

6.4 Uniformly Continuous Implies Continuous, Rule by Rule

Chapter 5 opened with a question. Continuity and uniform continuity differ only in where one quantifier stands; why does one imply the other in a few lines, while the converse is false? Section 5.1 gave the informal answer: a variable may depend on the variables quantified to its left, and only on them. Exercise (C1: Where the $\forall x$ stands) of Section 6.1 asked for both translations into formulas, and Section 6.2 negated uniform continuity (Example (Three negations from analysis) (c)). This section answers the question with the rules. It derives “uniformly continuous implies continuous” in full, line by line, with the bounded rules of Proposition 6.1.1. It then reads off the derivation the one thing this section adds: **which boxes are open where a constant is chosen**, and what that says about what the constant may depend on. The same reading locates the exact lines at which the converse breaks down, and the function x^2 shows that the converse is not merely unproved but false.

6.4.1 Two definitions that differ in one place

Let A be a set of real numbers and f a function defined on A . We work over the domain $\mathbb{R}$, with f a one-place function symbol, $\in A$ a one-place predicate symbol (Section 5.2), and $|\cdot|$ the absolute value. Here the two definitions are only **quantifier patterns**, and nothing is used about f except that $f(x)$ is a term. (A function symbol has a value at every real number; the values outside A are never used, because f is only applied to variables bounded by $\in A$, as with sequences in Section 5.1.)

f is **uniformly continuous on A** (UC) when

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in A \forall y \in A (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon).$$

f is **continuous at every point of A** (C) when

$$\forall x \in A \forall \varepsilon > 0 \exists \delta > 0 \forall y \in A (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon).$$

The two formulas have the same matrix and the same four bounded quantifiers. Only $\forall x \in A$ has moved: in C it stands **before** $\exists \delta > 0$, in UC **after** it. In the game of Section 5.1, the defender chooses δ in C after seeing x and ε , and in UC after seeing only ε . So in UC a single δ must serve every point of A at once: that is what **uniformly** means.

Quick check. Swap the first two quantifiers of UC, to get $\exists \delta > 0 \forall \varepsilon > 0 \forall x \in A \forall y \in A (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)$. What does the new formula say about two points of A closer than δ ? Does $f(x) = x$ on $\mathbb{R}$ satisfy it?

Solution. Now δ is chosen before ε , so it must work for every $\varepsilon > 0$ at once. For two points $x, y \in A$ with $|x - y| < \delta$, this says $|f(x) - f(y)| < \varepsilon$ for every $\varepsilon > 0$, and by Proposition 6.2.5, $f(x) = f(y)$. So the formula says: for some $\delta > 0$, any two points of A closer than δ have the **same** value. The function $f(x) = x$ on $\mathbb{R}$ fails it: whatever $\delta > 0$ is proposed, the points 0 and $\delta/2$ are closer than δ and have different values. Yet $f(x) = x$ is uniformly continuous, with $\delta = \varepsilon$. Swapping $\forall \varepsilon$ and $\exists \delta$ asks far too much.

6.4.2 The skeleton first

Hold ε fixed, and write $K(x, \delta)$ for “ δ works at x ”, the formula $\forall y \in A (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)$. Drop the bounds on δ and x for a moment. Then UC at this ε says $\exists \delta \forall x K(x, \delta)$, one δ for every point, and C at this ε says $\forall x \exists \delta K(x, \delta)$, a δ for each point. The step from the first to the second is Proposition 5.6.2 with other letters:

1		$\exists \delta \forall x K(x, \delta)$	
2	x_0 δ_0	$\forall x K(x, \delta_0)$	asm, for $\exists E$
3		$K(x_0, \delta_0)$	$\forall E$ 2
4		$\exists \delta K(x_0, \delta)$	$\exists I$ 3
5		$\exists \delta K(x_0, \delta)$	$\exists E$ 1, 2–4
6		$\forall x \exists \delta K(x, \delta)$	$\forall I$ 2–5

Fix a point, obtain the one δ that works everywhere, choose it for this point. The converse, from $\forall x \exists \delta K$ to $\exists \delta \forall x K$, is the swap whose search stalled in Section 5.6. The rest of this section puts ε and the bounds back and writes K out. That costs five lines: the premise UC itself (line 1), the assumptions $x_0 \in A$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ (lines 2, 3 and 5), and the line that closes the box of ε_0 (line 10). The skeleton’s line 1 becomes line 4 of the derivation below (UC at ε_0), its lines 2–5 become lines 6–9, and its line 6 becomes line 11, which closes the box of x_0 .

Quick check. Why may line 5 of the skeleton close the box of δ_0 ?

Solution. δ_0 does not appear in line 5, $\exists \delta K(x_0, \delta)$, nor in the existential of line 1, which is also the only premise, and the one box still open, that of x_0 , has no assumption. Every clause of $\exists E$ holds.

6.4.3 The derivation

Example. Uniformly continuous implies continuous

Derive C from UC, with the bounded rules of Proposition 6.1.1.

Idea. Run the search method of Section 6.3. The goal C begins $\forall x \in A$, then $\forall \varepsilon > 0$: the **goal-directed** steps open two bounded $\forall I$ boxes, “let $x_0 \in A$ ” and “let $\varepsilon_0 > 0$ ”, and write the last two lines. The new goal begins $\exists \delta > 0$, and no witness is in sight, so it is **postponed** (Section 5.5). Turn to the premise UC. It begins $\forall \varepsilon > 0$, and the term the goal mentions is ε_0 , whose bound $\varepsilon_0 > 0$ is on a line: $\forall E$ at ε_0 (Section 5.3). That gives an $\exists \delta > 0$ line, and its $\exists E$ box is opened **at once**, with a witness δ_0 (Section 5.6), aiming for the postponed goal as its last line. Inside, the premise has become “ δ_0 works for all $x, y \in A$ ”; instantiate it at the point we have, x_0 . Now the postponed goal has its witness: δ_0 itself.

Before reading on: which lines does the line introducing δ_0 rest on, and does any of them mention x_0 ?

Solution.

1	$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	
2	$x_0 \mid x_0 \in A$	asm, for $\forall I$
3	$\varepsilon_0 \mid \varepsilon_0 > 0$	asm, for $\forall I$
4	$\exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon_0)$	$\forall E$ 1, 3
5	$\delta_0 \mid \delta_0 > 0$	asm, for $\exists E$
6	$\forall x \in A \forall y \in A (x - y < \delta_0 \rightarrow f(x) - f(y) < \varepsilon_0)$	asm, for $\exists E$
7	$\forall y \in A (x_0 - y < \delta_0 \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\forall E$ 6, 2
8	$\exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\exists I$ 5, 7
9	$\exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\exists E$ 4, 5–8
10	$\forall \varepsilon > 0 \exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon)$	$\forall I$ 3–9
11	$\forall x \in A \forall \varepsilon > 0 \exists \delta > 0 \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	$\forall I$ 2–10

Every line is tagged with its rule; lines 4, 7 and 8 are the one-line bounded rules, and lines 9, 10 and 11 close bounded boxes. The terms and the side conditions:

- Line 4: $\forall E$ at ε_0 , with its bound on line 3. The term ε_0 is a variable that does not appear in line 1, so it is free for ε .
- Line 7: $\forall E$ at x_0 , with its bound on line 2, which is available because the box of line 2 is still open.
- Line 8: $\exists I$ with the witness δ_0 , its bound on line 5 and the instance on line 7.
- Line 9 ($\exists E$, flag δ_0): δ_0 does not appear in line 9 (the witness was forgotten at line 8), in line 4, in the premise, or in the assumptions still open at line 9, lines 2 and 3.
- Line 10 ($\forall I$, flag ε_0): ε_0 does not appear in the premise, in line 10, or in the assumption still open at line 10, line 2.
- Line 11 ($\forall I$, flag x_0): x_0 does not appear in the premise or in line 11, and no assumption is open.

The answer to the prompt: line 5 is the assumption of the $\exists E$ box, so it rests on line 4, which rests on lines 1 and 3. None of the three mentions x_0 . The witness δ_0 was obtained from the premise at ε_0 alone, before x_0 played any part. That is uniformity, read off the lines.

The derivation has 11 lines. Written with the unbounded rules it has 17, since each of its six bounded steps (lines 4, 7, 8, 9, 10 and 11) costs one line more (Section 6.1). The bounded rules keep it at the level at which a mathematician thinks, and the same argument in prose is three sentences, each aligned with the lines it abbreviates:

Let $x \in A$ and $\varepsilon > 0$ [lines 2, 3, 10, 11]. By uniform continuity there is $\delta > 0$ such that $|u - v| < \delta$ implies $|f(u) - f(v)| < \varepsilon$ for all $u, v \in A$; choose such a δ [4, 5, 6, 9]. Applied with $u = x$, this gives $|f(x) - f(y)| < \varepsilon$ for every $y \in A$ with $|x - y| < \delta$, so this δ works for x and ε [7, 8].

The prose renames the bound variables of UC to u, v , so that “ x ” can name the fixed point; a derivation needs no such care, because $\forall E$ handles bound variables by substitution.

6.4.4 Reading dependence off the boxes

The derivation shows something the informal proof leaves unsaid: the exact place where δ is chosen, and what surrounds it there.

A constant chosen in a box may depend on the names of the boxes open around that box, and it does depend only on the names in the line it comes from.

Quick check. In the derivation above, which flagged boxes surround the box of δ_0 , opened at line 5? What may δ_0 therefore depend on, and what does it in fact depend on?

Solution. The boxes of x_0 (opened at line 2) and of ε_0 (opened at line 3). So δ_0 may depend on x_0 and on ε_0 , as C allows. In fact it depends only on ε_0 : it comes from line 4, which mentions ε_0 and not x_0 .

Why “may”? Inside the boxes of x_0 and ε_0 , every term built from x_0 and ε_0 is available, so a premise could have been instantiated at such a term, and a witness obtained from the resulting $\exists$ line would depend on both names. The side conditions permit that here, because the goal C puts $\exists\delta$ inside both $\forall$ ’s: the witness is forgotten by $\exists$ I (line 8) before either box closes. Why “does”? Because everything known about δ_0 is what its $\exists$ line says, and line 4 mentions ε_0 alone.

◉ **Second reading** · *Choosing the delta before the point*

So δ_0 could have been obtained before x_0 entered at all: fix ε_0 , obtain δ_0 , and only then let $x_0 \in A$. That is the order of UC itself. A derivation in that order nests the boxes as ε_0 , δ_0 , x_0 (the box of δ_0 rests on line 4, so it stays inside the box of ε_0), and it ends in $\forall\varepsilon > 0 \forall x \in A \exists\delta > 0 \forall y \in A (\dots)$, which is C with its first two quantifiers exchanged. C follows by exchanging them back, as in the swap of Example (Flagged boxes at work) (c), here with the bounded rules: boxes for $x_1 \in A$ and $\varepsilon_1 > 0$, $\forall E$ at ε_1 and then at x_1 , and two $\forall I$ lines. C’s order of choices allows more dependence than this proof uses.

The limit derivation of Section 6.1 showed the same thing on a smaller scale: its witness N_0 came from a line obtained at $\varepsilon_0/2$, inside the box of ε_0 only, and so it depended on ε_0 .

6.4.5 The converse breaks at a side condition

The converse, from C to UC, is the interesting direction. The search for it fails, and the failure has an exact address.

Example. The search from C to UC

Run the search method on the sequent with premise C and goal UC, and find where it stops.

Try it first (ten minutes): start the search. Where does a δ come from, and at what moment can you get one?

Idea. The obvious plan copies the derivation above: fix ε_0 , obtain δ_0 from the premise, choose it as the witness. It runs into an obstacle at the second step. In C, $\exists\delta$ stands behind $\forall x \in A$, so to reach it the premise must first be instantiated at a point, and the bounded $\forall E$ needs a line $t \in A$. No such line exists until we assume one, in a box for some x_0 . The δ_0 obtained inside that box comes from a line that mentions x_0 , and both side conditions then hold it in: it cannot leave its own box while a line about it still mentions it, and x_0 cannot be generalized while a line about δ_0 still mentions x_0 .

Solution. The goal UC begins $\forall\varepsilon > 0$, so open a box for ε_0 . The new goal $\exists\delta > 0 \forall x \in A \forall y \in A (\dots)$ is postponed. The premise C begins $\forall x \in A$, and no line $t \in A$ is available; the only way to get one is to assume it, so open a bounded box “let $x_0 \in A$ ”. Then the premise yields a δ . Here is how far the search gets, every line legal:

1	$\forall x \in A \forall \varepsilon > 0 \exists \delta > 0 \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	
2	$\varepsilon_0 \mid \varepsilon_0 > 0$	asm, for $\forall I$
3	$x_0 \mid x_0 \in A$	asm, for $\forall I$
4	$\forall \varepsilon > 0 \exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon)$	$\forall E$ 1, 3
5	$\exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\forall E$ 4, 2
6	$\delta_0 \mid \delta_0 > 0$	asm, for $\exists E$
7	$\forall y \in A (x_0 - y < \delta_0 \rightarrow f(x_0) - f(y) < \varepsilon_0)$	asm, for $\exists E$
8	$\exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\exists I$ 6, 7
9	$\exists \delta > 0 \forall y \in A (x_0 - y < \delta \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\exists E$ 5, 6–8
10	$\forall x \in A \exists \delta > 0 \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon_0)$	$\forall I$ 3–9
11	$\forall \varepsilon > 0 \forall x \in A \exists \delta > 0 \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	$\forall I$ 2–10

The goal inside the box of ε_0 needs a δ that works for **all** points: the line $\forall x \in A \forall y \in A (|x - y| < \delta_0 \rightarrow |f(x) - f(y)| < \varepsilon_0)$, so that $\exists I$ at δ_0 could finish. Line 7 says this only for the point x_0 . There are two ways to try to generalize it, and each is stopped by a side condition.

- **Generalize inside the box of δ_0 .** After line 7, open a new flagged box with the name x_0 and the assumption $x_0 \in A$, repeat line 7 in it, and close it by $\forall I$ with the conclusion $\forall x \in A \forall y \in A (|x - y| < \delta_0 \rightarrow \dots)$. That line breaks the side condition of $\forall I$: x_0 appears in assumptions still open there, line 3 and line 7. (With a different name x_1 the box is legal, but line 7 says nothing about x_1 , so the box cannot reach its last line.)
- **Leave the box of δ_0 first, carrying line 7.** Close the $\exists E$ box with line 7 itself as its conclusion, in order to generalize over x_0 outside it. That line breaks the side condition of $\exists E$: its conclusion mentions the witness δ_0 .

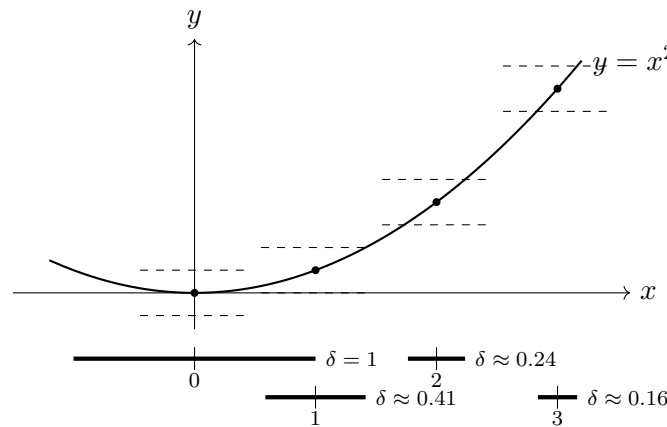
The only legal way out of the box of δ_0 is to forget δ_0 , as line 8 does with $\exists I$ before line 9 closes the box. Then $\forall I$ on x_0 is legal, and line 10 is what it gives: $\forall x \in A \exists \delta > 0$, with δ still behind x . That is C at ε_0 again, not UC, and closing the box of ε_0 gives line 11, which is C with its first two quantifiers exchanged. The search has gone round in a circle.

These are the same two blocks that stopped the converse of Proposition 5.6.2: a $\forall I$ blocked by an open assumption that mentions its name, and an $\exists E$ box that cannot close on a line mentioning its witness. That is no coincidence: with the bounds and the matrix stripped away, C to UC is the swap $\forall x \exists \delta$ to $\exists \delta \forall x$.

A failed search proves nothing about all derivations; a counter-reading does, and the next subsection gives one: a function for which C is true and UC false. Section 7.2 explains why that settles the question.

6.4.6 A function that is continuous and not uniformly continuous

Here is the picture first; the example proves it.



With $\epsilon = 1$, the largest δ that works at $x \geq 0$ is $\sqrt{x^2 + 1} - x$: 1 at 0, about 0.41 at 1, 0.24 at 2, 0.16 at 3. It shrinks as x grows, and no single positive δ works at every x .

Example. The square function on the real line

Let $f(x) = x^2$ on $A = \mathbb{R}$. Show that f satisfies C and does not satisfy UC.

Solution. C holds. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Let

$$\delta := \min\left(1, \frac{\epsilon}{2|x| + 1}\right),$$

which is positive, since $\epsilon > 0$ and $2|x| + 1 \geq 1 > 0$. It depends on ϵ **and on** x . Let $y \in \mathbb{R}$ with $|x - y| < \delta$. By the triangle inequality (Example (The triangle inequality by cases)), used twice, and since $\delta \leq 1$, $|y| \leq |x| + |y - x| < |x| + 1$, so $|x + y| \leq |x| + |y| < 2|x| + 1$. Hence

$$|x^2 - y^2| = |x - y| \cdot |x + y| \leq |x - y| (2|x| + 1) < \delta (2|x| + 1) \leq \epsilon,$$

where the strict step uses $2|x| + 1 > 0$, and the last step uses $\delta \leq \epsilon/(2|x| + 1)$. So this δ works for x and ϵ .

UC fails. By the negation of uniform continuity in Example (Three negations from analysis)

(c), with the bounds $x \in \mathbb{R}$ and $y \in \mathbb{R}$ dropped because they always hold, we must show

$$\exists \varepsilon > 0 \forall \delta > 0 \exists x \exists y (|x - y| < \delta \wedge |x^2 - y^2| \geq \varepsilon).$$

Take $\varepsilon = 1$. Let $\delta > 0$ be arbitrary, and take $x := 1/\delta$ and $y := 1/\delta + \delta/2$, which depend on δ . Then $|x - y| = \delta/2 < \delta$, and, since $y > x > 0$ (so $y - x$ and $y + x$ are both positive),

$$|x^2 - y^2| = (y - x)(y + x) = \frac{\delta}{2} \left(\frac{2}{\delta} + \frac{\delta}{2} \right) = 1 + \frac{\delta^2}{4} \geq 1.$$

Since δ was arbitrary, UC fails.

The δ for C grows smaller as $|x|$ grows, and no single δ serves the whole line. The proof of failure says why: however small δ is, far enough out (at $x = 1/\delta$) the graph is so steep that two points $\delta/2$ apart have values more than 1 apart. Read in the real numbers, with f as squaring, the premise C is true and the conclusion UC is false: a counter-reading, and by Section 7.2 no derivation of UC from C exists.

Warning. “**Depending on ε** ” is a claim, and it can be false. When a proof says “choose δ (depending on ε)”, the parenthesis records which boxes are open where δ is chosen, and which names its formula uses. This is the logical origin of the book’s rule that every chosen constant says what it depends on: the dependence is a record of the open boxes, and a reader can check it. A proof of uniform continuity must be able to write “depending on ε ” truthfully. A “proof” that x^2 is uniformly continuous on $\mathbb{R}$ that takes $\delta = \min(1, \varepsilon/(2|x|+1))$ (“(depending on ε)” has a formula for δ that mentions x : the δ was chosen inside the box of x , so the proof shows only C, and [Example \(The square function on the real line\)](#) shows that UC is false.

The proof that UC fails for x^2 established, with numbers, the positive form of the negation from Section 6.2. That form, taken before its last order fact, refutes UC by logic alone, and a derivation shows how; one TAUT line (Section 6.3) carries its propositional core.

Example. The negation at work, in one derivation

Let E be $\exists \varepsilon > 0 \forall \delta > 0 \exists x \in A \exists y \in A (|x - y| < \delta \wedge \neg(|f(x) - f(y)| < \varepsilon))$, the form of Example (Three negations from analysis) (c) before its last order fact. Derive $E \vdash \neg\text{UC}$, with the bounded rules of Section 6.1.

Solution.

1	$\exists \varepsilon > 0 \forall \delta > 0 \exists x \in A \exists y \in A (x - y < \delta \wedge \neg(f(x) - f(y) < \varepsilon))$	
2	$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	asm, for $\neg I$
3	$\varepsilon_1 \quad \varepsilon_1 > 0$	asm, for $\exists E$
4	$\forall \delta > 0 \exists x \in A \exists y \in A (x - y < \delta \wedge \neg(f(x) - f(y) < \varepsilon_1))$	asm, for $\exists E$
5	$\exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon_1)$	$\forall E$ 2, 3
6	$\delta_1 \quad \delta_1 > 0$	asm, for $\exists E$
7	$\forall x \in A \forall y \in A (x - y < \delta_1 \rightarrow f(x) - f(y) < \varepsilon_1)$	asm, for $\exists E$
8	$\exists x \in A \exists y \in A (x - y < \delta_1 \wedge \neg(f(x) - f(y) < \varepsilon_1))$	$\forall E$ 4, 6
9	$x_1 \quad x_1 \in A$	asm, for $\exists E$
10	$\exists y \in A (x_1 - y < \delta_1 \wedge \neg(f(x_1) - f(y) < \varepsilon_1))$	asm, for $\exists E$
11	$y_1 \quad y_1 \in A$	asm, for $\exists E$
12	$ x_1 - y_1 < \delta_1 \wedge \neg(f(x_1) - f(y_1) < \varepsilon_1)$	asm, for $\exists E$
13	$\forall y \in A (x_1 - y < \delta_1 \rightarrow f(x_1) - f(y) < \varepsilon_1)$	$\forall E$ 7, 9
14	$ x_1 - y_1 < \delta_1 \rightarrow f(x_1) - f(y_1) < \varepsilon_1$	$\forall E$ 13, 11
15	$\perp$	TAUT 12, 14
16	$\perp$	$\exists E$ 10, 11–15
17	$\perp$	$\exists E$ 8, 9–16
18	$\perp$	$\exists E$ 5, 6–17
19	$\perp$	$\exists E$ 1, 3–18
20	$\neg \forall \varepsilon > 0 \exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	$\neg I$ 2–19

Line 15's propositional skeleton is $p \wedge \neg q$, $p \rightarrow q \models \perp$, so the line replaces $\wedge E$ twice, $\rightarrow E$ and $\neg E$, four lines. The boxes on the witness lines 5, 8 and 10 are opened as soon as those lines exist, the move “name the witness early” (Section 5.6), which Section 6.3 puts first. Every closing line's witness (y_1 , x_1 , δ_1 , ε_1) is absent from $\perp$, from the $\exists$ line it cites, from the premise, and from the assumptions still open.

6.4.7 Witness functions

UC says that for every $\varepsilon > 0$ some $\delta > 0$ works for all points. A rule that picks such a δ for each ε is a function, $\delta(\varepsilon)$: a **witness function** for UC. For $f(x) = x/2$ on $\mathbb{R}$, $\delta(\varepsilon) = 2\varepsilon$ works, since $|x/2 - y/2| = |x - y|/2 < \varepsilon$ when $|x - y| < 2\varepsilon$.

For C the witness may depend on the point too: $\delta(x, \varepsilon)$. For x^2 , [Example \(The square function on the real line\)](#) gave $\delta(x, \varepsilon) = \min(1, \varepsilon/(2|x| + 1))$.

A derivation shows which arguments a witness function needs. A witness obtained inside the boxes of ε_0 and x_0 may be a function of both, and one obtained inside the box of ε_0 alone is a function of ε_0 alone. “May depend” and “does depend” are exactly the two readings of “Reading dependence off the boxes”. So UC holds when some witness function for C ignores its

point argument, and the derivation of [Example \(Uniformly continuous implies continuous\)](#) does the reverse: it turns a $\delta(\varepsilon)$ for UC into a $\delta(x, \varepsilon) := \delta(\varepsilon)$ for C, one that ignores x .

In the game of Section 5.1, a witness function is a strategy for the defender: a rule that answers every challenge, and the defender wins by following it.

A formula of this book cannot say “there is a function”: its variables range over numbers. A function symbol can stand for one chosen function, and that is how Sections 5.1 and 6.1 wrote sequences.

A tempting “proof” of the converse tries to build a witness function for UC out of one for C, by taking the smallest of its values.

Example. A “proof” that continuity gives uniform continuity

Find the error in this argument, which claims that every function continuous at every point of A is uniformly continuous on A .

Let $\varepsilon > 0$. For each $x \in A$, continuity at x gives $\delta_x > 0$ with $|f(x) - f(y)| < \varepsilon$ whenever $y \in A$ and $|x - y| < \delta_x$. Let δ be the smallest of the numbers δ_x . Then $\delta > 0$, and it works at every point, so f is uniformly continuous.

Solution.

- The conclusion is false for x^2 on $\mathbb{R}$ ([Example \(The square function on the real line\)](#)), so some step fails, and it is the third sentence.
- Infinitely many positive numbers need not have a smallest, and they may come arbitrarily close to 0. With $\varepsilon = 1$, the largest δ that works for x^2 at $x \geq 0$ is $\sqrt{x^2 + 1} - x$ (the figure above), which is $1/(\sqrt{x^2 + 1} + x) < 1/(2x)$ for $x > 0$, and so smaller than any given positive number once x is large.
- In the rules: each δ_x is the witness of an $\exists E$ box inside the box of x_0 , and “the smallest of them” would be a line about all the witnesses at once. Using it needs a δ that works at every point, the line $\forall x \in A \forall y \in A (|x - y| < \delta \rightarrow \dots)$ whose two routes [Example \(The search from C to UC\)](#) found blocked by side conditions.
- For finite $A = \{a_1, \dots, a_k\}$ the proof is right: k uses of $\exists E$ give $\delta_1, \dots, \delta_k$ (the subsection “Two witnesses” of Section 5.6, repeated), and $\min(\delta_1, \dots, \delta_k) > 0$ works at every point.

The search stalls at the same side condition that stopped [Example \(The search from C to UC\)](#); that x^2 is a counter-reading for the converse is the evidence that no other route exists (Section 7.2).

Quick check. For which sets A does the step “let δ be the smallest of the δ_x ” make sense, and give a positive δ ?

Solution. For finite non-empty A . Finitely many positive numbers have a smallest, and it is positive. For empty A , UC holds outright, since its $\forall x \in A$ is vacuous.

Remark (Looking ahead). Chapter 8 shows how the eleven lines of [Example \(Uniformly continuous implies continuous\)](#) become the three sentences of a written proof, at several levels of detail.

6.4.8 Exercises

6.4.8.1 A. Check your understanding

Exercise. A1

These parts are about the derivation of [Example \(Uniformly continuous implies continuous\)](#).

- Why is line 7 legal? Name the rule, the term, and the lines it cites, and say why those lines are available.
- At which line is the premise's "for all $x \in A$ " used, and at which term?
- State the idea of the derivation in two sentences.
- Run the derivation for $f(x) = 3x + 1$ on $A = \mathbb{R}$, where UC holds with $\delta = \varepsilon/3$. For a given x_0 and $\varepsilon_0 > 0$, which δ does the derivation hand to C at line 8, and does it depend on x_0 ? Check that it works.

Solution. (a) Line 7 is the bounded $\forall E$ at the term x_0 . It cites line 6, $\forall x \in A \forall y \in A (\dots)$, and line 2, $x_0 \in A$, the bound's instance. Line 6 is the assumption of the $\exists E$ box that contains line 7, and line 2 is the assumption of the box of x_0 , which also contains line 7; both boxes are still open at line 7. The term x_0 is a variable not appearing in line 6, so it is free for x .

(b) At line 7, at the term x_0 . Line 6 says that δ_0 works for **all** pairs of points of A , and line 7 uses that for pairs with first point x_0 .

(c) Uniform continuity at ε supplies one δ that works for all points at once, and it is supplied before any point is named. Continuity at a point x then takes that same δ , specialized to the point x .

(d) Line 4 says that some $\delta > 0$ works at ε_0 , and line 8 hands C that witness δ_0 , whatever it is; the derivation knows only lines 5 and 6 about it. Here $\varepsilon_0/3$ is such a witness (so is any δ with $0 < \delta \leq \varepsilon_0/3$). Reading the derivation with the witness the exercise names, $\delta_0 = \varepsilon_0/3$, the δ handed to C is $\varepsilon_0/3$. It does not depend on x_0 . It works: for every real y with $|x_0 - y| < \varepsilon_0/3$,

$$|f(x_0) - f(y)| = |(3x_0 + 1) - (3y + 1)| = 3|x_0 - y| < 3 \cdot \frac{\varepsilon_0}{3} = \varepsilon_0.$$

Exercise. A2: Drill

Each part takes a line or two. The parts run from single steps, through a missing line, a justification column and a step to judge, to witness functions, a negation and one short derivation. UC and C are the formulas of "Two definitions that differ in one place".

- Line 1 is UC and line 2 is $1 > 0$. Write the line that the bounded $\forall E$ 1, 2 gives.

- (b) The available lines are $\delta_1 > 0$ and $\forall y \in A (|x_0 - y| < \delta_1 \rightarrow |f(x_0) - f(y)| < \varepsilon_0)$. Write the line that the bounded $\exists I$ with witness δ_1 gives.
- (c) In [Example \(The negation at work, in one derivation\)](#), line 13 is the bounded $\forall E$ from line 7 at x_1 . Which second line must be available?
- (d) In the six-line derivation of “The skeleton first”, give the rule and the cited lines of lines 3 and 4.
- (e) In [Example \(The search from C to UC\)](#), give the rule and the cited lines of line 10.
- (f) Is this correct? Inside the boxes of ε_0 and x_0 , a witness δ_0 obtained from C at x_0 is used in $\exists I$ to write $\exists \delta > 0 \forall x \in A \forall y \in A (\dots)$.
- (g) Give a witness function for UC of $f(x) = 5x$ on $\mathbb{R}$.
- (h) For x^2 , what does the witness function $\min(1, \varepsilon/(2|x| + 1))$ give at $x = 3$, $\varepsilon = 1$?
- (i) For x^2 and $\varepsilon = 1$, what is the largest δ that works at $x = 4$, by the formula under the figure?
- (j) Negate C, “continuous at every point of A ”, into positive form.
- (k) Which quantifier of C must move, and past which, to turn C into UC?
- (l) Derive $\forall \varepsilon > 0 \exists \delta > 0 (\delta \leq \varepsilon)$ with the bounded rules, given the arith line $\varepsilon_0 \leq \varepsilon_0$.

Solution.

- (a) $\exists \delta > 0 \forall x \in A \forall y \in A (|x - y| < \delta \rightarrow |f(x) - f(y)| < 1)$.
- (b) $\exists \delta > 0 \forall y \in A (|x_0 - y| < \delta \rightarrow |f(x_0) - f(y)| < \varepsilon_0)$.
- (c) Line 9, $x_1 \in A$.
- (d) Line 3: $\forall E$ 2, at x_0 . Line 4: $\exists I$ 3, with witness δ_0 .
- (e) The bounded $\forall I$, 3–9.
- (f) No. That $\exists I$ needs the line $\forall x \in A \forall y \in A (|x - y| < \delta_0 \rightarrow \dots)$, and that line needs $\forall I$ over x_0 while an open assumption about δ_0 mentions x_0 : the side condition of $\forall I$.
- (g) $\delta(\varepsilon) = \varepsilon/5$, since $|5x - 5y| = 5|x - y| < \varepsilon$ when $|x - y| < \varepsilon/5$.
- (h) $\min(1, 1/7) = 1/7$.
- (i) $\sqrt{17} - 4 \approx 0.123$.
- (j) $\exists x \in A \exists \varepsilon > 0 \forall \delta > 0 \forall y \in A (|x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon)$.
- (k) $\forall x \in A$ moves to the right, past $\exists \delta > 0$. Passing $\forall \varepsilon > 0$ is harmless (two $\forall$ ’s exchange); passing $\exists \delta$ is the swap that fails.
- (l) The witness function is $\delta(\varepsilon) = \varepsilon$:

1	ε_0	$\varepsilon_0 > 0$	asm, for $\forall I$
2		$\varepsilon_0 \leq \varepsilon_0$	arith
3		$\exists \delta > 0 (\delta \leq \varepsilon_0)$	$\exists I$ 1, 2
4		$\forall \varepsilon > 0 \exists \delta > 0 (\delta \leq \varepsilon)$	$\forall I$ 1–3

6.4.8.2 B. Practice

Exercise. B1: Lipschitz implies uniformly continuous

A function f is **Lipschitz** on A when

$$\exists L \geq 0 \forall x \in A \forall y \in A (|f(x) - f(y)| \leq L \cdot |x - y|).$$

Prove that a function that is Lipschitz on A is uniformly continuous on A , first as a derivation with the bounded rules (lines of arithmetic justified “arith”), then as a short proof in prose. Say what your δ depends on. (The bound $L \geq 0$ is part of the definition: if A has at most one point, then $x = y$ in the matrix, and any L , even $L = -1$, would satisfy it. The bound rules this out, so a proof may use $L \geq 0$, for instance to divide by $L + 1$.)

Solution. The goal UC begins $\forall \varepsilon > 0$: open a box for ε_0 . The new goal $\exists \delta > 0 \dots$ is postponed until a witness is in hand. The premise begins $\exists L \geq 0$: open its $\exists E$ box at once, with a witness L_0 , aiming for the postponed goal, which does not mention L_0 . Work backwards from what the boxes for x_0 and y_0 will need: from $|x_0 - y_0| < \delta$ we want $|f(x_0) - f(y_0)| < \varepsilon_0$, and the premise gives $|f(x_0) - f(y_0)| \leq L_0|x_0 - y_0| < L_0\delta$ when $L_0 > 0$. So $\delta = \varepsilon_0/L_0$ is the natural guess, but $L_0 = 0$ is allowed and then ε_0/L_0 is undefined. Take $\delta = \varepsilon_0/(L_0 + 1)$ instead: $L_0 + 1 \geq 1 > 0$, and $L_0\delta = \varepsilon_0 \cdot L_0/(L_0 + 1) < \varepsilon_0$.

1	$\exists L \geq 0 \forall x \in A \forall y \in A (f(x) - f(y) \leq L \cdot x - y)$	
2	$\varepsilon_0 \mid \varepsilon_0 > 0$	asm, for $\forall I$
3	$L_0 \mid L_0 \geq 0$	asm, for $\exists E$
4	$\forall x \in A \forall y \in A (f(x) - f(y) \leq L_0 \cdot x - y)$	asm, for $\exists E$
5	$\varepsilon_0/(L_0+1) > 0$	arith 2, 3
6	$x_0 \mid x_0 \in A$	asm, for $\forall I$
7	$y_0 \mid y_0 \in A$	asm, for $\forall I$
8	$ x_0 - y_0 < \varepsilon_0/(L_0+1)$	asm, for $\rightarrow I$
9	$\forall y \in A (f(x_0) - f(y) \leq L_0 \cdot x_0 - y)$	$\forall E$ 4, 6
10	$ f(x_0) - f(y_0) \leq L_0 \cdot x_0 - y_0 $	$\forall E$ 9, 7
11	$ f(x_0) - f(y_0) < \varepsilon_0$	arith 2, 3, 8, 10
12	$ x_0 - y_0 < \varepsilon_0/(L_0+1) \rightarrow f(x_0) - f(y_0) < \varepsilon_0$	$\rightarrow I$ 8–11
13	$\forall y \in A (x_0 - y < \varepsilon_0/(L_0+1) \rightarrow f(x_0) - f(y) < \varepsilon_0)$	$\forall I$ 7–12
14	$\forall x \in A \forall y \in A (x - y < \varepsilon_0/(L_0+1) \rightarrow f(x) - f(y) < \varepsilon_0)$	$\forall I$ 6–13
15	$\exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon_0)$	$\exists I$ 5, 14
16	$\exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	$\exists E$ 1, 3–15
17	$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in A \forall y \in A (x - y < \delta \rightarrow f(x) - f(y) < \varepsilon)$	$\forall I$ 2–16

Line 11 is arithmetic from lines 2, 3, 8 and 10: $|f(x_0) - f(y_0)| \leq L_0|x_0 - y_0| \leq L_0 \cdot \varepsilon_0/(L_0 + 1)$, where the second step uses $L_0 \geq 0$, and $L_0/(L_0 + 1) < 1$ with $\varepsilon_0 > 0$ gives $L_0\varepsilon_0/(L_0 + 1) < \varepsilon_0$. Line 15 is the bounded $\exists I$ at the term $\varepsilon_0/(L_0 + 1)$, with its bound on line 5. At line 16, L_0

does not appear in line 16 (the witness term was forgotten at line 15), in line 1, which is the premise, or in the assumption $\varepsilon_0 > 0$ still open there. At lines 13, 14 and 17 the flags y_0, x_0, ε_0 do not appear in the premise, in the line itself, or in the assumptions then open (among lines 2, 3, 4 and 6).

In prose: Let $L \geq 0$ be a constant with $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in A$. Let $\varepsilon > 0$, and let $\delta := \varepsilon/(L + 1)$, which is positive and depends on ε (and on the constant L , which is fixed before ε), but not on any point. For $x, y \in A$ with $|x - y| < \delta$, $|f(x) - f(y)| \leq L|x - y| \leq L\delta = \varepsilon L/(L + 1) < \varepsilon$. So f is uniformly continuous on A . The box of L_0 is open where δ is chosen, so δ may depend on L_0 . That is harmless: L_0 is a constant of the function, not a point, and its box could as well have been opened first, outside the box of ε_0 .

Exercise. B2: Put the proof in order

Below are ten sentences. Eight of them, in a suitable order, form a proof that a function f uniformly continuous on A is continuous at every point of A ; two are distractors. Write down an order for the eight, and say why each distractor is wrong.

- (i) Since x and y lie in A and $|x - y| < \delta$, the choice of δ , applied to $u = x$ and $v = y$, gives $|f(x) - f(y)| < \varepsilon$.
- (ii) Let $\delta > 0$ be arbitrary.
- (iii) Let $\varepsilon > 0$.
- (iv) Since $\varepsilon > 0$ and $x \in A$ were arbitrary, f is continuous at every point of A .
- (v) Suppose f is uniformly continuous on A .
- (vi) By uniform continuity, applied to ε , choose $\delta > 0$ such that for all $u, v \in A$, $|u - v| < \delta$ implies $|f(u) - f(v)| < \varepsilon$.
- (vii) Choose $\delta > 0$ for this x and this ε , and then let $x \in A$ be arbitrary.
- (viii) Let $x \in A$.
- (ix) So for every $y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$: this δ works for x and ε .
- (x) Let $y \in A$ with $|x - y| < \delta$.

Solution. An admissible order: (v), (viii), (iii), (vi), (x), (i), (ix), (iv).

- (v) is the hypothesis of the implication being proved, so it opens the proof (its $\rightarrow$ I box is the outermost one). After it, other orders are admissible as long as each sentence comes after what it uses. (vi) uses the ε of (iii); (x) uses the δ of (vi) and the x of (viii); then (i), (ix), (iv) follow in that order. So exactly three orders are admissible: (viii), (iii), (vi); or (iii), (viii), (vi); or (iii), (vi), (viii), each preceded by (v) and followed by (x), (i), (ix), (iv). Only the first matches the derivation of [Example \(Uniformly continuous implies continuous\)](#) directly. The second and the third fix ε before x ; the third even chooses δ before any point is given, which is possible because uniform continuity supplies δ from ε alone. They correspond to variants of that derivation in which the box of ε_0 (in the third, also that of δ_0) is opened outside the box of x_0 . Both variants end in $\forall \varepsilon > 0 \forall x \in A \exists \delta > 0 \forall y \in A (\dots)$, and C follows by exchanging the two bounded universal quantifiers (the swap of Example (Flagged boxes at work) (c), with the bounded rules); sentence (iv), “since ε and x were arbitrary”, covers both steps in prose.

Distractor (ii) introduces δ as an arbitrary object, by $\forall I$. But in “continuous at x ” the δ is under $\exists$: the proof must **produce** it, by a choice from the hypothesis, not receive it from a challenger. A proof that took δ arbitrary would have to show that every $\delta > 0$ works, which is false in general.

Distractor (vii) chooses δ and only afterwards lets x be arbitrary. It looks legitimate, because when (vii) is read after (viii), x is already fixed, and continuity at x does allow δ to depend on x . But the sentence reverses the order of the two steps: a δ fitted to one particular x is followed by “let x be arbitrary”, which claims that the same δ works for every x . In the rules: if δ really was chosen for this x , its $\exists E$ box assumes a line that mentions x , and generalizing over x while that assumption is open breaks the side condition of $\forall I$, the first blocked line of [Example \(The search from C to UC\)](#). If δ does not depend on x , as here, then “for this x ” is idle and the sentence should be (vi).

Exercise. B3: The square root

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $A = [0, \infty)$. You may use the school inequality $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$ for all $x, y \geq 0$. State what your δ depends on.

Solution. Let $\varepsilon > 0$, and let $\delta := \varepsilon^2$, which is positive and depends on ε only. Let $x, y \in [0, \infty)$ with $|x - y| < \delta$. Then, by the inequality given and because the square root is strictly increasing on $[0, \infty)$,

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} < \sqrt{\varepsilon^2} = \varepsilon.$$

Since x and y were arbitrary, this δ works for all pairs of points; since ε was arbitrary, f is uniformly continuous on $[0, \infty)$. The δ was chosen before the points: in a derivation of this proof, its $\exists I$ line would lie inside the box of ε_0 only.

Exercise. B4: Witness functions

For each function, give a witness function for uniform continuity on the set named, say what it depends on, and prove that it works.

- (a) $f(x) = |x|$ on $\mathbb{R}$.
- (b) $f(x) = x^2$ on $[0, 3]$.
- (c) Say in one sentence why the formula of (b) gives no witness function on all of $\mathbb{R}$.

Solution. (a) $\delta(\varepsilon) = \varepsilon$, which depends on ε only. Let $x, y \in \mathbb{R}$ with $|x - y| < \varepsilon$. By the triangle inequality (Example (The triangle inequality by cases)), $|x| \leq |x - y| + |y|$ and $|y| \leq |y - x| + |x|$, so $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$; hence $||x| - |y|| \leq |x - y| < \varepsilon$.

(b) $\delta(\varepsilon) = \varepsilon/6$. For $x, y \in [0, 3]$, $|x + y| = x + y \leq 6$, so if $|x - y| < \varepsilon/6$, then

$$|x^2 - y^2| = |x - y| \cdot |x + y| \leq 6|x - y| < \varepsilon.$$

It depends on ε and on the bound 3 of the set, and on no point.

(c) On $\mathbb{R}$, $|x + y|$ has no bound, so no constant can replace the 6, and [Example \(The square](#)

function on the real line) shows that no witness function exists there at all.

Marking scheme. (10 points) - 3: (a), with $||x| - |y|| \leq |x - y|$ proved (for instance by the triangle inequality twice). - 4: (b): 1 for the bound $|x + y| \leq 6$, 2 for the chain, 1 for the dependence. - 3: (c). Fatal error (0 for the step): a “witness” that mentions x or y .

6.4.8.3 C. Going deeper

Exercise. C1: Composition, half done

Let f be uniformly continuous on A , with $f(x) \in B$ for every $x \in A$, and let g be uniformly continuous on B . Here is the start of a proof that the composite $x \mapsto g(f(x))$ is uniformly continuous on A :

Fix $\varepsilon > 0$. By the uniform continuity of g , choose $\eta > 0$ such that $|u - v| < \eta$ implies $|g(u) - g(v)| < \varepsilon$ for all $u, v \in B$.

The rest is blank. *Name the move, then complete it.*

Solution (Hint 1). The goal asks for a δ . Which hypothesis can produce one, and at which positive number should it be applied?

Solution (Hint 2). “By the uniform continuity of f , applied to η , choose $\delta > 0$ such that ...”

Solution. The move is fix, obtain, choose: ε was fixed ($\forall I$), η obtained from the hypothesis on g ($\forall E$ at ε , then $\exists E$); now obtain δ from the hypothesis on f at the term the goal needs, η ($\forall E$ at η , then $\exists E$), and choose that δ as the witness ($\exists I$).

Completion. Since $\eta > 0$, the uniform continuity of f , applied to η , gives $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \eta$ for all $x, y \in A$; choose such a δ . It depends on η , hence on ε , and on no point. Let $x, y \in A$ with $|x - y| < \delta$. Then $|f(x) - f(y)| < \eta$, and $f(x), f(y) \in B$, so the choice of η , applied to $u = f(x)$ and $v = f(y)$, gives $|g(f(x)) - g(f(y))| < \varepsilon$. Since x, y were arbitrary, this δ works for all pairs of points of A ; since $\varepsilon > 0$ was arbitrary, $x \mapsto g(f(x))$ is uniformly continuous on A .

Marking scheme. (10 points) - 2: names the move (fix, obtain, choose), or the rules $\forall E$ at η , $\exists E$, $\exists I$. - 3: applies the uniform continuity of f at η (not at ε), using $\eta > 0$, and chooses δ before any point. - 3: for arbitrary $x, y \in A$ with $|x - y| < \delta$, gets $|f(x) - f(y)| < \eta$ and then $|g(f(x)) - g(f(y))| < \varepsilon$, using $f(x), f(y) \in B$. - 2: concludes with the quantifiers in the order ε, δ , points, and says δ depends on ε only. Fatal errors (0 for the step): δ chosen after x or depending on x ; the uniform continuity of g applied to points not known to lie in B .

Exercise. C2: Gluing two sets

Let A and B be sets of real numbers and S the set of numbers that lie in A or in B . Suppose f is uniformly continuous on A and uniformly continuous on B .

- (a) Show that f need not be uniformly continuous on S : take $A = (0, 1)$, $B = (1, 2)$, and $f(x) = 0$ for $x < 1$, $f(x) = 1$ for $x \geq 1$.
- (b) Prove that f is uniformly continuous on S when $A = [a, b]$ and $B = [b, c]$ with $a < b < c$.

Solution (Hint 1). For (b), the two sets share the point b . Two points on opposite sides of b are each close to b .

Solution (Hint 2). For (b), apply the two hypotheses at $\varepsilon/2$ and take $\delta = \min(\delta_1, \delta_2)$.

Solution. (a) On $A = (0, 1)$, f is constantly 0, so every $\delta > 0$ works for every $\varepsilon > 0$: $|f(x) - f(y)| = 0 < \varepsilon$. The same holds on $B = (1, 2)$, where f is constantly 1. On S we prove the negation of uniform continuity (Example (Three negations from analysis) (c)). Take $\varepsilon = 1$. Let $\delta > 0$, and let $\eta := \min(\delta/4, \frac{1}{2})$, $x := 1 - \eta$, $y := 1 + \eta$. Then $\frac{1}{2} \leq x < 1$, so $x \in A \subseteq S$; $1 < y \leq \frac{3}{2}$, so $y \in B \subseteq S$; $|x - y| = 2\eta \leq \delta/2 < \delta$; and $|f(x) - f(y)| = |0 - 1| = 1 \geq \varepsilon$. So f is not uniformly continuous on S . The gluing fails because the two sets come arbitrarily close at 1 but have no point in common: points close to 1 on opposite sides are each controlled only by their own set, and no common point links them.

(b) Let $\varepsilon > 0$. By the uniform continuity on A at $\varepsilon/2$, choose $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for all $x, y \in A$ with $|x - y| < \delta_1$; likewise choose $\delta_2 > 0$ for B . Let $\delta := \min(\delta_1, \delta_2) > 0$, which depends on ε only. Let $x, y \in S$ with $|x - y| < \delta$. There are three cases.

- *Case 1:* $x, y \in A$. Then $|f(x) - f(y)| < \varepsilon/2 < \varepsilon$, since $|x - y| < \delta \leq \delta_1$.
- *Case 2:* $x, y \in B$. The same, with δ_2 .
- *Case 3:* one point in each set, say $x \in A$ and $y \in B$ (the other way round is the same with the letters exchanged). Then $x \leq b \leq y$, so $|x - b| = b - x \leq y - x = |x - y| < \delta \leq \delta_1$, and $b \in A$; hence $|f(x) - f(b)| < \varepsilon/2$. Likewise $|b - y| \leq |x - y| < \delta_2$ and $b \in B$, so $|f(b) - f(y)| < \varepsilon/2$. By the triangle inequality (Example (The triangle inequality by cases)),

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(b) - f(y)| < \varepsilon.$$

Every point of S lies in A or in B , so the cases cover all pairs. So f is uniformly continuous on S . The common point b is what (a) lacked: there, A and B came arbitrarily close at 1 but had no point in common.

Marking scheme. (10 points) - 3: (a) uniform continuity on A and on B checked, and the negation on S proved: ε first, then for arbitrary δ two points depending on δ , with all three conditions checked. - 2: (b) δ_1 for A at some $\varepsilon_1 > 0$ and δ_2 for B at some $\varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$ (the solution takes $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$), and $\delta = \min(\delta_1, \delta_2)$, depending on ε only. - 2: the cases with both points in one set. - 3: the mixed case, through the point b , with $|x - b|, |b - y| \leq |x - y|$ justified and the triangle inequality cited. Fatal errors (0 for the step): a δ depending on x or y ; the mixed case omitted; in (a), points chosen before δ .

Exercise. C3: Exit check: a quantifier derivation

(a) Give a full derivation of

$$\exists x (P(x) \wedge \forall y R(x, y)) \vdash \forall y \exists x (P(x) \wedge R(x, y)),$$

and check the side condition of every line that closes a flagged box.

(b) For the converse, $\forall y \exists x (P(x) \wedge R(x, y)) \vdash \exists x (P(x) \wedge \forall y R(x, y))$, give a domain with two elements and a reading of P and R under which the premise is true and the conclusion false, checking both with the informal reading of Section 5.1.

Solution. (a) The goal is a $\forall$: open a flagged box for y_0 . The premise is an $\exists$: open its $\exists E$ box at once, with a witness x_0 , aiming for the goal's instance. Inside, take the conjunction apart, instantiate $\forall y$ at y_0 , and choose x_0 as the witness:

1	$\exists x (P(x) \wedge \forall y R(x, y))$		
2	y_0	x_0	$P(x_0) \wedge \forall y R(x_0, y)$ asm, for $\exists E$
3			$P(x_0)$ $\wedge E$ 2
4			$\forall y R(x_0, y)$ $\wedge E$ 2
5			$R(x_0, y_0)$ $\forall E$ 4
6			$P(x_0) \wedge R(x_0, y_0)$ $\wedge I$ 3, 5
7			$\exists x (P(x) \wedge R(x, y_0))$ $\exists I$ 6
8			$\exists x (P(x) \wedge R(x, y_0))$ $\exists E$ 1, 2–7
9	$\forall y \exists x (P(x) \wedge R(x, y))$		

- Line 8 ($\exists E$, flag x_0): x_0 does not appear in line 8, in line 1 (the premise), or in any assumption still open at line 8 (there is none: the box of y_0 has no assumption). Line 7, the last line of the box of x_0 , is a line of that box itself and is the formula $\exists x (P(x) \wedge R(x, y_0))$ that line 8 repeats (Definition 5.6.1).
- Line 9 ($\forall I$, flag y_0): y_0 does not appear in the premise or in line 9, and no assumption is open. Line 8, the last line of the box of y_0 , is the instance at y_0 of $\exists x (P(x) \wedge R(x, y))$.

The other nesting, with the box of y_0 inside the box of x_0 , works too: $\forall I$ at the end of the inner box (its name y_0 is in no open assumption, since the assumption on line 2 does not mention it), then $\exists E$ on a conclusion without x_0 .

(b) Take a domain with two elements a, b ; let P be true of both, and let $R(x, y)$ mean “ x is y ”. The premise is true: for $y = a$ the witness $x = a$ has $P(a)$ and $R(a, a)$, and for $y = b$ the witness $x = b$ works. The conclusion is false: $x = a$ fails $\forall y R(a, y)$ at $y = b$, and $x = b$ fails it at $y = a$. The witness in the premise depends on y ; the conclusion asks for one x that serves every y . (Section 7.2 makes this informal reading a structure, and shows that it proves the converse is not derivable.)

Marking scheme. (10 points) - 2: a flagged box with a fresh name y_0 for the $\forall I$, closed correctly. - 3: an $\exists E$ box with a fresh witness x_0 , whose conclusion does not contain x_0 , with the side condition checked. - 2: $\wedge E$, $\forall E$ at y_0 and $\exists I$ with the witness x_0 , with the correct terms. - 3: the two-element reading: domain, meaning of P and R , the premise checked for each y and the conclusion refuted for each x . Either nesting of the two boxes earns full marks. Fatal errors (0 for the step): $\forall I$ on a name that appears in an open assumption; an

$\exists$ E conclusion containing the witness.

6.4.8.4 R. Review

Exercise. R1

Derive $\forall x (f(x) = x) \vdash \forall x (f(f(x)) = x)$.

Solution. The goal is a $\forall$: open a flagged box. The premise is instantiated twice, at the flag and at the compound term $f(x_0)$, the term the goal's instance contains.

1		$\forall x (f(x) = x)$	
2	x_0	$f(x_0) = x_0$	$\forall$ E 1
3		$f(f(x_0)) = f(x_0)$	$\forall$ E 1
4		$f(f(x_0)) = x_0$	$=$ E 2, 3
5		$\forall x (f(f(x)) = x)$	$\forall$ I 2–4

Line 3 instantiates at $f(x_0)$, which is free for x . Line 4 is transitivity, Proposition 5.7.1 (b), written as its one $=$ E step: the equation of line 2 rewrites the last $f(x_0)$ of line 3. At line 5, x_0 is not in the premise or in line 5, and no assumption is open. The method is Section 5.3's “instantiate at the term you need”, here a compound term, with the equality of Section 5.7.

Exercise. R2

Show that the list $P \leftrightarrow \neg Q$, $Q \leftrightarrow \neg R$, $R \leftrightarrow \neg P$ is inconsistent.

Solution. A list is inconsistent when $\perp$ is derivable from it (Definition 3.2.1). The derivation below uses no RAA: three $\neg$ I boxes give $\neg R$, $\neg P$ and $\neg Q$, and each $\leftrightarrow$ E is used in whichever of its two forms (Definition 2.4.1) the line needs.

1	$P \leftrightarrow \neg Q$	
2	$Q \leftrightarrow \neg R$	
3	$R \leftrightarrow \neg P$	
4	P	asm, for $\neg$ I
5	$\neg Q$	$\leftrightarrow$ E 1, 4
6	R	asm, for $\neg$ I
7	$\neg P$	$\leftrightarrow$ E 3, 6
8	$\perp$	$\neg$ E 4, 7
9	$\neg R$	$\neg$ I 6–8
10	Q	$\leftrightarrow$ E 2, 9
11	$\perp$	$\neg$ E 10, 5
12	$\neg P$	$\neg$ I 4–11
13	R	$\leftrightarrow$ E 3, 12
14	Q	asm, for $\neg$ I
15	$\neg R$	$\leftrightarrow$ E 2, 14
16	$\perp$	$\neg$ E 13, 15
17	$\neg Q$	$\neg$ I 14–16
18	P	$\leftrightarrow$ E 1, 17
19	$\perp$	$\neg$ E 18, 12

A TAUT line (Section 4.8) would do it in one step, since the three formulas have no row that makes them all true: an odd cycle of “is the negation of” is impossible.

Exercise. R3

You have derived $P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)$. Derive

$$\exists x (P(x) \wedge (Q(x) \vee R(x))) \vdash \exists x (P(x) \wedge Q(x)) \vee \exists x (P(x) \wedge R(x)),$$

and say which line is the propositional law.

Solution.

1	$\exists x (P(x) \wedge (Q(x) \vee R(x)))$	
2	$x_0 \mid P(x_0) \wedge (Q(x_0) \vee R(x_0))$	asm, for $\exists E$
3	$(P(x_0) \wedge Q(x_0)) \vee (P(x_0) \wedge R(x_0))$	TAUT 2
4	$\mid P(x_0) \wedge Q(x_0)$	asm, for $\vee E$
5	$\mid \exists x (P(x) \wedge Q(x))$	$\exists I$ 4
6	$\mid \exists x (P(x) \wedge Q(x)) \vee \exists x (P(x) \wedge R(x))$	$\vee I$ 5
7	$\mid P(x_0) \wedge R(x_0)$	asm, for $\vee E$
8	$\mid \exists x (P(x) \wedge R(x))$	$\exists I$ 7
9	$\mid \exists x (P(x) \wedge Q(x)) \vee \exists x (P(x) \wedge R(x))$	$\vee I$ 8
10	$\exists x (P(x) \wedge Q(x)) \vee \exists x (P(x) \wedge R(x))$	$\vee E$ 3, 4–6, 7–9
11	$\exists x (P(x) \wedge Q(x)) \vee \exists x (P(x) \wedge R(x))$	$\exists E$ 1, 2–10

Line 3, justified by TAUT on the skeleton $p \wedge (q \vee r) \models (p \wedge q) \vee (p \wedge r)$, is the propositional law Proposition 1.6.1 of Section 1.6; citing it as a derived rule (Definition 2.3.1) gives the same line. At line 11, x_0 is not in line 11 or in line 1, the premise, and no assumption is open. The quantifier steps frame the law: this is the propositional core of Section 6.3.