in F_{>0}$. Then $\varepsilon/2 \in F_{>0}$ (Proposition 4.1.2 (b)). Since $a_n \rightarrow a$, choose N_1 (depending on ε) with $|a_n - a| < \varepsilon/2$ for every $n \geq N_1$; since $b_n \rightarrow b$, choose N_2 (depending on ε) with $|b_n - b| < \varepsilon/2$ for every $n \geq N_2$. Let $N := \max\{N_1, N_2\}$. For $n \geq N$, both estimates hold, and the triangle inequality (Proposition 4.1.3 (e)) gives

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where the strict inequality is Proposition 2.3.5 (b) and the last equality Proposition 4.1.2 (b).

- (b) If $c = 0$, then $ca_n = 0 = ca$ for every n (Proposition 2.3.2 (b)), a constant sequence, which converges to 0 (Example (First Limits) (a)). Let $c \neq 0$, so $|c| > 0$ (Proposition 4.1.3 (a)), and let $\varepsilon \in F_{>0}$. Then $\varepsilon/|c| \in F_{>0}$ (Proposition 2.5.2 (a) and (O4)). Choose N (depending on ε and c) with $|a_n - a| < \varepsilon/|c|$ for every $n \geq N$. For such n , by distributivity and Proposition 4.1.3 (b),

$$|ca_n - ca| = |c(a_n - a)| = |c||a_n - a| < |c| \frac{\varepsilon}{|c|} = \varepsilon,$$

multiplying by $|c| > 0$ (Proposition 2.3.5 (c)). With $c = -1$, and $(-1)x = -x$ (Proposition 2.3.2 (c)), this gives $-a_n \rightarrow -a$. Then (a), applied to (a_n) and $(-b_n)$, gives $a_n - b_n = a_n + (-b_n) \rightarrow a + (-b) = a - b$.

- (c) By Proposition 4.3.5 there is $M \in F_{>0}$ with $|a_n| \leq M$ for every $n \in \mathbb{N}$. We fix this M now; it depends on the sequence (a_n) only. Now let $\varepsilon \in F_{>0}$. Since $|b| \geq 0$, the element $|b| + 1$ is $\geq 1 > 0$ (Proposition 2.3.5 (b), (e)). Since $2 > 0$ (Proposition 4.1.2 (b)), $2M > 0$ and $2(|b| + 1) > 0$ by (O4), so Proposition 2.5.2 (a) and (O4) give

$$\frac{\varepsilon}{2M} \in F_{>0} \quad \text{and} \quad \frac{\varepsilon}{2(|b| + 1)} \in F_{>0}.$$

Since $b_n \rightarrow b$, choose N_1 with $|b_n - b| < \varepsilon/(2M)$ for every $n \geq N_1$. Since $a_n \rightarrow a$, choose N_2 with $|a_n - a| < \varepsilon/(2(|b| + 1))$ for every $n \geq N_2$. Both depend only on ε , M and b . Let $N := \max\{N_1, N_2\}$, and let $n \geq N$. Using the identity of the Idea, the triangle inequality and the product rule (Proposition 4.1.3 (e), (b)),

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n||b_n - b| + |b||a_n - a| \\ &\leq M|b_n - b| + (|b| + 1)|a_n - a| \\ &< M \frac{\varepsilon}{2M} + (|b| + 1) \frac{\varepsilon}{2(|b| + 1)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The second line uses Proposition 4.1.2 (d) twice, with $0 \leq |a_n| \leq M$ and $0 \leq |b| \leq |b| + 1$, and adds the two estimates (Proposition 2.3.5 (b)). The third multiplies the two estimates by the positive elements M and $|b| + 1$ (Proposition 2.3.5 (c)) and adds them (Proposition 2.3.5 (b)); the last equality is Proposition 4.1.2 (b). So $a_n b_n \rightarrow ab$.

- (d) First we show $1/b_n \rightarrow 1/b$. By Lemma 4.4.2 there is N_0 (depending on (b_n) and b) with $|b_n| > |b|/2$ for every $n \geq N_0$. For such n , Lemma 4.1.1 (c) and Proposition 4.1.3 (b) give

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b_n b} \right| = \frac{|b_n - b|}{|b_n||b|}.$$

Multiplying $|b_n| > |b|/2$ by $|b| > 0$ gives $|b_n||b| > |b|^2/2 > 0$, so $1/(|b_n||b|) < 2/|b|^2$ (Proposition 2.5.2 (b)). Multiplying by $|b_n - b| \geq 0$,

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| \leq \frac{2}{|b|^2} |b_n - b| \quad (n \geq N_0).$$

Now let $\varepsilon \in F_{>0}$. Then $\varepsilon|b|^2/2 \in F_{>0}$ ((O4), Proposition 4.1.2 (b)); choose N_1 (depending on ε and b) with $|b_n - b| < \varepsilon|b|^2/2$ for every $n \geq N_1$, and let $N := \max\{N_0, N_1\}$. For $n \geq N$, the last display and multiplication of $|b_n - b| < \varepsilon|b|^2/2$ by $2/|b|^2 > 0$ (Proposition 2.5.2 (a), (O4); Proposition 2.3.5 (c)) give $|1/b_n - 1/b| < (2/|b|^2)(\varepsilon|b|^2/2) = \varepsilon$. So

$1/b_n \rightarrow 1/b$. Finally $a_n/b_n = a_n \cdot (1/b_n)$, and (c) gives $a_n/b_n \rightarrow a \cdot (1/b) = a/b$. This proves the theorem. ■

The answer to the prompt: the tolerance $\varepsilon/(2M)$ contains M , so M must exist before that tolerance is written down. And M must not depend on n : it has to bound $|a_n|$ for **every** $n \geq N$ at once, including indices not yet chosen. That is why it comes from Proposition 4.3.5, a single bound for the whole sequence, and not from “ $|a_n|$ is some element of F ”. The $+1$ in $|b| + 1$ is there because b may be 0, and $\varepsilon/(2|b|)$ would then mean nothing.

Part (a) is the move we call **split** ε : to make a sum of k small terms smaller than ε , give each term the tolerance ε/k , and combine the finitely many thresholds by a maximum. Part (c) is **bound, then shrink**: replace the factor you cannot control by a bound fixed in advance, then make the other factor small.

Alternatively, (c) follows from the earlier tools without new estimates. By Proposition 4.3.3, $(b_n - b)$ and $(a_n - a)$ are null; $a_n(b_n - b)$ is null by Proposition 4.4.1, with the bound M ; $b(a_n - a)$ is null by (b); so their sum $a_n b_n - ab$ is null by (a), and Proposition 4.3.3 again gives $a_n b_n \rightarrow ab$. The bound M is still fixed first: it is the hypothesis of Proposition 4.4.1.

In (d), the hypothesis “ $b_n \neq 0$ for every n ” is there only so that a_n/b_n is defined for every n . By Lemma 4.4.2, $b_n \neq 0$ holds eventually anyway once $b \neq 0$. If some early b_n vanish, the quotient is a sequence from some N_0 on, and Proposition 4.3.2 (b) applies the law to the shifted sequences (a_{n+N_0}) and (b_{n+N_0}) .

Warning. Part (d) needs $b \neq 0$, even when every $b_n \neq 0$. In $\mathbb{Q}$, $a_n = 1/(n+1)$ and $b_n = 1/(n+1)^2$ are both null (the first by Example (First Limits) (b), the second by (c), since $b_n = a_n a_n$), with every $b_n \neq 0$, and $a_n/b_n = n+1$. Since $n+1 > n$, the sequence $(n+1)$ is not bounded (the non-example after Definition 4.3.4), and so has no limit (Proposition 4.3.5). The “law” $\lim(a_n/b_n) = 0/0$ names nothing.

Quick check. Find $\lim(3n^2 - n)/(n^2 + 5)$ in $\mathbb{Q}$, citing a law, or another result, for each step.

Solution. The limit is 3. Let $u_0 := 0$ and $u_n := 1/n$ for $n \geq 1$. The shifted sequence $(u_{n+1})_n = (1/(n+1))_n$ is null (Example (First Limits) (b); this is where Archimedes enters), so (u_n) is null by Proposition 4.3.2 (b). By Theorem 4.4.3: $3 - u_n \rightarrow 3 - 0 = 3$ (the constant 3 converges to 3 by Example (First Limits) (a); then (b) and (a)); $u_n^2 \rightarrow 0$ ((c)); $1 + 5u_n^2 \rightarrow 1$ ((b), (a)). Every $1 + 5u_n^2 \geq 1 > 0$: $u_n^2 \geq 0$ (Proposition 2.3.5 (e) if $u_n \neq 0$, and $u_0^2 = 0$), so $5u_n^2 \geq 0$ (Proposition 4.1.2 (d)), and $1 + 5u_n^2 \geq 1$ by Proposition 2.3.5 (b). Also the limit $1 \neq 0$, so (d) gives

$$c_n := \frac{3 - u_n}{1 + 5u_n^2} \rightarrow \frac{3}{1} = 3.$$

For $n \geq 1$, multiplying numerator and denominator by n^2 shows $c_n = (3n^2 - n)/(n^2 + 5)$. The two sequences agree from $N_0 = 1$ on (at $n = 0$ they differ: 3 versus 0), so Proposition 4.3.2 (a) gives the limit 3.

4.4.3 Limits and order

Inequalities between terms pass to the limits, provided we accept that a strict inequality may become an equality.

Proposition 4.4.4. Limits Preserve Weak Inequalities

Let $a_n \rightarrow a$ and $b_n \rightarrow b$ in F , and let $N_0 \in \mathbb{N}$.

- (a) If $a_n \leq b_n$ for every $n \geq N_0$, then $a \leq b$.
- (b) In particular, for $c \in F$: if $a_n \leq c$ for every $n \geq N_0$, then $a \leq c$; and if $a_n \geq c$ for every $n \geq N_0$, then $a \geq c$.
- (c) (*Squeeze.*) Let (c_n) be a sequence in F . If $a = b$ and $a_n \leq c_n \leq b_n$ for every $n \geq N_0$, then $c_n \rightarrow a$.

Idea. For (a), suppose instead that $b < a$. The midpoint $(a + b)/2$ separates them, and from some index on a_n lies above it and b_n below it, contradicting $a_n \leq b_n$. The tolerance that achieves this is half the gap, $(a - b)/2$. For (c), from some index on a_n and b_n both lie in the window $(a - \varepsilon, a + \varepsilon)$, and c_n is trapped between them.

Proof.

- (a) Suppose, for a contradiction, that $b < a$. Let

$$\varepsilon := \frac{a - b}{2},$$

which lies in $F_{>0}$ by Proposition 2.3.5 (a) and Proposition 4.1.2 (b). Choose N_1 (depending on ε) with $|a_n - a| < \varepsilon$ for every $n \geq N_1$, and N_2 (depending on ε) with $|b_n - b| < \varepsilon$ for every $n \geq N_2$. Let $n := \max\{N_0, N_1, N_2\}$. By Proposition 4.1.3 (d), at this n ,

$$a_n > a - \varepsilon = \frac{a + b}{2} = b + \varepsilon > b_n.$$

This contradicts $a_n \leq b_n$, which holds because $n \geq N_0$. So $b < a$ is false, and $a \leq b$ by trichotomy (O1).

- (b) A constant sequence c converges to c (Example (First Limits) (a)). Apply (a) to (a_n) and the constant sequence c for the first claim, and to the constant sequence c and (a_n) for the second.
- (c) Let $\varepsilon \in F_{>0}$. Choose N_1 (depending on ε) with $|a_n - a| < \varepsilon$ for $n \geq N_1$ and N_2 (depending on ε) with $|b_n - a| < \varepsilon$ for $n \geq N_2$ (recall $b = a$), and let $N := \max\{N_0, N_1, N_2\}$. For $n \geq N$, Proposition 4.1.3 (d) gives

$$a - \varepsilon < a_n \leq c_n \leq b_n < a + \varepsilon,$$

so $|c_n - a| < \varepsilon$ by the same part (d). Hence $c_n \rightarrow a$. ■

The two midpoint equalities in (a) are field algebra: $a - (a - b)/2 = (2a - a + b)/2 = (a + b)/2$,

and likewise $b + (a - b)/2 = (a + b)/2$. Chapter 5 uses the squeeze (c) repeatedly, mostly with $F = \mathbb{Q}$ to show that a sequence is null. In the squeeze (c), the middle sequence is not assumed to converge: its convergence is the conclusion.

Warning. Strict inequalities do not pass to the limit. In $\mathbb{Q}$, $1/(n+1) > 0$ for every n , and the limit is 0, not positive (Example (First Limits) (b)). From $a_n < b_n$ for every n , (a) gives only $a \leq b$.

4.4.4 Exercises

4.4.4.1 A. Check your understanding

Exercise. A1

These questions are about the proof of the product law, [Theorem 4.4.3](#) (c).

- (a) Why is M fixed before ε , and what does it depend on?
- (b) Which hypothesis supplies N_1 , which supplies N_2 , and which supplies M ?
- (c) Why does the proof use $|b| + 1$ rather than $|b|$?
- (d) State the idea of the proof in two sentences.
- (e) Run the proof in $\mathbb{Q}$ on $a_n = 1 + 1/(n+1)$, $b_n = 2$ and $\varepsilon = 1/10$: give a valid M , the least N_1 , the least N_2 , and N .

Solution.

- (a) The tolerance $\varepsilon/(2M)$ is built from M , so M must be available first. It depends only on the sequence (a_n) : it bounds $|a_n|$ for every n , independently of ε and of n . A “bound” depending on n would be useless, since the tolerance for $|b_n - b|$ must be named before n is.
- (b) N_1 comes from $b_n \rightarrow b$, applied with the tolerance $\varepsilon/(2M)$; N_2 from $a_n \rightarrow a$, with the tolerance $\varepsilon/(2(|b| + 1))$; M also from $a_n \rightarrow a$, through Proposition 4.3.5.
- (c) b may be 0, and then $\varepsilon/(2|b|)$ is undefined. Since $|b| + 1 \geq 1 > 0$, the tolerance $\varepsilon/(2(|b| + 1))$ always exists, and $|b| \leq |b| + 1$ keeps the estimate valid.
- (d) Write $a_n b_n - ab = a_n(b_n - b) + b(a_n - a)$, so that each term has a factor tending to 0. The other factor of the first term is not small, but it is bounded by a constant M fixed in advance, so making $|b_n - b| < \varepsilon/(2M)$ and $|a_n - a| < \varepsilon/(2(|b| + 1))$ makes each term smaller than $\varepsilon/2$.
- (e) Here $a = 1$ and $b = 2$. Since $0 < 1/(n+1) \leq 1$, $|a_n| \leq 2$ for every n , so $M = 2$ is valid. Then $\varepsilon/(2M) = 1/40$, and $|b_n - b| = 0 < 1/40$ for every n : the least N_1 is 0. Next $\varepsilon/(2(|b| + 1)) = (1/10)/6 = 1/60$, and $|a_n - a| = 1/(n+1) < 1/60$ if and only if $n+1 > 60$, that is, $n \geq 60$: the least N_2 is 60. So $N = \max\{0, 60\} = 60$. Check at $n = 60$: $a_{60}b_{60} - ab = 2(1 + 1/61) - 2 = 2/61 < 1/10$.

4.4.4.2 B. Practice

Exercise. B1

Find the following limits in $\mathbb{Q}$ by the limit laws, naming the result used at each step.

- (a) $\frac{n+2}{3n+1}$.
- (b) $\frac{n^2+1}{(n+1)^2}$.
- (c) $\left(\frac{1}{n+1} - 2\right)\left(3 + \frac{(-1)^n}{n+1}\right)$.

Hint: write each term through $u_n := 1/(n+1)$.

Solution. Let $u_n := 1/(n+1)$, so $u_n \rightarrow 0$ (Example (First Limits) (b)), and constant sequences converge to their value (part (a) of that example). All laws cited are parts of [Theorem 4.4.3](#).

- (a) With $m := n+1 \geq 1$: $(n+2)/(3n+1) = (m+1)/(3m-2) = (1+u_n)/(3-2u_n)$, dividing numerator and denominator by m . Now $1+u_n \rightarrow 1$ by (a), and $3-2u_n \rightarrow 3$ by (b) and (a). Since $0 < u_n \leq 1$, $3-2u_n \geq 1 > 0$ for every n , and the limit $3 \neq 0$, so (d) gives the limit $1/3$.
- (b) With $m = n+1$: $(n^2+1)/(n+1)^2 = ((m-1)^2+1)/m^2 = 1-2u_n+2u_n^2$. By (c), $u_n^2 \rightarrow 0$; by (b) and (a), $1-2u_n+2u_n^2 \rightarrow 1-0+0 = 1$. No quotient law is needed.
- (c) $u_n - 2 \rightarrow -2$ by (b) and (a). The sequence $(-1)^n u_n$ is null by [Proposition 4.4.1](#), since u_n is null and $|(-1)^n| = 1$ (Example (First Limits) (d)); so $3 + (-1)^n u_n \rightarrow 3$ by (a). By (c), the product tends to $(-2) \cdot 3 = -6$.

In each part Archimedes enters once, through $u_n \rightarrow 0$; every other step is a law valid in any ordered field.

Exercise. B2: Put the proof in order

Here are lines of the proof of [Proposition 4.4.4](#) (a), shuffled, together with two lines that do not belong. Recall the hypotheses: $a_n \rightarrow a$, $b_n \rightarrow b$, and $a_n \leq b_n$ for every $n \geq N_0$. Write the seven lines of a correct proof in a correct order, say which other orders are also admissible, and say why each of the other two lines fails.

- (i) Let $n := \max\{N_0, N_1, N_2\}$.
- (ii) Let $\varepsilon := a - b$, which lies in $F_{>0}$.
- (iii) Choose N_2 with $|b_n - b| < \varepsilon$ for every $n \geq N_2$.
- (iv) Suppose, for a contradiction, that $b < a$.
- (v) This contradicts $a_n \leq b_n$; so $b < a$ is false, and $a \leq b$ by trichotomy.
- (vi) Let $\varepsilon := (a - b)/2$, which lies in $F_{>0}$ by [Proposition 4.1.2](#).
- (vii) Let $n := N_1$.
- (viii) Then $a_n > a - \varepsilon = (a + b)/2 = b + \varepsilon > b_n$.
- (ix) Choose N_1 with $|a_n - a| < \varepsilon$ for every $n \geq N_1$.

Solution. An admissible order is

(iv), (vi), (ix), (iii), (i), (viii), (v).

The only other admissible order exchanges (ix) and (iii), which use only ε and one hypothesis each. Nothing else moves: (vi) uses the assumption $b < a$ of (iv) to make ε positive; (ix) and (iii) use ε ; (i) uses N_1 and N_2 ; (viii) uses the two estimates at this n , and needs $n \geq N_1$ and $n \geq N_2$; (v) uses (viii) and $n \geq N_0$, and closes the assumption of (iv).

Line (ii) is an ε of the wrong size. It is positive, but with it (viii) becomes $a_n > a - (a - b) = b$ and $b_n < b + (a - b) = a$, and “ $a_n > b$ and $b_n < a$ ” is no contradiction with $a_n \leq b_n$: for instance $a_n = b_n = (a + b)/2$ satisfies all three at once. The two windows of half-width $a - b$ around a and b overlap; half-width $(a - b)/2$ makes them touch only at the midpoint, which neither open window contains.

Line (vii) chooses an index that depends on the wrong thing. With $n := N_1$, the estimate for a_n holds, but nothing makes $n \geq N_2$ or $n \geq N_0$: the estimate for b_n and the hypothesis $a_n \leq b_n$ may both fail at this n . The index must lie beyond **all three** thresholds, which the maximum in (i) ensures.

Exercise. B3: Close to a convergent sequence

Let $a_n \rightarrow a$ in F , and let (b_n) be a sequence in F such that $(b_n - a_n)$ is null. Prove that $b_n \rightarrow a$. Here is the start of a proof.

Let $\varepsilon \in F_{>0}$. Then $\varepsilon/2 \in F_{>0}$. Since $a_n \rightarrow a$, choose N_1 with $|a_n - a| < \varepsilon/2$ for every $n \geq N_1$. Since $(b_n - a_n)$ is null, choose N_2 with $|b_n - a_n| < \varepsilon/2$ for every $n \geq N_2$. Let $N := \max\{N_1, N_2\}$.

...

Name the move, then complete it.

Solution. The move is **split** ε , with two terms: each gets $\varepsilon/2$, and the two thresholds are combined by a maximum.

Completion. Let $n \geq N$. Then $n \geq N_1$ and $n \geq N_2$, and the three-point triangle inequality (Proposition 4.1.3 (e)) gives

$$|b_n - a| \leq |b_n - a_n| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon \in F_{>0}$ was arbitrary, $b_n \rightarrow a$. (The same result also follows from the laws: $b_n = (b_n - a_n) + a_n \rightarrow 0 + a$ by [Theorem 4.4.3](#) (a). The exercise is the law’s proof, run once more.)

4.4.4.3 C. Going deeper

Exercise. C1: Exit check: the product law

Prove, from Definition 4.3.2, that if $a_n \rightarrow a$ and $b_n \rightarrow b$ in an ordered field F , then $a_n b_n \rightarrow ab$. You may use that convergent sequences are bounded (Proposition 4.3.5), the rules

for the absolute value (Proposition 4.1.3), Proposition 4.3.3 and [Proposition 4.4.1](#), but not [Theorem 4.4.3](#).

Solution. By Proposition 4.3.5 there is $M \in F_{>0}$ with $|a_n| \leq M$ for every $n \in \mathbb{N}$. It depends only on (a_n) , and we fix it before any tolerance is named.

Let $\varepsilon \in F_{>0}$. Since $|b| + 1 \geq 1 > 0$, both $\varepsilon/(2M)$ and $\varepsilon/(2(|b| + 1))$ lie in $F_{>0}$. By $b_n \rightarrow b$ there is N_1 with $|b_n - b| < \varepsilon/(2M)$ for every $n \geq N_1$; by $a_n \rightarrow a$ there is N_2 with $|a_n - a| < \varepsilon/(2(|b| + 1))$ for every $n \geq N_2$. These depend only on ε , M and b . Let $N := \max\{N_1, N_2\}$.

Let $n \geq N$. Since

$$a_n b_n - ab = a_n(b_n - b) + b(a_n - a),$$

the triangle inequality and $|xy| = |x||y|$ give

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n||b_n - b| + |b||a_n - a| \\ &\leq M|b_n - b| + (|b| + 1)|a_n - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where the second line uses $0 \leq |a_n| \leq M$ and $0 \leq |b| \leq |b| + 1$ (Proposition 4.1.2 (d)), and the third multiplies the two estimates by $M > 0$ and $|b| + 1 > 0$. So for every $\varepsilon \in F_{>0}$ there is $N \in \mathbb{N}$ with $|a_n b_n - ab| < \varepsilon$ for every $n \geq N$: $a_n b_n \rightarrow ab$.

Marking scheme. (10 points) - 2: fixes a bound $M \in F_{>0}$ for $(|a_n|)$, justified (convergent sequences are bounded), before ε . - 2: writes $a_n b_n - ab$ as a sum of two terms, each with a factor tending to 0. - 3: chooses N_1, N_2 depending only on ε, M and b , with the case $b = 0$ handled, and N as their maximum. - 2: carries out the estimate with the triangle inequality and the product rule for $|\cdot|$. - 1: concludes with the quantifiers in the order $\forall \varepsilon \exists N \forall n \geq N$. Fatal errors (0 for the step): a threshold or tolerance that depends on n ; the bound “ $|a_n| \leq |a| + 1$ for every n ” without the finite part for $n < N$; dividing by $|b|$ without $b \neq 0$. A route through null sequences ($(b_n - b)$ and $(a_n - a)$ are null by Proposition 4.3.3; $a_n(b_n - b)$ and $b(a_n - a)$ are null by [Proposition 4.4.1](#), with bounds M and $|b| + 1$; the sum of two null sequences is null, proved from Definition 4.3.2 by splitting ε) earns the same points: decomposition 2, the bound 2, the two null products and the proved sum 5, conclusion 1. Citing [Theorem 4.4.3](#) for the sum or for $b(a_n - a)$ gets 0 for that step. A three-term route ($a_n b_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a)$, with a tolerance such as $\delta = \min\{1, \varepsilon/(1 + |a| + |b|)\}$ chosen from ε, a and b only) earns the same points: decomposition 2; the choice of δ , independent of n and with $\delta \leq 1$ controlling the product term (in place of the bound M), 2; a threshold N for both sequences at tolerance δ (the case $a = 0$ or $b = 0$ covered by the 1 in the denominator) 3; the estimate 2; conclusion 1.

Exercise. C2: Non-negative terms and reciprocals

Let $a_n \rightarrow a$ in F , with $a_n \geq 0$ for every n .

- (a) Prove that $a \geq 0$.
- (b) Suppose moreover $a > 0$. Prove that there is $N_0 \in \mathbb{N}$ with $a_n > 0$ for every $n \geq N_0$,

and that the sequence $(1/a_{n+N_0})_{n \in \mathbb{N}}$ converges to $1/a$.

- (c) Give an example in $\mathbb{Q}$ where $a_n > 0$ for every n and $(1/a_n)$ does not converge.

Solution (Hint 1). For (a), compare with a constant sequence. For (b), which result keeps terms away from a non-zero limit?

Solution (Hint 2). For (b), start: “By [Lemma 4.4.2](#), since $a > 0$, there is N_0 with $a_n > a/2$ for every $n \geq N_0$.”

Solution.

- (a) Since $a_n \geq 0$ for every $n \geq 0$, [Proposition 4.4.4](#) (b) with $c = 0$ gives $a \geq 0$.
- (b) Since $a > 0$, [Lemma 4.4.2](#) gives N_0 with $a_n > a/2 > 0$ for every $n \geq N_0$. The shifted sequence $(a_{n+N_0})_n$ converges to a by [Proposition 4.3.2](#) (b), and each of its terms is non-zero. By [Theorem 4.4.3](#) (d), applied to the constant sequence 1 and to $(a_{n+N_0})_n$, whose limit a is non-zero, $1/a_{n+N_0} \rightarrow 1/a$.
- (c) $a_n = 1/(n+1)$: every term is positive, and $1/a_n = n+1$ is not bounded in $\mathbb{Q}$: for any $M \in \mathbb{Q}_{>0}$, the non-example after [Definition 4.3.4](#) gives n with $M < n < n+1 = |1/a_n|$. So it does not converge ([Proposition 4.3.5](#)). The hypothesis $a > 0$ of (b) fails here: $a = 0$.

Marking scheme. (10 points) - 3: (a), by comparison with the constant sequence 0 (or by contradiction with $\varepsilon = -a/2$). - 3: (b), an index N_0 beyond which a_n is bounded away from 0, from $a > 0$. - 2: (b), $1/a_{n+N_0} \rightarrow 1/a$, either by the quotient law ([Theorem 4.4.3](#) (d)) applied to the shift, with its convergence to a and $a \neq 0$ checked, or directly from $|1/a_{n+N_0} - 1/a| \leq (2/a^2)|a_{n+N_0} - a|$, which uses $a_{n+N_0} > a/2$. - 2: (c), an example with the failure of convergence justified. Fatal errors (0 for the step): concluding $a > 0$ in (a) from $a_n > 0$; applying the quotient law without checking that the limit is non-zero.