

Ordered Fields and Sequences in an Ordered Field

In $\mathbb{Q}$ the numbers $2, 3/2, 17/12, 577/408, \dots$ crowd together, and their squares crowd toward 2; yet Chapter 2 proved that no rational squares to 2 (Theorem 2.6.1). Chapter 5 will make the missing number out of sequences like this one. Before it can, we must say what “crowd together” means when the only yardsticks are rationals, and prove the rules for such sequences without presupposing the number they are aiming at. This chapter does that once, for an **arbitrary ordered field** F , with every tolerance ε taken from F itself. It adds to the inequality rules of Chapter 2 what limits need (halving, fractions, the absolute value and the triangle inequality), places a copy of $\mathbb{N}$ and of $\mathbb{Q}$ inside every ordered field, and names the Archimedean property, which $\mathbb{Q}$ has and an ordered field need not have. Then come limits (uniqueness, boundedness, the limit laws, limits and order) and Cauchy sequences (convergent implies Cauchy, Cauchy implies bounded, sums and products, and the sign of a Cauchy sequence that is not null). Chapter 5 applies all of it with $F = \mathbb{Q}$.

The chapter ends at the gap it was written for. The sequence above is Cauchy in $\mathbb{Q}$ and has no limit in $\mathbb{Q}$, so in $\mathbb{Q}$ “Cauchy” does not imply “convergent”. If you have met limits in a course on $\mathbb{R}$, most statements here will look familiar. What is new is that they are proved in any ordered field, from the axioms alone. Watch where the Archimedean property is needed (each use names it), and note that completeness never appears: no supremum, no monotone convergence theorem and no Bolzano–Weierstrass theorem is used, as Section 4.6 checks.

What you need. Chapters 1 and 2, chiefly Sections 2.3, 2.5 and 2.6, and Section 3.4 for review exercise R1 only; *Logic and Proof*, Chapters 2–8, chiefly Chapter 6; and *Sets, Relations, and Cardinality*, Chapters 1 and 2, with Chapter 3 for one exercise. From Chapter 2, Section 2.3: commutative rings and fields, ordered rings and fields and the rules for inequalities (Definition 2.3.1, Definition 2.3.2, Proposition 2.3.2, Proposition 2.3.5, Proposition 2.3.3), the integers as an ordered ring with least elements of sets bounded below (Theorem 2.3.1, Theorem 2.3.4, Proposition 2.3.6), and the two-element field that cannot be ordered (Example (A field with two elements that cannot be ordered)). From Sections 2.2, 2.4 and 2.5: the copies of $\mathbb{N}$ in $\mathbb{Z}$ and of $\mathbb{Z}$ in $\mathbb{Q}$ (Proposition 2.2.3, Proposition 2.4.4), $\mathbb{Q}$ as an ordered field with reciprocals reversing order (Theorem 2.5.1, Definition 2.5.1, Proposition 2.5.2), and the Archimedean property of $\mathbb{Q}$, its form $1/N < \varepsilon$, and the integer part (Theorem 2.5.3, Corollary 2.5.4, Proposition 2.5.5). From Section 2.6: no rational squares to 2, and the Newton sequence with the term-wise facts proved there (Theorem 2.6.1, Example (A Rational Sequence Aimed at the Gap)), which Sections 4.3, 4.5 and 4.6 use. From Chapter 1: the citing rule of Section 1.7 (Corollary 1.7.3), by which induction (*Logic and Proof*, Axiom 4.2.1) and the recursion theorem (*Sets, Relations, and Cardinality*, Theorem 2.3.1, Theorem 1.5.1) apply to $\mathbb{N}$; the arithmetic and order of $\mathbb{N}$ (Definition 1.5.1, Theorem 1.5.3, Theorem 1.6.1, Proposition 1.6.3); and the powers of Section 1.6 (Proposition 1.6.4), which Section 4.2 compares with powers in F and Exercise B2 of Section 4.6 uses as indices. From *Logic and Proof*, Chapter 6: the negation laws (*Logic and Proof*, Proposition 6.2.1, *Logic and Proof*, Proposition 6.2.3, *Logic and Proof*, Proposition 6.2.2), used to negate the definitions of limit, Cauchy sequence and Archimedean field, and the statement “eventually keeps one sign” in review exercise R3; the translation of a limit statement and the bounded quantifier rules (*Logic and Proof*, Example (Five definitions, translated), *Logic and Proof*, Proposition 6.1.1) in Section 4.3; and the argument that numbers arbitrarily close are equal (*Logic and Proof*, Proposition 6.2.5), which Section 4.3 reruns in F . From *Logic and Proof*, Chapter 2, the rule RAA (*Logic and Proof*, Definition 2.5.1), and De Morgan’s laws (*Logic and Proof*, Proposition 2.5.5), which R3 also uses; from its Chapter 8, the catalogue of what a proof

may not do (Logic and Proof, Example (What is not allowed)), for the find-the-error exercises C2 of Section 4.3 and C3 of Section 4.5. From *Sets, Relations, and Cardinality*: sequences as functions (Sets, Relations, and Cardinality, Definition 1.4.1), from Chapter 1; total orders (Sets, Relations, and Cardinality, Definition 2.6.1) and finite maxima (Sets, Relations, and Cardinality, Proposition 2.3.8), from Sections 2.6 and 2.3. A few items reach further. Exercise C1 of Section 4.2 uses finite and infinite sets (Sets, Relations, and Cardinality, Definition 3.1.2, Sets, Relations, and Cardinality, Definition 3.1.3, Sets, Relations, and Cardinality, Corollary 3.1.3) from *Sets, Relations, and Cardinality*, Section 3.1. Review exercise R1 uses Chapter 3's ideals and quotient ring (Definition 3.4.1, Theorem 3.4.1), from Section 3.4: the null sequences form an ideal of the ring of Cauchy sequences in $\mathbb{Q}$, and the quotient by it, which Chapter 3 built with Chapter 2's lemma on operations on classes (Theorem 2.2.1), gives the classes and their sum. Nothing else from Chapter 3 is used. Section 4.6 names suprema (Sets, Relations, and Cardinality, Definition 2.6.5) without using them. One example, the non-Archimedean field $\mathbb{Q}(t)$ of Section 4.2, uses school algebra of polynomials, which *Algebraic Preliminaries*, Chapter 4 builds; it appears only in examples, remarks, warnings and exercises, and no proof of a numbered result cites it. Nothing from *Analysis on the Real Line* is used, and no fact about $\mathbb{R}$.

Time. About 25 hours on the core track (all of A, one B and one C per section, the exit checks); about 40 hours for everything.

4.-1.1 Roadmap

- **Working in an Ordered Field.** The standing setup of an arbitrary ordered field F , the fraction rules of a field and the absence of zero divisors, halving a tolerance and the midpoint, comparing reciprocals and multiplying non-negative inequalities, and the absolute value with the triangle and reverse triangle inequalities, ending in the form $|x - a| < \varepsilon \iff a - \varepsilon < x < a + \varepsilon$ that the rest of the chapter uses.
- **The Rationals inside an Ordered Field; the Archimedean Property.** The elements $n \cdot 1_F$ and powers in F , which add, multiply and compare as in $\mathbb{N}$; the unique embedding ι_F of $\mathbb{Q}$ into every ordered field, strictly increasing and order-reflecting; the Archimedean property as a property of F , with its reciprocal form $1/(n \cdot 1_F) < \varepsilon$; the integer part and the density of the rationals in an Archimedean field; and $\mathbb{Q}(t)$, an ordered field that is not Archimedean.
- **Sequences and Limits in an Ordered Field.** Sequences indexed from 0 and “eventually”, the limit with tolerances from F and its negation, first limits in $\mathbb{Q}$ and the divergence of $((-1)^n)$ in every ordered field, null and bounded sequences, $1/(n+1) \rightarrow 0$ as a restatement of the Archimedean property, changing finitely many terms, uniqueness of limits, and the boundedness of convergent sequences.
- **The Algebra and Order of Limits.** Null times bounded is null, terms near a non-zero limit stay away from 0, the limit laws for sums, multiples, products and quotients, and weak inequalities passing to the limit, with the squeeze; no result of the section uses the Archimedean property.
- **Cauchy Sequences.** Terms close to one another instead of to a point, convergent sequences Cauchy and Cauchy sequences bounded, sums and products of Cauchy sequences, the sign theorem (a Cauchy sequence that is not null eventually stays at least some $c > 0$ above 0, or at least c below it), sign stability under null differences, reciprocals, and a sequence of rationals whose steps shrink to 0 and which is not Cauchy.
- **A Cauchy Sequence with No Rational Limit.** The Newton sequence for $\sqrt{2}$ is decreasing with overshoot $e_n \leq 1/(4n)$, Cauchy in $\mathbb{Q}$, and convergent to no rational, since a limit L would satisfy $L^2 = 2$; the section then lists what the chapter did not prove (suprema, monotone convergence, Bolzano–Weierstrass), each of which fails in $\mathbb{Q}$, and the review exercises return to Chapter 3's quotient by an ideal, to induction, and to the

negation laws of *Logic and Proof*, Chapter 6.

4.-1.2 Named moves

- **Split ε .** To make a sum of k small terms smaller than ε , give each term the tolerance ε/k , and combine the finitely many thresholds by a maximum. It proves the sum law in Section 4.4 and the sum of two Cauchy sequences in Section 4.5; Exercise B3 of Section 4.4 and Exercise C1 of Section 4.5 leave part of it to you.
- **Bound, then shrink.** In a product, replace the factor you cannot control by a bound fixed in advance, then make the other factor small enough to absorb the bound. The bound comes from the whole sequence, before ε is named. It proves the product law in Section 4.4 and the product of two Cauchy sequences in Section 4.5.
- **Fix the bad ε .** “Not null” says: some ε is missed beyond every threshold. It hands you one ε , to keep for the rest of the proof; ask the Cauchy condition for a threshold with $\varepsilon/2$ first, and only then take a term beyond it that misses ε . This is the sign theorem of Section 4.5.
- **Locate the missing limit.** Suppose a limit L exists, and use the limit laws and uniqueness to find an equation that L must satisfy; if the field has no solution, there is no limit. In Section 4.6 the equation is $L^2 = 2$.