

8.8 Stages of Knowledge: Kripke Models

This section is optional, and nothing outside it depends on it. Section 8.7 named several classical laws that have no derivation without RAA. For two of them, excluded middle and double-negation elimination, Chapter 4's optional Theorem 4.7.4 gave a proof by a three-valued reading; for the rest, Section 8.7 states the fact and points here. Truth tables cannot show it: every row makes $p \vee \neg p$ true. Here is a reading that covers them all. A statement is not true or false once and for all, but *known* or *not yet known* at a stage of an inquiry that grows. The rules without RAA are sound for this reading, and RAA is not; two or three stages then suffice to show which laws need it.

8.8.1 Stages of knowledge

An inquiry passes through stages. At each stage some atomic statements are known: established, never to be withdrawn. From a stage, the inquiry may go on in several ways, to later stages where more is known. Nobody knows which way it will go. The atomic statements are the letters of Chapter 4's formulas (Definition 4.1.1), its atoms, and the formulas of this section are those formulas: letters, $\perp$, and the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

Definition 8.8.1. Kripke Model

A **Kripke model** consists of:

- a finite non-empty set of **stages**;
- for each stage w , a set of stages **reachable** from w (written $w \leq w'$), such that $w \leq w$ for every stage, and $w \leq w'$ and $w' \leq w''$ give $w \leq w''$;
- for each stage w , a set of letters **known at** w , such that a letter known at w is known at every stage reachable from w .

We draw a model as dots for the stages, with an arrow from a stage to each stage that comes directly after it. A stage is reachable from w when it can be reached by following arrows, none or several; so w reaches itself, and arrows are never drawn for that. Beside each dot stands the set of letters known there.



The model M_1 : at w_0 nothing is known yet; at the later stage w_1 , p is known.

8.8.2 Forcing

What a stage knows about a compound formula is computed from its parts, as a row computes a truth value, except that some connectives look ahead to later stages.

Definition 8.8.2. Forcing

Let w be a stage of a Kripke model. w **forces** a formula φ , written $w \Vdash \varphi$, as follows; $w \nVdash \varphi$

says that it does not. The definition is a recursion on φ (Theorem 4.3.4) that defines, for each formula, the set of stages that force it; clauses (K5) and (K6) need the sets of its parts, at other stages.

- (K1) $w \Vdash p$, for a letter p , if p is known at w ;
- (K2) $w \Vdash \perp$ never;
- (K3) $w \Vdash \varphi \wedge \psi$ if $w \Vdash \varphi$ and $w \Vdash \psi$;
- (K4) $w \Vdash \varphi \vee \psi$ if $w \Vdash \varphi$ or $w \Vdash \psi$;
- (K5) $w \Vdash \varphi \rightarrow \psi$ if every stage w' reachable from w with $w' \Vdash \varphi$ has $w' \Vdash \psi$;
- (K6) $w \Vdash \neg\varphi$ if no stage reachable from w forces φ ;
- (K7) $w \Vdash \varphi \leftrightarrow \psi$ if $w \Vdash \varphi \rightarrow \psi$ and $w \Vdash \psi \rightarrow \varphi$.

A stage **refutes** a sequent $\Gamma \vdash \psi$ if it forces every formula of Γ and does not force ψ .

In words: a conjunction or a disjunction is decided at the stage itself, and a disjunction is forced only when one of its sides is: the inquiry must already know which. An implication and a negation look ahead. To force $\varphi \rightarrow \psi$ is to know that, however the inquiry goes on, ψ will be known whenever φ is. To force $\neg\varphi$ is to know that φ will never be known. Not forcing φ is weaker than forcing $\neg\varphi$.

A model with **one stage** is a row of a truth table. With one stage, the stages reachable from w are w alone, so (K5) says “ φ is not forced, or ψ is”, and (K6) says “ φ is not forced”: forcing becomes Chapter 4’s truth (Definition 4.4.2). Kripke models are rows that can grow.

A stage that refutes $\Gamma \vdash \psi$ forces every member of Γ and not ψ , which is weaker than forcing $\neg\psi$. It is Section 1.1’s counterexample set, read at a stage: where truth tables make $\neg\psi$ true, a stage only fails to know ψ .

Quick check. In a one-stage model in which p is not known, does the stage force $\neg p$? Does it force $p \vee \neg p$?

Solution. Yes to both. The only stage reachable is the stage itself, which does not force p , so (K6) gives $\neg p$, and (K4) gives $p \vee \neg p$. One stage is one row.

8.8.3 A first countermodel

Example. Two stages refute excluded middle

In M_1 , decide whether w_0 forces p , $\neg p$, $p \vee \neg p$, $\neg\neg p$ and $\neg\neg p \rightarrow p$.

Solution.

- p is not known at w_0 , so $w_0 \not\Vdash p$ (K1).
- w_1 is reachable from w_0 and forces p , so $w_0 \not\Vdash \neg p$ (K6).
- Hence $w_0 \not\Vdash p \vee \neg p$ (K4): w_0 refutes $\vdash p \vee \neg p$.
- Next, $\neg\neg p$. No stage reachable from w_0 forces $\neg p$: not w_0 , as just shown, and not w_1 , which forces p . So $w_0 \Vdash \neg\neg p$ (K6).

- So w_0 forces $\neg\neg p$ and not p : it refutes $\neg\neg p \vdash p$, and by (K5), applied with $w' = w_0$, $w_0 \not\models \neg\neg p \rightarrow p$.

At w_0 the inquiry knows that p cannot be refuted, since p becomes known later on every way it can go, but it does not yet know p . Excluded middle asks it to know now which of p and $\neg p$ holds.

8.8.4 Knowledge persists

Letters stay known by the definition of a Kripke model. The same holds for every formula.

Lemma 8.8.1. Heredity

In every Kripke model, if $w \Vdash \varphi$ and $w \leq w'$, then $w' \Vdash \varphi$.

In words: what a stage forces, every later stage forces. The proof is a structural induction on φ (Theorem 4.3.1); it is in the fold.

🔄 **Second reading** · *The proof of heredity*

Proof. We prove, by Theorem 4.3.1, that every formula φ has the property: for all stages $w \leq w'$ of every Kripke model, $w \Vdash \varphi$ gives $w' \Vdash \varphi$.

Letters. A letter forced at w is known at w (K1), hence known at w' by the condition on known letters in Definition 8.8.1, hence forced at w' .

$\perp$. No stage forces it (K2), so there is nothing to show.

$\wedge$ and $\vee$. If $w \Vdash \varphi \wedge \psi$, then w forces both parts (K3), so w' forces both by the induction hypothesis, and $w' \Vdash \varphi \wedge \psi$. For $\vee$, w forces one part (K4), and w' forces the same part by the induction hypothesis.

$\rightarrow$. Let $w \Vdash \varphi \rightarrow \psi$, and let w'' be reachable from w' with $w'' \Vdash \varphi$. Since $w \leq w'$ and $w' \leq w''$, transitivity gives $w \leq w''$, so $w'' \Vdash \psi$ by (K5) at w . Hence $w' \Vdash \varphi \rightarrow \psi$ by (K5). No induction hypothesis is used.

$\neg$. The same: every stage reachable from w' is reachable from w , and none of those forces φ (K6), so $w' \Vdash \neg\varphi$.

$\leftrightarrow$. By (K7), w forces the two implications, so w' forces both by the case $\rightarrow$, and $w' \Vdash \varphi \leftrightarrow \psi$.

Every clause of Definition 4.1.1 is covered, so every formula has the property. This proves the lemma. ■

Quick check. A stage w forces $\neg\varphi$, and w' is reachable from w . Why does w' force $\neg\varphi$, whatever φ is?

Solution. $w \Vdash \neg\varphi$ says that no stage reachable from w forces φ . The stages reachable from w' are among those, by transitivity, so none of them forces φ either. Nothing is assumed about φ itself.

8.8.5 The rules without RAA are sound for stages

Theorem 8.8.2. Soundness without RAA

Let Γ be a list of formulas and ψ a formula. If $\Gamma \vdash \psi$ by a derivation every line of which is a formula of Definition 4.1.1, in which no line uses RAA or TAUT, nor a derived rule or cited result whose derivation uses one of them or a line outside Definition 4.1.1, then every stage of every Kripke model that forces every formula of Γ forces ψ .

In words: a derivation without RAA never leads from what a stage knows to what it does not know.

TAUT (Section 4.8) is not among the rules here: it is licensed by completeness, whose proof uses LEM (Section 8.7), so a derivation with a TAUT line is not a derivation without RAA.

The proof is Chapter 4’s proof of Theorem 4.7.2, Steps 1–4, with “every valuation making the context true” replaced by “every stage forcing the context”. Each rule’s case, in one line:

Rule	Why its case holds at a stage w that forces the context
R	the cited line itself, forced at w
$\wedge$ I, $\wedge$ E, $\vee$ I, $\leftrightarrow$ E	the clause (K3), (K4) or (K7) at w itself, with (K5) at $w' = w$ for $\leftrightarrow$ E
$\rightarrow$ E, $\neg$ E	(K5) or (K6), applied at $w' = w$, which is reachable from w ; for $\neg$ E no stage forces both cited lines, so there is nothing to show
$\perp$ E	no stage forces $\perp$, so there is nothing to show
$\vee$ E	w forces one side, by (K4); the box of that side ends in its last line, forced at w
$\rightarrow$ I, $\neg$ I, $\leftrightarrow$ I	a later stage w' forces the context too, by heredity , so the box applies at w'
RAA	fails: its box on $\neg\varphi$ shows only that no stage reachable from w forces $\neg\varphi$, that is, $w \Vdash \neg\neg\varphi$; in M_1 , w_0 forces $\neg\neg p$ and not p

Proof. Step 1 (reduction to primitive rules). As in Step 1 of the proof of Theorem 4.7.2, a derived rule or cited result derived without RAA and TAUT abbreviates a pasted derivation in the propositional primitive rules (Chapters 1 and 2) other than RAA. So fix a derivation of ψ whose premises all belong to Γ and which uses only those rules, with lines $1, \dots, n$; let φ_k be the formula on line k and Γ_k its context.

Step 2 (the induction). For $k \in \mathbb{N}$, let $S(k)$ be the statement: *if $1 \leq k \leq n$, then every stage of every Kripke model that forces every formula of Γ_k forces φ_k .* We prove $S(k)$ for every k by Theorem 4.2.2. Let $1 \leq k \leq n$ (otherwise there is nothing to prove), assume $S(j)$ for every $j < k$, and let w force every formula of Γ_k . Every line and box that line k cites lies before it (Definition 1.4.2), so the induction hypothesis holds for each cited line and for the last line of each cited box.

Step 3 (the cases). *Premises and assumptions:* φ_k belongs to Γ_k , so w forces it. *One-step rules:* a cited line i is available at line k , so $\Gamma_i \subseteq \Gamma_k$ (Lemma 4.7.1 (a)), and the hypothesis gives $w \Vdash \varphi_i$. Then R is the hypothesis for the cited line, and $\wedge I$, $\wedge E$ and $\vee I$ follow from (K3) and (K4) at w . For $\rightarrow E$, $w \Vdash \varphi \rightarrow \psi$ and $w \Vdash \varphi$, and (K5) with $w' = w$ gives $w \Vdash \psi$; $\leftrightarrow E$ is the same after (K7). For $\neg E$, $w \Vdash \neg \varphi$ would say by (K6) that w , reachable from itself, does not force φ , against $w \Vdash \varphi$; so no stage forces Γ_k , and there is nothing to show. For $\perp E$, $w \Vdash \perp$ is impossible by (K2), and again there is nothing to show.

Box rules: if line k cites the box from line a to line b , then $\Gamma_b = \Gamma_k, \varphi_a$ (Lemma 4.7.1 (b)), so the hypothesis for line b says that every stage forcing Γ_k and φ_a forces φ_b . For $\rightarrow I$, line k is $\varphi \rightarrow \psi$. Let w' be reachable from w with $w' \Vdash \varphi$. By Lemma 8.8.1, w' forces every formula of Γ_k , so it forces Γ_k and φ , and hence ψ . So $w \Vdash \varphi \rightarrow \psi$ by (K5). $\neg I$ is the same with $\psi = \perp$: a stage w' reachable from w that forced φ would force $\perp$, against (K2), so none does, and $w \Vdash \neg \varphi$ by (K6). $\leftrightarrow I$ is two cases of $\rightarrow I$, joined by (K7). For $\vee E$, $w \Vdash \varphi \vee \psi$ by the hypothesis for the cited line, so w forces φ or ψ (K4); if φ , then w forces Γ_k and φ , and the first box gives $w \Vdash \chi$; if ψ , the second box does. For RAA the case fails, as the table says, and no derivation here uses it.

So $S(k)$ holds, and by Theorem 4.2.2 every line k satisfies it.

Step 4 (the last line). The last line lies in no box (Definition 1.4.2), so its context Γ_n consists of premises only, each of which belongs to Γ (Definition 1.2.3). A stage forcing every formula of Γ forces every formula of Γ_n , so by Steps 2 and 3 it forces $\varphi_n = \psi$. This proves the theorem. ■

Corollary 8.8.3. Kripke Countermodel

If some stage of some Kripke model refutes $\Gamma \vdash \psi$, then $\Gamma \vdash \psi$ has no derivation without RAA in the propositional rules whose lines are all formulas of Definition 4.1.1.

Proof. Suppose it had one. By Theorem 8.8.2 the stage would force ψ , and it does not; so it has none. ■

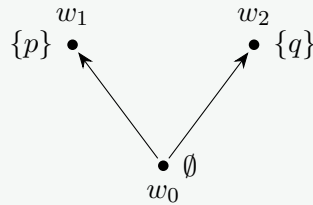
Quick check. Which case of the proof fails for RAA, and which stage of M_1 shows that it must?

Solution. RAA's own case: its box assumes $\neg\varphi$ and the rule writes φ . w_0 forces $\neg\neg p$ and not p . If RAA were sound for stages, the derivation of $\neg\neg p \vdash p$ (Proposition 2.5.1 (b)) would make w_0 force p .

8.8.6 Two branches

Example. A fork

In the model M_2 the inquiry can go two ways: one learns p , the other q .



Show that w_0 refutes $\vdash (p \rightarrow q) \vee (q \rightarrow p)$ and $\neg(p \wedge q) \vdash \neg p \vee \neg q$.

Solution.

- w_1 is reachable from w_0 , forces p and does not force q , so $w_0 \not\models p \rightarrow q$ (K5). Likewise w_2 gives $w_0 \not\models q \rightarrow p$. By (K4), w_0 does not force the disjunction.
- No stage forces both p and q , so $w_0 \models \neg(p \wedge q)$ (K6).
- w_1 forces p , so $w_0 \not\models \neg p$; w_2 forces q , so $w_0 \not\models \neg q$. Hence $w_0 \not\models \neg p \vee \neg q$.
- By [Corollary 8.8.3](#), neither $\vdash (p \rightarrow q) \vee (q \rightarrow p)$ nor $\neg(p \wedge q) \vdash \neg p \vee \neg q$ has a derivation without RAA.

Classically each is derivable (Section 2.5, and Corollary 4.8.5 for the first, a tautology). The branch is what defeats them. At w_0 the inquiry knows that p and q will never both be known, but not which one will fail.

8.8.7 What two or three stages settle

Here is the propositional right column of the table in Section 8.7, “Which laws survive without RAA”, each sequent with a stage that refutes it. Two more models are described in words: M_3 has stages $w_0 \rightarrow w_1$, nothing known at w_0 , and p and q known at w_1 ; M_4 has stages $w_0 \rightarrow w_1$, p known at both, and q known at w_1 only.

Sequent (letters p, q)	A refuting stage
$\vdash p \vee \neg p$; $\neg\neg p \vdash p$; $\vdash \neg\neg p \rightarrow p$	w_0 of M_1 (Example (Two stages refute excluded middle))

Sequent (letters p, q)	A refuting stage
$\vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$	w_0 of M_1 : no stage forces $p \rightarrow q$, so both stages force $(p \rightarrow q) \rightarrow p$; $w_0 \not\vdash p$
$\neg(p \rightarrow q) \vdash p \wedge \neg q$	w_0 of M_1 : no stage forces $p \rightarrow q$, so $w_0 \Vdash \neg(p \rightarrow q)$; $w_0 \not\vdash p$
$(p \rightarrow q) \rightarrow q \vdash p \vee q$ (Section 8.7, R3)	w_0 of M_1 : as above, both stages force $(p \rightarrow q) \rightarrow q$; w_0 forces neither p nor q
$p \rightarrow q \vdash \neg p \vee q$	w_0 of M_3 : $w_0 \Vdash p \rightarrow q$; $w_0 \not\vdash \neg p$ since $w_1 \Vdash p$, and $w_0 \not\vdash q$
$\neg q \rightarrow \neg p \vdash p \rightarrow q$	w_0 of M_4 : no stage forces $\neg q$, so $w_0 \Vdash \neg q \rightarrow \neg p$; w_0 forces p and not q
$\vdash (p \rightarrow q) \vee (q \rightarrow p); \neg(p \wedge q) \vdash \neg p \vee \neg q$	w_0 of M_2 (Example (A fork))

With [Corollary 8.8.3](#), each row proves that the sequent has no derivation without RAA in the propositional rules; that lines outside Definition 4.1.1 and the rules for $\forall, \exists$ and $=$ do not change this is stated without proof (Section 8.7). This covers every propositional entry of Section 8.7's right column, and the claims stated without proof in its Exercises B1 and R3.

8.8.8 Searching for a countermodel

How were these models found? Here is the search for $(p \rightarrow q) \rightarrow q \vdash p \vee q$, run backwards from what must fail, in the manner of Section 4.6's search for a refuting row.

- Start with a stage w_0 that must not force $p \vee q$. So neither p nor q is known there.
- It must force $(p \rightarrow q) \rightarrow q$: every later stage that forces $p \rightarrow q$ must force q . The cheapest way is to make sure no stage forces $p \rightarrow q$.
- w_0 itself must then fail $p \rightarrow q$, so some stage reachable from it forces p and not q . Add w_1 after w_0 , with p known.
- Check every clause at both stages: this is M_1 .

In general:

- to refute a disjunction at w , refute both sides at w itself;
- to refute $\varphi \rightarrow \psi$ or $\neg\varphi$ at w , add a later stage that forces φ (and, for $\rightarrow$, not ψ);
- keep letters known once known.

The search is a guide, not a decision procedure as stated here.

Remark. Three facts, stated without proof.

- The converse of [Theorem 8.8.2](#) holds: a sequent forced at every stage of every Kripke model that forces its premises has a derivation without RAA. This is completeness for this semantics.
- From it, by placing a new first stage below two countermodels, follows the **disjunction property**: if $\vdash \varphi \vee \psi$ without RAA, then $\vdash \varphi$ or $\vdash \psi$ without RAA, the BHK clause for “or” made a theorem about derivations.
- Kripke models extend to the quantifiers, with a domain at each stage that may grow. In them $\neg\forall x \varphi \vdash \exists x \neg\varphi$ and the two passage directions of Section 8.7's table can be refuted.

Nothing in this book depends on these facts.

Warning (Not allowed: misreadings of forcing). Each sentence below is wrong; the tag names the clause it misreads.

- “ $w \not\models p$, so $w \models \neg p$ ”: (K6) looks at every stage reachable from w ; in M_1 , w_0 forces neither p nor $\neg p$.
- “ $w \models \varphi \rightarrow \psi$, because $w \not\models \varphi$ ”: (K5) looks at every later stage, not only at w .
- “ $w \models \varphi \vee \psi$, because every stage reachable from w , other than w itself, forces φ or ψ ”: (K4) is decided at w itself; in M_2 , every stage reachable from w_0 other than w_0 forces p or q , and w_0 forces neither.
- “A picture with p known at w_0 and not at a later w_1 ”: it is not a Kripke model, since a known letter stays known ([Definition 8.8.1](#)).

8.8.9 Exercises

8.8.9.1 A. Check your understanding

Exercise. A1

- In M_1 , does w_1 force $\neg\neg p \rightarrow p$?
- Why is a one-stage Kripke model a row of a truth table?
- True or false: if w does not force φ , then w forces $\neg\varphi$.
- Which rule’s case fails in the proof of [Theorem 8.8.2](#), and how does M_1 show that it must fail?

Solution.

- Yes. The only stage reachable from w_1 is w_1 , which forces p , so it forces the implication by (K5).
- With one stage, (K5) and (K6) look only at that stage, so the clauses become those of Definition 4.4.2, with “forced” for “true”.
- False: w_0 of M_1 , with $\varphi = p$, forces neither p nor $\neg p$.
- RAA’s. w_0 forces $\neg\neg p$ and not p , but $\neg\neg p \vdash p$ is derived with RAA; if RAA’s case held, w_0 would force p .

Exercise. A2: Drill

Decide each in a line, with the clause you use. M_1 and M_2 are the models of the text. M_5 is the chain $w_0 \rightarrow w_1 \rightarrow w_2$, with nothing known at w_0 , p known at w_1 , and p, q known at w_2 .

- M_1 : does w_1 force $\neg p$?
- M_1 , with q known nowhere: does w_0 force $\neg q$?
- M_2 : does w_0 force $p \vee q$?
- M_2 : does w_1 force $\neg q$?

- (e) M_5 : does w_0 force $p \rightarrow q$?
- (f) M_5 : does w_1 force $\neg\neg q$?
- (g) Is this a Kripke model: $w_0 \rightarrow w_1$, with p known at w_0 and not at w_1 ?
- (h) In M_1 , a student says “ w_0 forces $\neg p$, because p is not known at w_0 ”. What is wrong?
- (i) M_2 : does w_0 force $\neg p \rightarrow q$?
- (j) M_5 : does w_0 force $q \vee \neg q$?
- (k) M_5 : does w_0 force $q \rightarrow p$?
- (l) Can a one-stage model refute $\neg\neg p \vdash p$?

Solution.

- (a) No: w_1 is reachable from itself and forces p (K6).
- (b) Yes: no stage reachable from w_0 forces q (K6).
- (c) No: neither letter is known at w_0 (K4).
- (d) Yes: the only stage reachable from w_1 is w_1 , where q is not known (K6).
- (e) No: w_1 is reachable, forces p , and does not force q (K5).
- (f) Yes. The stages reachable from w_1 are w_1 and w_2 ; neither forces $\neg q$, since w_2 forces q and is reachable from both (K6). And $w_1 \not\models q$, so $\neg\neg q$ is forced where q is not.
- (g) No: a letter known at w_0 must be known at every stage reachable from it ([Definition 8.8.1](#)).
- (h) (K6) needs p to fail at every reachable stage, and w_1 forces p .
- (i) Yes (K5). Of the stages reachable from w_0 , only w_2 forces $\neg p$ (w_0 and w_1 reach w_1 , which knows p), and w_2 forces q .
- (j) No: q is not known at w_0 , and w_2 , reachable from w_0 , forces q , so $w_0 \not\models \neg q$ (K4, K6).
- (k) Yes: the only stage that forces q is w_2 , which forces p (K5).
- (l) No. At a single stage forcing is truth, and $\neg\neg p$ is true exactly when p is. A refutation needs a later stage.

8.8.9.2 B. Practice

Exercise. B1: A chain of three stages

Let M_6 be the chain $w_0 \rightarrow w_1 \rightarrow w_2$, with nothing known at w_0 , q known at w_1 , and p and q known at w_2 .

- (a) Decide whether w_0 forces $\neg\neg p$, $p \vee \neg p$, $q \rightarrow p$, $(q \rightarrow p) \vee (p \rightarrow q)$ and $\neg p \rightarrow \neg q$.
- (b) Which sequents among $\vdash p \vee \neg p$, $\neg\neg p \vdash p$ and $\vdash (p \rightarrow q) \vee (q \rightarrow p)$ does w_0 refute?
- (c) A **chain** is a Kripke model whose stages can be listed $u_0, \dots, u_m$ so that $u_i \leq u_j$ exactly when $i \leq j$. Show that no stage of a chain refutes $\vdash (p \rightarrow q) \vee (q \rightarrow p)$.

Solution.

- (a)
 - $\neg\neg p$: yes. Every stage reachable from w_0 reaches w_2 , which forces p , so none forces $\neg p$ (K6).
 - $p \vee \neg p$: no. p is not known at w_0 , and $w_0 \not\models \neg p$ because of w_2 (K4).

- $q \rightarrow p$: no. w_1 forces q and not p (K5).
- $(q \rightarrow p) \vee (p \rightarrow q)$: yes. The only stage forcing p is w_2 , which forces q , so $w_0 \Vdash p \rightarrow q$ (K5), and (K4) gives the disjunction.
- $\neg p \rightarrow \neg q$: yes. No stage reachable from w_0 forces $\neg p$, so the implication holds with nothing to check (K5).

- (b) $\vdash p \vee \neg p$ and $\neg \neg p \vdash p$, since w_0 forces $\neg \neg p$ and not p . Not $\vdash (p \rightarrow q) \vee (q \rightarrow p)$, by (a).
- (c) Suppose some stage w of a chain refuted it, that is, forced neither $p \rightarrow q$ nor $q \rightarrow p$. By (K5), some u reachable from w forces p and not q , and some v reachable from w forces q and not p . In a chain, $u \leq v$ or $v \leq u$. If $u \leq v$, then v forces p , since letters stay known, a contradiction. If $v \leq u$, then u forces q , a contradiction. So no stage of a chain refutes it, and the branch of M_2 is needed.

Marking scheme. (10 points) - 5: (a), one per formula, each with its clause. - 2: (b). - 3: (c), the two orders of u, v , each with persistence of letters.

Fatal errors (0 for the step): deciding (K5) or (K6) at w_0 alone; in (c), checking one chain only.

Exercise. B2: Derive without RAA, or break with a Kripke model

For each claim, either derive it without RAA and without TAUT (and without results derived with RAA), or give a stage of a Kripke model that refutes it.

- (a) $\neg \neg(p \wedge q) \vdash \neg \neg p \wedge \neg \neg q$.
- (b) $\neg \neg p \wedge \neg \neg q \vdash \neg \neg(p \wedge q)$.
- (c) $\neg \neg(p \vee q) \vdash \neg \neg p \vee \neg \neg q$.
- (d) $(p \rightarrow q) \rightarrow p \vdash \neg \neg p$.
- (e) $\neg p \rightarrow p \vdash p$.
- (f) $(p \vee q) \wedge \neg p \vdash q$.
- (g) $(p \rightarrow q) \wedge (\neg p \rightarrow q) \vdash q$.

Solution. The verdict comes first in each part.

- (a) **Derive.** Each conjunct is a negation, so each takes a $\neg$ I box, whose clash builds $\neg(p \wedge q)$ against the premise:

1	$\neg\neg(p \wedge q)$	
2	$\neg p$	asm, for $\neg$ I
3	$p \wedge q$	asm, for $\neg$ I
4	p	$\wedge$ E 3
5	$\perp$	$\neg$ E 4, 2
6	$\neg(p \wedge q)$	$\neg$ I 3–5
7	$\perp$	$\neg$ E 6, 1
8	$\neg\neg p$	$\neg$ I 2–7
9	$\neg q$	asm, for $\neg$ I
10	$p \wedge q$	asm, for $\neg$ I
11	q	$\wedge$ E 10
12	$\perp$	$\neg$ E 11, 9
13	$\neg(p \wedge q)$	$\neg$ I 10–12
14	$\perp$	$\neg$ E 13, 1
15	$\neg\neg q$	$\neg$ I 9–14
16	$\neg\neg p \wedge \neg\neg q$	$\wedge$ I 8, 15

(b) **Derive.** A $\neg$ I box on $\neg(p \wedge q)$; inside it, $\neg q$ and then $\neg p$ are built, each clashing with a conjunct of the premise:

1	$\neg\neg p \wedge \neg\neg q$	
2	$\neg\neg p$	$\wedge$ E 1
3	$\neg\neg q$	$\wedge$ E 1
4	$\neg(p \wedge q)$	asm, for $\neg$ I
5	p	asm, for $\neg$ I
6	q	asm, for $\neg$ I
7	$p \wedge q$	$\wedge$ I 5, 6
8	$\perp$	$\neg$ E 7, 4
9	$\neg q$	$\neg$ I 6–8
10	$\perp$	$\neg$ E 9, 3
11	$\neg p$	$\neg$ I 5–10
12	$\perp$	$\neg$ E 11, 2
13	$\neg\neg(p \wedge q)$	$\neg$ I 4–12

(c) **Break,** at w_0 of M_2 . Every stage reachable from w_0 is, or reaches, w_1 or w_2 , each of which forces $p \vee q$, so none forces $\neg(p \vee q)$: $w_0 \Vdash \neg\neg(p \vee q)$. But $w_2 \Vdash \neg p$ (only w_2 is reachable from it, and p is not known there), so $w_0 \nVdash \neg\neg p$. Likewise $w_1 \Vdash \neg q$, so

$w_0 \not\models \neg\neg q$, and w_0 does not force the disjunction (K4).

(d) **Derive.** A $\neg$ I box on $\neg p$, in which $p \rightarrow q$ is built by $\perp$ E and fed to the premise:

1		$(p \rightarrow q) \rightarrow p$	
2		$\neg p$	asm, for $\neg$ I
3		p	asm, for $\rightarrow$ I
4		$\perp$	$\neg$ E 3, 2
5		q	$\perp$ E 4
6		$p \rightarrow q$	$\rightarrow$ I 3–5
7		p	$\rightarrow$ E 1, 6
8		$\perp$	$\neg$ E 7, 2
9		$\neg\neg p$	$\neg$ I 2–8

(e) **Break**, at w_0 of M_1 . No stage forces $\neg p$, so $w_0 \Vdash \neg p \rightarrow p$ with nothing to check, and $w_0 \not\models p$.

(f) **Derive.** The disjunction is used by cases; the case p clashes with $\neg p$:

1		$(p \vee q) \wedge \neg p$	
2		$p \vee q$	$\wedge$ E 1
3		$\neg p$	$\wedge$ E 1
4		p	asm, for $\vee$ E
5		$\perp$	$\neg$ E 4, 3
6		q	$\perp$ E 5
7		q	asm, for $\vee$ E
8		q	R 7
9		q	$\vee$ E 2, 4–6, 7–8

(g) **Break**, at w_0 of M_3 (stages $w_0 \rightarrow w_1$, nothing known at w_0 , p and q known at w_1). The only stage forcing p is w_1 , which forces q , so $w_0 \Vdash p \rightarrow q$. No stage forces $\neg p$, since both reach w_1 , which forces p ; so $w_0 \Vdash \neg p \rightarrow q$ with nothing to check. Hence w_0 forces the conjunction (K3), and $w_0 \not\models q$.

The look-alikes: (a) and (b) are converses, and both derivable. (c) is (a)'s look-alike with $\vee$ for $\wedge$, and it breaks. (e) is (d)'s look-alike and breaks: from $\neg p \rightarrow p$ one derives only $\neg\neg p$ (a box on $\neg p$, then $\rightarrow$ E, $\neg$ E and $\neg$ I). (g) is (f)'s look-alike: a proof by cases on p and $\neg p$ with no disjunction to split, which needs $p \vee \neg p$. It is the pattern of Lemma 4.8.3 (Section 4.8) with no further premises, which is why completeness, and with it TAUT, rests on LEM.

Marking scheme. (10 points) - 6: the four derivations, 1.5 each. - 3: the three counter-models, 1 each, every clause checked. - 1: one correct remark on a look-alike pair.

A wrong verdict scores 0 for that item. Fatal errors (0 for that item): a derivation that uses RAA, $\neg\neg$ E or LEM, or cites a result derived with one; a TAUT line (every claim is a tautology, so a TAUT line would “derive” each, the breakable (c), (e) and (g) included; TAUT rests on completeness); a “model” with a known letter lost at a later stage.

8.8.9.3 C. Going deeper

Exercise. C1: Peirce's law and one instance of excluded middle

- (a) Derive $((p \rightarrow q) \rightarrow p) \rightarrow p$ from the premise $p \vee \neg p$ without RAA or TAUT.
- (b) Derive without RAA or TAUT, or refute with a Kripke model: $\vdash (((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow (p \vee \neg p)$.
- (c) Compare (a), (b) and Exercise (B5: Behind two negations, and one instance) (c) with Theorem 2.5.7, in one sentence.

Solution.

- (a) A $\rightarrow$ I box on $(p \rightarrow q) \rightarrow p$, and inside it the premise used by cases. In the case $\neg p$, the implication $p \rightarrow q$ is built by $\perp$ E and fed to the box's assumption:

1	$p \vee \neg p$	
2	$(p \rightarrow q) \rightarrow p$	asm, for $\rightarrow$ I
3	p	asm, for $\vee$ E
4	p	R 3
5	$\neg p$	asm, for $\vee$ E
6	p	asm, for $\rightarrow$ I
7	$\perp$	$\neg$ E 6, 5
8	q	$\perp$ E 7
9	$p \rightarrow q$	$\rightarrow$ I 6–8
10	p	$\rightarrow$ E 2, 9
11	p	$\vee$ E 1, 3–4, 5–10
12	$((p \rightarrow q) \rightarrow p) \rightarrow p$	$\rightarrow$ I 2–11

- (b) **Refute**, at w_0 of the fork M_7 : stages $w_0 \rightarrow w_1$ and $w_0 \rightarrow w_2$, with p known at w_1 only, and q known nowhere.

- At w_2 no stage reachable forces p , so $w_2 \Vdash p \rightarrow q$, and $w_2 \nVdash p$. So $w_2 \nVdash (p \rightarrow q) \rightarrow p$, and neither does w_0 , which reaches w_2 . w_1 forces p , hence $(p \rightarrow q) \rightarrow p$. The only stage forcing $(p \rightarrow q) \rightarrow p$ is w_1 , and it forces p : so $w_0 \Vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$ (K5).
- $w_0 \nVdash p$, and $w_0 \nVdash \neg p$ because of w_1 . So $w_0 \nVdash p \vee \neg p$. Since w_0 is reachable from itself, forces $((p \rightarrow q) \rightarrow p) \rightarrow p$ and does not force $p \vee \neg p$, (K5) gives $w_0 \nVdash (((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow (p \vee \neg p)$, so w_0 refutes the claim; by [Corollary 8.8.3](#) it has no derivation without RAA.
- M_1 would not do: there no stage forces $p \rightarrow q$, so w_0 forces $(p \rightarrow q) \rightarrow p$ and not p , and so does not force Peirce's law; the only stage that does, w_1 , forces $p \vee \neg p$. So the second branch is needed.

- (c) Theorem 2.5.7 compares rules, which act on every statement at once. Instance by instance the three differ: one instance of excluded middle gives Peirce's law (a) and double-negation elimination (Exercise (B5: Behind two negations, and one instance) (c)) for that letter, while that instance of Peirce's law does not give the instance of

excluded middle back (b).

Marking scheme. (10 points) - 5: (a). The $\vee E$ on the premise, with both cases inside the box of line 2 (2); the case $\neg p$ reaching p without RAA (2), for instance by building $p \rightarrow q$ with $\perp E$, or via $\neg\neg p$ as in B2 (d) (citing B2 (d) and Exercise (B5: Behind two negations, and one instance) (c) is allowed); the closing $\rightarrow I$ (1). - 4: (b). A model whose stage forces $((p \rightarrow q) \rightarrow p) \rightarrow p$ and not $p \vee \neg p$, every clause checked. - 1: (c).

Fatal errors (0 for the step): using M_1 in (b), where the formula is forced; using RAA or TAUT in (a).

Exercise. C2: Challenge: three negations are one

Prove that in every Kripke model, for every stage w and every formula φ : $w \Vdash \neg\neg\neg\varphi$ if and only if $w \Vdash \neg\varphi$. Compare with Example (Three negations become one).

Solution. ($\Leftarrow$) Let $w \Vdash \neg\varphi$, and let w' be reachable from w . By [Lemma 8.8.1](#), $w' \Vdash \neg\varphi$. Since w' is reachable from itself, $w' \not\Vdash \neg\neg\varphi$ (K6). So no stage reachable from w forces $\neg\neg\varphi$, and $w \Vdash \neg\neg\neg\varphi$.

($\Rightarrow$) Let $w \Vdash \neg\neg\neg\varphi$, and suppose some w' reachable from w forces φ . Every w'' reachable from w' forces φ by [Lemma 8.8.1](#), so no such w'' forces $\neg\varphi$ (K6, with w'' reachable from itself), and $w' \Vdash \neg\neg\varphi$. But w' is reachable from w , and $w \Vdash \neg\neg\neg\varphi$ says no such stage forces $\neg\neg\varphi$: a contradiction. So no stage reachable from w forces φ , and $w \Vdash \neg\varphi$.

The comparison: Example (Three negations become one) derives $\neg\neg\neg P \vdash \neg P$ without RAA, and $\neg P \vdash \neg\neg\neg P$ is Proposition 2.5.1 (a) for $\neg P$. By [Theorem 8.8.2](#) both directions hold at every stage. The exercise proves the same fact directly, from the clauses.

Marking scheme. (10 points) - 4: ($\Leftarrow$), with heredity and reflexivity named. - 5: ($\Rightarrow$), with heredity of φ used to get $w' \Vdash \neg\neg\varphi$. - 1: the comparison. - Second route: both directions from [Theorem 8.8.2](#), applied to the derivations without RAA of Example (Three negations become one) with φ for P and Proposition 2.5.1 (a) with $\neg\varphi$ for P , earn the 9 points for the two directions; the 1 point for the comparison then goes to naming where the direct proof uses heredity.

Fatal errors (0 for the step): “not forced, so the negation is forced”; heredity used without being cited.