

5.4 Proving “For Every”: Let x Be Arbitrary

Section 5.3 gave the rule for using a universal statement, and ended at a stall: from $\forall x (P(x) \wedge Q(x))$ it could derive $P(c)$ for every term c we cared to name, but never $\forall x P(x)$ itself. No list of instances is a statement about everything. This section gives the rule that proves a universal statement, $\forall I$. It needs one new device, a box that carries a **name** instead of an assumption, and one new side condition, which turns the mathematician’s “let x be arbitrary” into something a reader can check line by line.

5.4.1 One number we know nothing about

Here is a fact of school algebra.

For every real number x , $x^2 - 2x + 1 \geq 0$.

We cannot check it number by number; there are too many numbers. The proof everyone writes takes **one** real number and argues about it: *Let $x \in \mathbb{R}$. Then $x^2 - 2x + 1 = (x - 1)^2$, and a square is never negative. So $x^2 - 2x + 1 \geq 0$. Then, often without saying so: since x was arbitrary, this holds for every real x .*

Chapter 1 adopted a convention for sentences like “let x be a real number”: the letter names **one fixed** number, not further specified (Section 1.1). It promised that Chapter 5 would explain why an argument of that kind proves a statement about every x at once. Section 2.2 met the same step at the end of the proof of Theorem 2.2.1, passing from an arbitrary rational r to every rational; Section 4.2 met it in induction, passing from “let $n \in \mathbb{N}$ ” to “for every n ”. Each time the step was taken on trust. Here is why it is correct, and when it is not.

The argument about x used **nothing** about x except that it is a real number. It never said “suppose $x > 0$ ” or “take $x = 3$ ”. So it could be repeated word for word with 0, with -7 , with π , with any real number in place of x , and it would prove the claim for that number. One argument about a number we know nothing about is a recipe for an argument about each number, and that is what “for every” asks for.

Everything depends on “nothing”. If the argument had used a fact about x that not every number has, the recipe would break for some numbers. “Let x be a real number with $x > 0$. Then $x > 0$. Since x was arbitrary, every real number is positive” is the same shape, and its conclusion is false. The step “since x was arbitrary” is legitimate only when x **was** arbitrary: when no premise and no assumption still in force says anything about it. The rule of this section makes that condition exact.

5.4.2 The flagged box

To prove “for every x ”, take a fresh name that nothing is known about, prove the statement for that name, and generalize.

In a derivation the argument about the arbitrary x is drawn as a box, as Chapter 1 drew the argument under an assumption. The box opens with no assumption at all. Instead it carries the arbitrary name, written at its top left, and is called a **flagged box**; the name is its **flag**.

Definition 5.4.1. Rules for the Universal Quantifier

$$\begin{array}{c}
i \quad \left| \begin{array}{c} x_0 \\ \vdots \end{array} \right. \\
j \quad \left| \begin{array}{c} \varphi[x_0/x] \end{array} \right. \\
k \quad \left| \begin{array}{c} \forall x \varphi \end{array} \right. \quad \forall I \ i-j
\end{array}
\qquad
\begin{array}{c}
i \quad \left| \begin{array}{c} \forall x \varphi \end{array} \right. \\
k \quad \left| \begin{array}{c} \varphi[t/x] \end{array} \right. \quad \forall E \ i
\end{array}$$

The rule on the left is $\forall I$ (**universal introduction**, or **generalization**). It cites a flagged box, with flag x_0 , from its first line i to its last line j , and writes $\forall x \varphi$ on line k . The flag x_0 is a **variable**. The box’s last line j is $\varphi[x_0/x]$, and it is a line of the flagged box **itself**, not of a box inside it. The side condition is on the name: at line k ,

- x_0 appears in **no premise**,
- x_0 appears in **no assumption of a box still open at line k** , and
- x_0 does **not** appear in $\forall x \varphi$.

The rule on the right is $\forall E$, as in Definition 5.3.1: t is any term free for x in φ .

In words: to prove that φ holds for every x , open a box flagged with a new name x_0 , prove $\varphi[x_0/x]$, “ φ holds of x_0 ”, as the box’s last line, close the box, and write $\forall x \varphi$. The side condition says that x_0 really was arbitrary.

5.4.2.1 A first flagged box: how the rule looks in a mathematical proof

Here is the opening proof as a derivation, with the general fact about squares as a premise and the algebra as one line justified “arith” (Section 5.3).

$$\begin{array}{c}
1 \quad \left| \begin{array}{c} \forall y (y^2 \geq 0) \end{array} \right. \\
\quad \left| \begin{array}{c} x_0 \\ \vdots \end{array} \right. \\
3 \quad \left| \begin{array}{c} x_0^2 - 2 \cdot x_0 + 1 \geq 0 \end{array} \right. \\
4 \quad \left| \begin{array}{c} \forall x (x^2 - 2 \cdot x + 1 \geq 0) \end{array} \right.
\end{array}$$

The goal is universal, so the box opens first, with the goal’s instance at x_0 as its last line; then the premise is instantiated at the term that last line needs, $x_0 - 1$.

$$\begin{array}{c}
1 \quad \left| \begin{array}{c} \forall y (y^2 \geq 0) \end{array} \right. \\
2 \quad \left| \begin{array}{c} x_0 \\ \vdots \end{array} \right. \quad (x_0 - 1)^2 \geq 0 \quad \forall E \ 1 \\
3 \quad \left| \begin{array}{c} x_0^2 - 2 \cdot x_0 + 1 \geq 0 \end{array} \right. \\
4 \quad \left| \begin{array}{c} \forall x (x^2 - 2 \cdot x + 1 \geq 0) \end{array} \right.
\end{array}$$

The finished derivation:

$$\begin{array}{c}
1 \quad \left| \begin{array}{c} \forall y (y^2 \geq 0) \end{array} \right. \\
2 \quad \left| \begin{array}{c} x_0 \\ \vdots \end{array} \right. \quad (x_0 - 1)^2 \geq 0 \quad \forall E \ 1 \\
3 \quad \left| \begin{array}{c} x_0^2 - 2 \cdot x_0 + 1 \geq 0 \end{array} \right. \quad \text{arith } 2 \\
4 \quad \left| \begin{array}{c} \forall x (x^2 - 2 \cdot x + 1 \geq 0) \end{array} \right. \quad \forall I \ 2-3
\end{array}$$

Line	Sentence
box x_0	“Let $x \in \mathbb{R}$.”
2	“Then $(x - 1)^2 \geq 0$, since a square is never negative.” ($\forall E$ at $x_0 - 1$)
3	“That is, $x^2 - 2x + 1 \geq 0$.” (arith)
4	“Since x was arbitrary, $x^2 - 2x + 1 \geq 0$ for every real x .” ($\forall I$)

Lines 2 and 3 are the derivation of Section 5.3 with the flag x_0 in place of the constant a . What changed is only the name: a was a constant that the language fixed, so nothing could be generalized over it; x_0 is a variable that no premise mentions, so line 4 may generalize. The written proof uses the same letter x for the flag and for the bound variable, and says “let $x \in \mathbb{R}$ ” where the derivation opens the box. Readers of mathematics are expected to supply the box.

Since the domain is $\mathbb{R}$, “let $x \in \mathbb{R}$ ” asks nothing of x . A proof that says “let $\varepsilon > 0$ ” does ask something: inside the box flagged ε_0 , a $\rightarrow I$ box opens with the assumption $\varepsilon_0 > 0$.

5.4.2.2 Reading the rule

Here are the two boxes side by side: the box of $\rightarrow I$ from Definition 1.4.1, and the flagged box.

i	$\left \begin{array}{c} \varphi \\ \vdots \end{array} \right $	asm, for $\rightarrow I$	i	$\left \begin{array}{c} x_0 \\ \vdots \end{array} \right $	
j	$\left \begin{array}{c} \psi \end{array} \right $		j	$\left \begin{array}{c} \varphi[x_0/x] \end{array} \right $	
k	$\varphi \rightarrow \psi$	$\rightarrow I$ $i-j$	k	$\forall x \varphi$	$\forall I$ $i-j$

Both draw a boundary and both are cited as a range $i-j$. The $\rightarrow I$ box opens with what we suppose; the flagged box opens with the name we have fixed. In the schema the flagged box begins with a gap $\vdots$ on line i , only so that the range has a first line; in a real derivation line i is whatever the box’s first line is.

This rule has conditions that matter; read each one. Read the rule part by part: the flag, the three clauses of the side condition, and the last line.

- **The flag.** Opening the box is the written “let x_0 be arbitrary”. The flag is always a variable, never a constant: a constant has a fixed meaning, while the whole point of x_0 is that it may stand for anything. The book writes flags with a subscript 0, x_0 , y_0 , ε_0 , so that they stand out.
- **No premise, no open assumption** (the first and second clauses). This is “we know nothing about x_0 ”. What is known at line k is the premises and the assumptions of the boxes still open there. If none of them mentions x_0 , then nothing that was used about x_0 inside the box is special to x_0 .
- **Not in the conclusion** (the third clause). The name must not leak out. Inside the box it stood for an arbitrary object, and $\forall x \varphi$ is meant to speak about all objects through the bound x . A conclusion that still mentions x_0 ties what it says to the one object x_0 names, while claiming it of everything.
- **The last line.** $\varphi[x_0/x]$ is what was proved about x_0 , and it must be a line of the flagged box itself, just as the last line of a $\rightarrow I$ box must be (Section 1.4). A line inside a further box would still depend on that box’s assumption.

“Appear” is the word of Section 5.2: written anywhere in the formula, including just after a

quantifier symbol. So a box flagged x can never be closed by $\forall I$ with $\forall x \varphi$, since x appears in $\forall x \varphi$. That is one more reason the book never uses the quantified letter itself as the flag.

In practice: choose as flag a letter written nowhere in the derivation so far, and as the quantified letter x one that does not appear in the box’s last line (if it does, rename that bound letter first, Section 5.2). When the box closes, put x in place of **every** occurrence of the flag in its last line. Then the last line is $\varphi[x_0/x]$ for the formula φ so obtained, and all three clauses hold at once: no premise or open assumption can contain a letter written nowhere, and the conclusion no longer contains the flag. The clauses are what a checker tests, and they allow a little more than this habit does (the fold below).

🕒 **Second reading** · *Fresh is not the same as new*

The condition asks about three places only: the premises, the open assumptions, and the conclusion. Lines **outside** the box may mention x_0 , before it or after it. An earlier line $\forall y R(x_0, y)$, obtained by $\forall E$ from a premise, is harmless; Exercise C1 builds a derivation with such a line. And once a flagged box has closed, its name may flag a later box. A name that satisfies the side condition at the line that closes its box is called **fresh** there. It need not be new to the whole derivation.

Quick check. A derivation has used the letters x_0 , y_0 and u_0 so far, and u_0 appears in an open assumption. Which flag can a new $\forall I$ box take without checking anything?

Solution. Any letter written nowhere yet, for example z_0 . (x_0 or y_0 may also be allowed, but only after checking the three clauses; u_0 is not, while its assumption is open.)

5.4.2.3 Why each clause

Each clause stops a derivation of a false conclusion from true premises. Here is one failure for each.

A name in a premise. If $\forall I$ could generalize over a name in a premise, then $P(x_0) \vdash \forall x P(x)$:

1	$P(x_0)$	
2	$x_0 \mid P(x_0)$	R 1
3	$\forall x P(x)$	$\forall I$ 2–2

Over $\mathbb{Z}$, read $P(x)$ as “ x is even” and let x_0 name 2. The premise is true and line 3, “every integer is even”, is false. Line 3 breaks the first clause: x_0 appears in the premise on line 1. The premise **does** say something about x_0 , so x_0 is not arbitrary.

A name in an open assumption. Suppose $x_0 > 0$, and inside that supposition generalize over x_0 (left):

1		$x_0 > 0$	asm, for $\rightarrow$ I
2		x_0 $x_0 > 0$	R 1
3		$\forall x (x > 0)$	$\forall$ I 2-2
4		$x_0 > 0 \rightarrow \forall x (x > 0)$	$\rightarrow$ I 1-3

1	x_0	$x_0 > 0$	asm, for $\rightarrow$ I
2		$x_0 + 1 > 0$	arith 1
3		$x_0 > 0 \rightarrow x_0 + 1 > 0$	$\rightarrow$ I 1-2
4		$\forall x (x > 0 \rightarrow x + 1 > 0)$	$\forall$ I 1-3

On the left, line 3 breaks the second clause, since x_0 appears in the assumption of line 1, whose box is still open at line 3. Read over $\mathbb{Z}$ with x_0 naming 1: line 4 says “if $1 > 0$, then every integer is positive”, which is false at 0, and it rests on no premise at all. This is the “let x be a real number with $x > 0 \dots$ since x was arbitrary” of the opening, made into lines. One more legal flagged box around everything turns line 4 into a false sentence; the fold shows it. On the right the boxes are nested the other way: the flagged box opens first, the supposition inside it, and $\forall$ I generalizes an implication that carries its hypothesis with it.

🕒 **Second reading** · *The same error, as a false sentence*

With one more flagged box, the error of the minimal derivation becomes a false **sentence**:

1	x_0	$x_0 > 0$	asm, for $\rightarrow$ I
2		x_0 $x_0 > 0$	R 1
3		$\forall x (x > 0)$	$\forall$ I 2-2
4		$x_0 > 0 \rightarrow \forall x (x > 0)$	$\rightarrow$ I 1-3
5		$\forall y (y > 0 \rightarrow \forall x (x > 0))$	$\forall$ I 1-4

Line 3 breaks the second clause: x_0 appears in the assumption on line 1, whose box is still open at line 3. Every other line is legal, including line 5, where the box of line 1 has closed. The inner box on line 2 is flagged x_0 while the outer box flagged x_0 is still open; the definition allows this, since its side condition asks only about premises, open assumptions and the conclusion. And line 5 says “every positive number makes every number positive”, which over $\mathbb{Z}$ is false: at $y = 1$ the hypothesis $1 > 0$ holds and $\forall x (x > 0)$ fails at $x = 0$. This is the “let x be a real number with $x > 0 \dots$ since x was arbitrary” of the opening, made into lines.

A name in the conclusion. From the premise $\forall x (x \leq x)$, true over $\mathbb{N}$, open a box flagged y_0 :

1		$\forall x (x \leq x)$	
2		y_0 $y_0 \leq y_0$	$\forall$ E 1
3		$\forall x (x \leq y_0)$	$\forall$ I 2-2

Line 2 is $(x \leq y_0)[y_0/x]$, so line 3 has the shape of $\forall$ I with $\varphi = (x \leq y_0)$. But y_0 appears in the

conclusion, which breaks the third clause. Line 3 says “every natural number is at most y_0 ”, and that is false whatever y_0 names: $y_0 + 1$ is a counterexample. (Exercise C1 of Section 5.5 turns this into a sentence, “there is a largest natural number”.)

The requirement on the last line is the one Chapter 1 already imposed on every box rule, for the same reason.

5.4.3 Derivations with flagged boxes

The search method of Section 3.1 gets a new goal-directed step: when the goal is $\forall x \varphi$, open a flagged box with a fresh name x_0 , and make $\varphi[x_0/x]$ the box’s last line, the new goal. The examples below use it, each with an Idea.

Example. Flagged boxes at work

Give a derivation of each sequent.

- (a) $\forall x \forall y R(x, y) \vdash \forall x R(x, x)$.
- (b) $\forall x (P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)$.
- (c) $\forall x \forall y R(x, y) \vdash \forall y \forall x R(x, y)$.

Solution.

- (a) *Idea.* The goal begins with $\forall x$, so the goal-directed step opens a box flagged x_0 , and the new goal is $R(x, x)[x_0/x] = R(x_0, x_0)$. That goal names the term for $\forall E$ (Section 5.3’s step): instantiate the premise at x_0 , twice.

1	$\forall x \forall y R(x, y)$	1	$\forall x \forall y R(x, y)$	
	$x_0 \mid \vdots$	2	$x_0 \mid \forall y R(x_0, y)$	$\forall E\ 1$
			$\vdots$	
3	$\mid R(x_0, x_0)$	3	$\mid R(x_0, x_0)$	
4	$\forall x R(x, x)$	4	$\forall x R(x, x)$	

1	$\forall x \forall y R(x, y)$	
2	$x_0 \mid \forall y R(x_0, y)$	$\forall E\ 1$
3	$\mid R(x_0, x_0)$	$\forall E\ 2$
4	$\forall x R(x, x)$	$\forall I\ 2-3$

The side condition at line 4: x_0 does not appear in the premise, no box is open at line 4, and x_0 does not appear in $\forall x R(x, x)$. Lines 2 and 3 instantiate at the variable x_0 , which is free for x and for y , since the formulas contain no quantifier on x_0 . Compare Section 5.3, (b), which reached only $R(c, c)$: there the term was a constant, a name fixed by the language, and nothing can be generalized over a constant; here it is a flag, a variable that no premise mentions.

- (b) *Idea.* The goal is universal: a box flagged x_0 , whose last line must be $Q(x_0)$. Both premises are universal; instantiate each at the term the box’s goal names, x_0 , and $\rightarrow E$ finishes.

Givens	Goal
$\forall x (P(x) \rightarrow Q(x)), \forall x P(x)$	$\forall x Q(x)$
the same, and the fresh name x_0 ($\forall$ I opens a box)	$Q(x_0)$
+ $P(x_0) \rightarrow Q(x_0), P(x_0)$ ($\forall$ E at x_0 , twice)	$Q(x_0)$
+ $Q(x_0)$ ($\rightarrow$ E)	$Q(x_0)$ ✓

1	$\forall x (P(x) \rightarrow Q(x))$	1	$\forall x (P(x) \rightarrow Q(x))$
2	$\forall x P(x)$	2	$\forall x P(x)$
3	$x_0 \mid \vdots$	3	$x_0 \mid P(x_0) \rightarrow Q(x_0) \quad \forall E\ 1$
4	$Q(x_0)$	4	$P(x_0) \quad \forall E\ 2$
5	$\forall x Q(x)$	5	$Q(x_0)$
6		6	$\forall x Q(x)$

1	$\forall x (P(x) \rightarrow Q(x))$
2	$\forall x P(x)$
3	$x_0 \mid P(x_0) \rightarrow Q(x_0) \quad \forall E\ 1$
4	$P(x_0) \quad \forall E\ 2$
5	$Q(x_0) \quad \rightarrow E\ 3, 4$
6	$\forall x Q(x) \quad \forall I\ 3-5$

At line 6, x_0 appears in neither premise nor in $\forall x Q(x)$, and no box is open.

- (c) *Idea.* Read the goal’s quantifiers from the outside: $\forall y$ first, so open a box flagged y_0 ; the new goal $\forall x R(x, y_0)$ is again universal, so open a box flagged x_0 inside it; the innermost goal is $R(x_0, y_0)$. Two boxes, opened in the goal’s order. Now the premise’s quantifiers are removed in **its** order: $\forall E$ at x_0 , then at y_0 .

1	$\forall x \forall y R(x, y)$	1	$\forall x \forall y R(x, y)$
2	$y_0 \mid x_0 \mid \vdots$	2	$y_0 \mid x_0 \mid \forall y R(x_0, y) \quad \forall E\ 1$
3	$R(x_0, y_0)$	3	$R(x_0, y_0)$
4	$\forall x R(x, y_0)$	4	$\forall x R(x, y_0)$
5	$\forall y \forall x R(x, y)$	5	$\forall y \forall x R(x, y)$

1	$\forall x \forall y R(x, y)$
2	$y_0 \mid x_0 \mid \forall y R(x_0, y) \quad \forall E\ 1$
3	$R(x_0, y_0) \quad \forall E\ 2$
4	$\forall x R(x, y_0) \quad \forall I\ 2-3$
5	$\forall y \forall x R(x, y) \quad \forall I\ 2-4$

The box flagged y_0 immediately contains the box flagged x_0 , so it has no line of its own before the inner box; following the convention for a flagged box that immediately contains another box (the fine print below), line 5 cites it as 2–4, from the first line of the inner box. Its last line is line 4, a line of the outer box itself. At line 4, x_0 is in no premise, the only open box (flagged y_0) has no assumption, and x_0 does not appear in $\forall x R(x, y_0)$. At line 5 the same holds for y_0 .

Nothing in (c) used that the formula under the quantifiers is $R(x, y)$. For any formula ψ and different variables x and y , with flags x_0 and y_0 that are different from each other and from x and y and that do not appear in ψ , the same five lines derive $\forall x \forall y \psi \vdash \forall y \forall x \psi$: line 3 becomes $(\psi[x_0/x])[y_0/y]$, which equals $(\psi[y_0/y])[x_0/x]$, since the two substitutions put two different fresh names in for two different variables and so do not interfere. Exchanging the letters x and y gives the other direction. So **two adjacent universal quantifiers may always be swapped**: $\forall x \forall y \psi \dashv\vdash \forall y \forall x \psi$. Section 5.1 read this off the game, where the challenger makes both moves; here it is derived.

Quick check. Suppose premise 2 of (b) were $P(x_0)$ in place of $\forall x P(x)$. The box still reaches $Q(x_0)$, by $\forall E$ 1 and $\rightarrow E$. May line 6 be written?

Solution. No: x_0 appears in a premise, so the first clause fails. A counter-reading shows the step is unsafe: over $\mathbb{Z}$, with P and Q both read as “even” and x_0 naming 2, both premises are true and $\forall x Q(x)$ is false.

Quick check. Could both boxes of (c) be flagged x_0 , the outer one standing for y ?

Solution. No. The inner box would end in $R(x_0, x_0)$, and the $\forall I$ line that closes it must be $\forall x R(x, x_0)$, the instance at x_0 of the outer goal; that line contains the inner flag x_0 , so the third clause fails. Here the nested flagged boxes need two names.

○ Second reading · The front renaming, derived

Section 5.2 took renaming of a bound variable as a convention and promised to derive it at the front of a formula. For a formula φ and variables y, z with z not appearing in φ , $\forall y \varphi \dashv\vdash \forall z \varphi[z/y]$:

1	$\forall y \varphi$		1	$\forall z \varphi[z/y]$	
2	z_0 $\varphi[z_0/y]$	$\forall E$ 1	2	z_0 $\varphi[z_0/y]$	$\forall E$ 1
3	$\forall z \varphi[z/y]$	$\forall I$ 2–2	3	$\forall y \varphi$	$\forall I$ 2–2

Take z_0 to be a variable that does not appear in φ and is different from y and z . Since z does not appear in φ , the formula $\varphi[z/y]$ is φ with each free y changed to z , and changing those z ’s on to z_0 gives the same result as changing each free y to z_0 at once: $(\varphi[z/y])[z_0/z] = \varphi[z_0/y]$. So in the left derivation, line 2 is a legal $\forall E$ (z_0 does not appear in φ , so it is free for y), and it is the instance $(\varphi[z/y])[z_0/z]$ that the $\forall I$ on line 3 needs. In the right derivation, line 2 is $(\varphi[z/y])[z_0/z]$, a legal $\forall E$ for the same reason, and it is $\varphi[z_0/y]$, the instance that $\forall y \varphi$ needs. At line 3 of each, z_0 appears neither in the premise nor in the conclusion, and no box is open.

Each direction has three lines.

Now the sequent at which Section 5.3 stalled, together with its converse. “For every x , $P(x)$ and $Q(x)$ ” says the same as “ $P(x)$ for every x , and $Q(x)$ for every x ”:

Proposition 5.4.1. Universal Quantifier and Conjunction

$$\forall x (P(x) \wedge Q(x)) \dashv\vdash \forall x P(x) \wedge \forall x Q(x).$$

Idea. $(\vdash)$ The goal is a conjunction, so (Section 1.2) two subgoals, $\forall x P(x)$ and $\forall x Q(x)$. Each is universal, so each gets its own flagged box, and inside each, $\forall E$ at the flag and $\wedge E$ give the box’s last line. $(\dashv)$ The goal is universal, so one flagged box, with the goal $P(x_0) \wedge Q(x_0)$; take the premise apart by $\wedge E$ and instantiate each half at x_0 .

The search, for $(\vdash)$:

1	$\forall x (P(x) \wedge Q(x))$		1	$\forall x (P(x) \wedge Q(x))$	
	$x_0 \mid \vdots$		2	$x_0 \mid P(x_0) \wedge Q(x_0)$	$\forall E 1$
3	$P(x_0)$		3	$P(x_0)$	
4	$\forall x P(x)$		4	$\forall x P(x)$	
	$x_0 \mid \vdots$		5	$x_0 \mid P(x_0) \wedge Q(x_0)$	$\forall E 1$
6	$Q(x_0)$			$\vdots$	
7	$\forall x Q(x)$		6	$Q(x_0)$	
8	$\forall x P(x) \wedge \forall x Q(x)$	$\wedge I 4, 7$	7	$\forall x Q(x)$	
			8	$\forall x P(x) \wedge \forall x Q(x)$	$\wedge I 4, 7$

For $(\dashv)$:

1	$\forall x P(x) \wedge \forall x Q(x)$		1	$\forall x P(x) \wedge \forall x Q(x)$	
2	$\forall x P(x)$	$\wedge E 1$	2	$\forall x P(x)$	$\wedge E 1$
3	$\forall x Q(x)$	$\wedge E 1$	3	$\forall x Q(x)$	$\wedge E 1$
	$x_0 \mid \vdots$		4	$x_0 \mid P(x_0)$	$\forall E 2$
6	$P(x_0) \wedge Q(x_0)$		5	$Q(x_0)$	$\forall E 3$
7	$\forall x (P(x) \wedge Q(x))$			$\vdots$	
			6	$P(x_0) \wedge Q(x_0)$	
			7	$\forall x (P(x) \wedge Q(x))$	

Proof. ($\vdash$)

1		$\forall x (P(x) \wedge Q(x))$	
2	x_0	$P(x_0) \wedge Q(x_0)$	$\forall E$ 1
3		$P(x_0)$	$\wedge E$ 2
4		$\forall x P(x)$	$\forall I$ 2–3
5	x_0	$P(x_0) \wedge Q(x_0)$	$\forall E$ 1
6		$Q(x_0)$	$\wedge E$ 5
7		$\forall x Q(x)$	$\forall I$ 5–6
8		$\forall x P(x) \wedge \forall x Q(x)$	$\wedge I$ 4, 7

At lines 4 and 7, x_0 appears neither in the premise nor in the conclusion of that line, and no box is open. The second flagged box reuses the name x_0 . That is allowed: the side condition at line 7 asks only about the premise, the open assumptions, and $\forall x Q(x)$, and the earlier box, with its lines 2 and 3, closed at line 3 and is not available at line 7. (A fresh y_0 would do equally well.)

($\neg$)

1		$\forall x P(x) \wedge \forall x Q(x)$	
2		$\forall x P(x)$	$\wedge E$ 1
3		$\forall x Q(x)$	$\wedge E$ 1
4	x_0	$P(x_0)$	$\forall E$ 2
5		$Q(x_0)$	$\forall E$ 3
6		$P(x_0) \wedge Q(x_0)$	$\wedge I$ 4, 5
7		$\forall x (P(x) \wedge Q(x))$	$\forall I$ 4–6

At line 7, x_0 is in neither the premise nor the conclusion, and no box is open. The two derivations prove the two sequents, as claimed. ■

The first derivation is the answer to Section 5.3’s stall: its lines 1–4 derive $\forall x P(x)$ from $\forall x (P(x) \wedge Q(x))$. The proposition says that $\forall$ distributes over $\wedge$.

Fine print: flagged boxes in a derivation. Definition 1.4.2 (Section 1.4) said that a box is a block of consecutive lines whose first line is an assumption. We now extend that definition, and two things change. First, a box may be **flagged**: it carries a name, a variable written at its top, and a flagged box need **not** open with an assumption. The box of $\forall I$ has none; the box of the rule for using “there exists” in Section 5.6 is flagged and has one. Second, the rules that close a flagged box carry side conditions about its name, such as the one above. Everything else in Definition 1.4.2 stays word for word: two boxes are disjoint or one is inside the other, a box ends at one of its lines, and the scope rule decides which lines and boxes are available where. A flagged box with no assumption adds nothing to what is assumed; it only marks where

the name is in use. When a flagged box immediately contains another box, as in the swap of two quantifiers in the examples above, the range starts at the first line of that inner box. The right-hand derivation of “Why each clause”, derivation (c) above and Exercise B1 below use the convention.

5.4.4 Every element of the empty set

Section 5.1 read $\forall x \in \emptyset \varphi$ as true whatever φ says, because there is no element to test, and promised a derivation. Section 2.1 met the same phenomenon for implications: $\neg P \vdash P \rightarrow Q$ (Proposition 2.1.2, the derived rule written VI). With $\forall I$ the two meet.

Let A be a set with no elements. What that means is the premise $\forall x \neg(x \in A)$: no object is in A . From it, “every element of A is greater than 5”, which by Definition 5.1.1 is $\forall x (x \in A \rightarrow x > 5)$, has a four-line derivation:

1		$\forall x \neg(x \in A)$	
		x_0 $\vdots$	
3		$x_0 \in A \rightarrow x_0 > 5$	
4		$\forall x (x \in A \rightarrow x > 5)$	

1		$\forall x \neg(x \in A)$	
2		x_0 $\neg(x_0 \in A)$	$\forall E$ 1
		$\vdots$	
3		$x_0 \in A \rightarrow x_0 > 5$	
4		$\forall x (x \in A \rightarrow x > 5)$	

Here is how the search grows it, and the result:

1		$\forall x \neg(x \in A)$	
2		x_0 $\neg(x_0 \in A)$	$\forall E$ 1
3		$x_0 \in A \rightarrow x_0 > 5$	VI 2
4		$\forall x (x \in A \rightarrow x > 5)$	$\forall I$ 2–3

Line 3 uses VI (Proposition 2.1.2) as one line, with the formulas $x_0 \in A$ and $x_0 > 5$ for P and Q ; every derived rule of Chapter 2 may be used in this way on first-order formulas. Nothing in the derivation used $x > 5$: with any formula in place of $x > 5$ in which x_0 does not appear, the same four lines go through, so “every element of an empty set has property φ ” is derivable for every φ , as Section 5.1 said.

🕒 **Second reading** · *Why a derived rule may stand in a flagged box*

Using VI inside the box is allowed because the derivation of Proposition 2.1.2 uses only $\neg E$, $\perp E$ and $\rightarrow I$, whose schemas apply to any formulas (Definition 1.2.1), and it has no flagged box, so pasting it into the box flagged x_0 gives a derivation in the rules themselves. From now on, any derived rule of Chapter 2 is used in this way on first-order formulas.

One more flagged box gives a statement about upper bounds:

1		$\forall x \neg(x \in A)$	
2		u_0 x_0 $\neg(x_0 \in A)$	$\forall E$ 1
3		$x_0 \in A \rightarrow x_0 \leq u_0$	$\forall I$ 2
4		$\forall x (x \in A \rightarrow x \leq u_0)$	$\forall I$ 2–3
5		$\forall u \forall x (x \in A \rightarrow x \leq u)$	$\forall I$ 2–4

Over $\mathbb{R}$, line 5 says that every real number u is an upper bound of A : every element of A is at most u . For an empty set this holds for every u , because there are no elements to check. At line 4, x_0 appears in neither the premise nor $\forall x (x \in A \rightarrow x \leq u_0)$, and the only open box (flagged u_0) has no assumption. The name u_0 does appear in that conclusion, which is allowed, since the condition at line 4 is about x_0 only. At line 5, u_0 appears in neither the premise nor $\forall u \forall x (x \in A \rightarrow x \leq u)$, and no box is open.

5.4.5 Common errors

Three errors with $\forall I$ are common enough to name now.

- **Checking instances.** Verifying φ at $x = 0, 1, 2$, or at a thousand values, proves $\varphi[0/x]$, $\varphi[1/x]$, $\varphi[2/x]$, and so on, and not $\forall x \varphi$. The rule needs one proof about a fresh name, not many proofs about particular terms. Exercise C2 shows how plausible this error can look.
- **Generalizing over a name in an open assumption.** Inside “suppose $x_0 > 0$ ”, the name x_0 is not arbitrary. The left-hand derivation of the pair in “Why each clause” shows where this leads.
- **Generalizing over a name in a premise.** A premise about x_0 , such as “ x_0 is even”, makes x_0 special. The first derivation of “Why each clause” shows it.

Warning. “Let x be arbitrary” is a promise about every line that follows, and the most common way to break it is an assumption made on the way. “Let $n \in \mathbb{N}$ and suppose n is even. Then n^2 is even. Since n was arbitrary, every n^2 is even” is false at $n = 1$. What the argument proves is “for every n , if n is even, then n^2 is even”: the $\rightarrow I$ box on “ n is even” must close **before** the $\forall I$ step, so that the assumption is no longer open.

Quick check. In the derivation below, line 6 is justified by $\forall I$. Is it legal? If not, which clause of the side condition fails, and can you find a reading in which the premises are true and line 6 is false?

1		$R(c, y_0)$	
2		$\forall x \forall y (R(x, y) \rightarrow R(y, x))$	
3		y_0 $\forall y (R(c, y) \rightarrow R(y, c))$	$\forall E$ 2
4		$R(c, y_0) \rightarrow R(y_0, c)$	$\forall E$ 3
5		$R(y_0, c)$	$\rightarrow E$ 4, 1
6		$\forall y R(y, c)$	$\forall I$ 3–5

Solution. No. Lines 1–5 are legal; in particular line 5 may cite line 1, which lies in no box. But at line 6 the flag y_0 appears in the premise on line 1, so the first clause fails: line 5 was proved about y_0 using a fact about y_0 , and y_0 was not arbitrary. A reading: over $\mathbb{Z}$, let $R(x, y)$ be $x \neq y$, which is symmetric, so premise 2 is true; let c name 0 and y_0 name 1, so premise 1, $0 \neq 1$, is true. Line 6 says that every integer differs from 0, which is false at $y = 0$.

Example. A Proof to Check

Here is a proof. Is its claim true? Which step fails?

Claim. Some real number is at least as large as every real number.

Proof. Let x be a real number, and put $M = x + 1$. Then $x \leq M$. Since x was arbitrary, $x \leq M$ for every real x , so M is such a number.

Solution. The claim is false: whatever M is proposed, $M + 1$ is a larger real number. The faulty step is “since x was arbitrary”. As a derivation, the box is flagged x_0 and ends in $x_0 \leq x_0 + 1$, a line of arithmetic; the proof then writes $\forall x (x \leq M)$, and M stands for $x_0 + 1$:

1	x_0	$x_0 \leq x_0 + 1$	arith
2		$\forall x (x \leq x_0 + 1)$	$\forall I$ 1–1

Line 2 breaks the third clause: the flag is still in the conclusion. The box earns $\forall x (x \leq x+1)$, with **every** occurrence of the flag replaced: true, and about no single M . The name M was chosen after x , so it may depend on x , which is the dependence Section 5.1 read off the order of quantifiers. With Section 5.5’s rule, $\exists I$ inside the box and then $\forall I$ give “every real number has one at least as large”, with the quantifiers in the safe order.

5.4.6 The rule card

Example. Instances of $\forall I$

In each row the box’s flag is in no premise and no open assumption.

flag	the box’s last line	the new line
x_0	$P(f(x_0)) \vee \neg R(x_0, c)$	$\forall x (P(f(x)) \vee \neg R(x, c))$
ε_0	$\varepsilon_0 > 0 \rightarrow \exists \delta (\delta > 0 \wedge \delta < \varepsilon_0)$	$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists \delta (\delta > 0 \wedge \delta < \varepsilon))$
y_0	$\forall x (R(x, y_0) \rightarrow R(x, g(y_0)))$	$\forall y \forall x (R(x, y) \rightarrow R(x, g(y)))$

Not an instance. A box flagged x_0 ends in $R(x_0, x_0)$. The line $\forall x R(x, x_0)$ is not written by $\forall I$: it keeps the flag in its conclusion (the third clause). The rule puts x in place of **every** occurrence of the flag: $\forall x R(x, x)$.

Chapters 1 and 2’s thirteen rules are on the card at the end of Section 3.1; here are this chapter’s so far.

Rule	Shape	Side condition	You write	Lean
$\forall E$	$\forall x \varphi \implies \varphi[t/x]$	t free for x in φ	“In particular, for $x = t, \dots$ ”; “applying the hypothesis to $t, \dots$ ”	<code>h t</code> (or <code>specialize h t</code>)
$\forall I$	$\text{box } [x_0] \dots$ $\varphi[x_0/x] \implies \forall x \varphi$	x_0 in no premise, in no open assumption, not in $\forall x \varphi$	“Let x be arbitrary. ... Since x was arbitrary, ...”	<code>intro x</code>

Other forms of the opening phrase are “let $x \in \mathbb{R}$ ”, “fix $\varepsilon > 0$ ”, “take any $n \in \mathbb{N}$ ”; the closing phrase is often left out, since the reader knows that a proof which began by fixing x proves a statement about every x . In Lean, `intro x` turns the goal `forall x, P x` into the goal `P x` with a new name `x` in context, about which nothing is assumed: Lean’s version of opening the flagged box. The same tactic opened the $\rightarrow I$ box in Section 1.4, and the two boxes are, to Lean, one kind of thing.

In the Givens/Goal table, $\forall I$ puts a fresh name among the Givens and makes $\varphi[x_0/x]$ the goal.

Warning (Not allowed: $\forall I$). Each step below is wrong; the tag names the clause of the side condition of [Definition 5.4.1](#) that it breaks.

- Generalize over a flag that a premise mentions: the first clause.
- Generalize over a flag that an open assumption mentions: the second clause (the pair in “Why each clause”).
- Write a conclusion that still mentions the flag: the third clause ([Example \(A Proof to Check\)](#)).
- Cite a line inside a flagged box that has closed: the scope rule of [Definition 1.4.2](#), as for every box.

5.4.7 The search method: one more step

- **A goal $\forall x \varphi$.** Open a flagged box with a fresh name x_0 , one that appears in no premise, in no open assumption, and not in the goal. The new goal, the box’s last line, is $\varphi[x_0/x]$.

This step is **safe**, in the sense of Section 3.1. If $\forall x \varphi$ can be derived at all, then so can $\varphi[x_0/x]$, by one more $\forall E$ at x_0 ; so opening the box never closes off a derivation that would otherwise exist. Take the step as soon as the goal is universal, and choose a flag that appears nowhere yet, so that the side condition cannot fail.

Remark (Looking ahead). Chapter 8 collects the errors of this section, with others, in a catalogue of steps a proof may not take.

5.4.8 Exercises

5.4.8.1 A. Check your understanding

Exercise. A1

Three derivations, D1, D2 and D3, are used in part (b).

D1:

1		$\forall x (P(x) \rightarrow Q(x))$	
2		$P(x_0)$	
3		x_0 $P(x_0) \rightarrow Q(x_0)$	$\forall E$ 1
4		$Q(x_0)$	$\rightarrow E$ 3, 2
5		$\forall x Q(x)$	$\forall I$ 3–4

D2:

1		$\forall x \forall y R(x, y)$	
2		$\forall y R(x_0, y)$	$\forall E$ 1
3		x_0 $R(x_0, x_0)$	$\forall E$ 2
4		$\forall x R(x, x)$	$\forall I$ 3–3

D3:

1		$R(x_0, c)$	asm, for $\rightarrow I$
2		x_0 $R(x_0, c)$	R 1
3		$\forall x R(x, c)$	$\forall I$ 2–2
4		$R(x_0, c) \rightarrow \forall x R(x, c)$	$\rightarrow I$ 1–3

- Write the schema of $\forall I$ with its side condition.
- For each of D1, D2, D3, decide whether its $\forall I$ line is legal. If not, name the clause of the side condition that fails.
- True or false, with a reason: “fresh” means that x_0 appears nowhere in the derivation outside its flagged box.
- In the derivation of the $(\vdash)$ half of [Proposition 5.4.1](#), the name x_0 flags two boxes. Why does the side condition allow this?

Solution.

- A box flagged x_0 , with no assumption, from line i to line j , whose last line j is $\varphi[x_0/x]$, a line of the flagged box itself; then line k : $\forall x \varphi$, justified “ $\forall I$ i – j ”. At line k the name x_0 appears in no premise, in no assumption of a box still open at line k , and not in $\forall x \varphi$ ([Definition 5.4.1](#)).
- D1: not legal. At line 5 the flag x_0 appears in the premise on line 2 (the first clause). Over $\mathbb{Z}$, with $P(x)$ and $Q(x)$ both read as “ x is even” and x_0 naming 2, both premises

are true and line 5, “every integer is even”, is false.

D2: legal. Line 2 mentions x_0 outside the box, and that is allowed: it is neither a premise nor an assumption, and the conclusion $\forall x R(x, x)$ does not contain x_0 . Line 3 cites line 2, which is available, since no box contains it. (It is also correct in meaning: line 2 came from a premise that says nothing about x_0 , so it holds whatever x_0 names.)

D3: not legal. At line 3 the assumption $R(x_0, c)$ on line 1 is still open, and x_0 appears in it (the second clause). Line 4, which closes the box legally, then says “if x_0 is below c , everything is below c ”: false over $\mathbb{Z}$ with R read as $<$, c as 5 and x_0 as 1.

- (c) False. The side condition asks about the premises, the assumptions of boxes still open at the $\forall I$ line, and the conclusion of that line. Other lines may mention x_0 , as line 2 of *D2* does, and a closed flagged box may share its name with a later one.
- (d) At line 7, which closes the second box, the side condition asks whether x_0 appears in the premise (no), in an assumption of an open box (no box is open), and in $\forall x Q(x)$ (no). The first box’s lines mention x_0 , but they are neither premises nor open assumptions, and the box closed at line 3, so its lines are not even available at line 7.

Exercise. A2: Drill

Each part takes a line or two. The parts run from single steps, through a missing line, a justification column and a judgement on one step, to whole derivations.

- (a) What does opening a flagged box say, in the words of a proof?
- (b) A box flagged x_0 ends in $P(x_0) \wedge Q(x_0)$, and x_0 is in no premise and no open assumption. Which line may $\forall I$ write?
- (c) Line 9 is $\forall x (P(x) \vee Q(f(x)))$, justified “ $\forall I$ 4–8”, the box flagged u_0 . What is line 8?
- (d) Justify each line after the premise: 1 $\forall x (P(x) \wedge Q(x))$, premise; a box flagged x_0 holding 2 $P(x_0) \wedge Q(x_0)$, 3 $Q(x_0)$, 4 $P(x_0)$, 5 $Q(x_0) \wedge P(x_0)$; then 6 $\forall x (Q(x) \wedge P(x))$.
- (e) The premise is $R(x_0)$. May a box flagged x_0 that ends in $R(x_0)$ be closed with $\forall x R(x)$? If not, which clause does it break?
- (f) May a flag be the constant c ?
- (g) A box flagged y_0 ends in $y_0 \leq y_0$. Is $\forall x (x \leq y_0)$ a legal $\forall I$ line? If not, which clause does it break?
- (h) Inside the box of an assumption $Q(x_0)$, a box flagged x_0 ends in $Q(x_0)$. May $\forall I$ close it? If not, which clause does it break?
- (i) Derive $\forall x P(x) \vdash \forall y P(y)$.
- (j) Derive $\forall x (P(x) \wedge Q(x)) \vdash \forall x Q(x)$.
- (k) Derive $\forall x P(x), \forall x Q(x) \vdash \forall x (Q(x) \wedge P(x))$.
- (l) Derive $\vdash \forall x (P(x) \rightarrow P(x))$.

Solution.

- (a) “Let x_0 be arbitrary.”
- (b) $\forall x (P(x) \wedge Q(x))$.
- (c) $P(u_0) \vee Q(f(u_0))$, the instance at the flag.
- (d) 2 $\forall E$ 1 (at x_0); 3 $\wedge E$ 2; 4 $\wedge E$ 2; 5 $\wedge I$ 3, 4; 6 $\forall I$ 2–5.

- (e) No, the first clause: x_0 appears in a premise.
- (f) No: a flag is always a variable.
- (g) No, the third clause: y_0 appears in the conclusion.
- (h) No, the second clause: x_0 appears in an open assumption.
- (i) A box flagged y_0 : 2 $P(y_0)$, $\forall E$ 1; then 3 $\forall y P(y)$, $\forall I$ 2–2.
- (j) A box flagged x_0 : 2 $P(x_0) \wedge Q(x_0)$, $\forall E$ 1; 3 $Q(x_0)$, $\wedge E$ 2; then 4 $\forall x Q(x)$, $\forall I$ 2–3.
- (k) A box flagged x_0 : 3 $Q(x_0)$, $\forall E$ 2; 4 $P(x_0)$, $\forall E$ 1; 5 $Q(x_0) \wedge P(x_0)$, $\wedge I$ 3, 4; then 6 $\forall x (Q(x) \wedge P(x))$, $\forall I$ 3–5.
- (l) A box flagged x_0 , and inside it the box of 1 $P(x_0)$; 2 $P(x_0) \rightarrow P(x_0)$, $\rightarrow I$ 1–1; then 3 $\forall x (P(x) \rightarrow P(x))$, $\forall I$ 1–2 (cited from line 1, the convention for a flagged box that immediately contains another).

5.4.8.2 B. Practice

Exercise. B1: A chain, for every x

A search for $\forall x (P(x) \rightarrow Q(x))$, $\forall x (Q(x) \rightarrow R(x)) \vdash \forall x (P(x) \rightarrow R(x))$ has reached the stage below. *Name the move, then complete it, and check the side condition at line 9.*

1		$\forall x (P(x) \rightarrow Q(x))$	
2		$\forall x (Q(x) \rightarrow R(x))$	
3		x_0	
			$P(x_0)$ asm, for $\rightarrow I$
			$\vdots$
8			$P(x_0) \rightarrow R(x_0)$?
9		$\forall x (P(x) \rightarrow R(x))$	?

Solution. The moves are *let x be arbitrary* and then the $\rightarrow I$ box; inside, instantiate both premises at x_0 and fire twice.

The goal is universal, so open a box flagged x_0 ; the new goal $P(x_0) \rightarrow R(x_0)$ is an implication, so open a $\rightarrow I$ box on $P(x_0)$; the new goal is $R(x_0)$. Now Section 5.3’s exercise B1 runs inside the boxes, with x_0 for a .

1		$\forall x (P(x) \rightarrow Q(x))$	
2		$\forall x (Q(x) \rightarrow R(x))$	
3			
3	x_0		$P(x_0)$ asm, for $\rightarrow$ I
4			$P(x_0) \rightarrow Q(x_0)$ $\forall$ E 1
5			$Q(x_0)$ $\rightarrow$ E 4, 3
6			$Q(x_0) \rightarrow R(x_0)$ $\forall$ E 2
7			$R(x_0)$ $\rightarrow$ E 6, 5
8			$P(x_0) \rightarrow R(x_0)$ $\rightarrow$ I 3–7
9			$\forall x (P(x) \rightarrow R(x))$ $\forall$ I 3–8

The flagged box immediately contains the $\rightarrow$ I box, so line 9 cites it from line 3. At line 9: x_0 appears in neither premise; the assumption $P(x_0)$ mentions x_0 , but its box closed at line 7, so it is not open at line 9; and x_0 does not appear in $\forall x (P(x) \rightarrow R(x))$. The order of the boxes matters: the $\rightarrow$ I box is inside the flagged box and closes first.

Exercise. B2: Contraposition under a quantifier, used later

- (a) Give a derivation of $\forall x (P(x) \rightarrow Q(x)) \vdash \forall x \neg Q(x) \rightarrow \forall x \neg P(x)$.
- (b) Try to derive the converse, $\forall x \neg Q(x) \rightarrow \forall x \neg P(x) \vdash \forall x (P(x) \rightarrow Q(x))$. Say where the search stalls, and which side condition stops it.

Solution.

- (a) The goal is an implication: a $\rightarrow$ I box on $\forall x \neg Q(x)$, with goal $\forall x \neg P(x)$. That is universal: a box flagged x_0 , with goal $\neg P(x_0)$. That is a negation: a $\neg$ I box on $P(x_0)$, with goal $\perp$. Inside, instantiate both universal lines at x_0 and look for a clash: $P(x_0)$ and the instance of the premise give $Q(x_0)$, and the assumption $\forall x \neg Q(x)$ at x_0 gives $\neg Q(x_0)$.

1	$\forall x (P(x) \rightarrow Q(x))$			
2		$\forall x \neg Q(x)$		asm, for $\rightarrow$ I
3	x_0	$P(x_0)$		asm, for $\neg$ I
4		$P(x_0) \rightarrow Q(x_0)$		$\forall$ E 1
5		$Q(x_0)$		$\rightarrow$ E 4, 3
6		$\neg Q(x_0)$		$\forall$ E 2
7		$\perp$		$\neg$ E 5, 6
8		$\neg P(x_0)$		$\neg$ I 3–7
9		$\forall x \neg P(x)$		$\forall$ I 3–8
10	$\forall x \neg Q(x) \rightarrow \forall x \neg P(x)$			$\rightarrow$ I 2–9

At line 9, the open box is the $\rightarrow$ I box of line 2, whose assumption $\forall x \neg Q(x)$ does

not contain x_0 ; the premise and the conclusion do not contain it either. So the side condition holds, even though the flagged box sits inside an open box.

- (b) The goal is universal, so open a box flagged x_0 , then a $\rightarrow$ I box on $P(x_0)$, with goal $Q(x_0)$. The only premise is an implication, $\forall x \neg Q(x) \rightarrow \forall x \neg P(x)$. Using it needs its hypothesis $\forall x \neg Q(x)$, which nothing supplies; and its conclusion, $\forall x \neg P(x)$, would clash with $P(x_0)$ and close the case by $\perp$ E. So everything turns on deriving $\forall x \neg Q(x)$. The goal $Q(x_0)$ is atomic and nothing produces it, so the method’s last resort is an RAA box on $\neg Q(x_0)$. Inside, the available facts about Q are $\neg Q(x_0)$, and to reach $\forall x \neg Q(x)$ from it we would close a flagged box over x_0 . The side condition forbids it: x_0 appears in the open assumptions $P(x_0)$ and $\neg Q(x_0)$. A fresh name y_0 does not help either, since nothing tells us anything about $\neg Q(y_0)$. The search stalls at the second clause of the side condition.

A stalled search proves nothing by itself; a cleverer derivation might exist. Section 7.2 settles the question with a structure in which the premise is true and the conclusion false, and shows that the converse does not follow.

Exercise. B3: A number is at most its absolute value, for every number

Example (A number is at most its absolute value) (Section 1.6) derived $|x| > x \vee |x| = x$ for one fixed real x from three premises about that x : $x \geq 0 \vee x < 0$, $x \geq 0 \rightarrow |x| = x$, and $x < 0 \rightarrow |x| > x$. Turn it into a derivation of

$$\forall x (|x| > x \vee |x| = x)$$

from premises that are universal statements. Explain why the premises of Example (A number is at most its absolute value), copied with x_0 in place of x , will not do.

Solution. The goal is universal, so the derivation of Section 1.6 must run inside a box flagged x_0 . Its three premises mention the number it is about. So here the premises are the three facts **for every** x , and inside the box $\forall$ E at x_0 gives each fact about x_0 .

1	$\forall x (x \geq 0 \vee x < 0)$	
2	$\forall x (x \geq 0 \rightarrow x = x)$	
3	$\forall x (x < 0 \rightarrow x > x)$	
4	$x_0 \mid x_0 \geq 0 \vee x_0 < 0$	$\forall E$ 1
5	$x_0 \geq 0 \rightarrow x_0 = x_0$	$\forall E$ 2
6	$x_0 < 0 \rightarrow x_0 > x_0$	$\forall E$ 3
7	$\mid x_0 \geq 0$	asm, for $\forall E$
8	$\mid x_0 = x_0$	$\rightarrow E$ 5, 7
9	$\mid x_0 > x_0 \vee x_0 = x_0$	$\vee I$ 8
10	$\mid x_0 < 0$	asm, for $\forall E$
11	$\mid x_0 > x_0$	$\rightarrow E$ 6, 10
12	$\mid x_0 > x_0 \vee x_0 = x_0$	$\vee I$ 11
13	$\mid x_0 > x_0 \vee x_0 = x_0$	$\forall E$ 4, 7–9, 10–12
14	$\forall x (x > x \vee x = x)$	$\forall I$ 4–13

Lines 7–13 are the derivation of Example (A number is at most its absolute value) with x_0 for x , its premises replaced by lines 4–6. At line 14, x_0 appears in none of the premises, no box is open (the two case boxes closed at lines 9 and 12), and x_0 is not in the conclusion. The derivation has 14 lines.

If instead the premises were $x_0 \geq 0 \vee x_0 < 0$, $x_0 \geq 0 \rightarrow |x_0| = x_0$ and $x_0 < 0 \rightarrow |x_0| > x_0$, the box could reach line 13 just as well, but line 14 would break the first clause: x_0 would appear in a premise. And rightly so. Premises about x_0 are facts about one number, and a derivation from them proves something about that number only. The universal premises are true facts about every real number, and that is what entitles the conclusion to speak about every real number.

Exercise. B4: Back to Exercise C2 of Section 5.3

- Derive $\forall x P(x) \vdash \forall x P(x)$ with $\forall E$ and $\forall I$, not R. Why can the flagged box not be flagged x ?
- Explain why no flag at all gives a derivation of $P(x) \vdash \forall x P(x)$ by this route, and give a counter-reading to the sequent.

Solution.

	1	$\mid \forall x P(x)$	
(a)	2	$x_0 \mid P(x_0)$	$\forall E$ 1
	3	$\mid \forall x P(x)$	$\forall I$ 2–2

With the flag x , the conclusion $\forall x P(x)$ contains x (the letter after the quantifier

counts: that is what “appear” means), so the third clause fails, whatever else holds.

- (b) With the flag x , the line $P(x)$ is available, but x appears in the premise and in the conclusion. With any other flag x_0 , the box needs the line $P(x_0)$, and the search finds nothing that gives it: the premise is about x , and no line mentions x_0 . A stalled search proves nothing by itself, since a cleverer derivation might exist; the counter-reading settles it. Over $\mathbb{Z}$, read P as “even” and let x name 2: the premise is true and $\forall x P(x)$ is false. (Section 7.2 makes this a counter-model.)

Marking scheme. (10 points) - 3: the derivation of (a). - 3: the flag x fails “not in the conclusion”, with the reason. - 2: (b) both cases of the flag. - 2: the counter-reading, with the value of x . Fatal errors (0 for the step): claiming the flag x is fine because x is bound in the conclusion; treating $P(x)$ and $P(x_0)$ as the same line.

Exercise. B5: Derive or break: properties of a relation, used later

A two-place predicate R is **reflexive** if $\forall x R(x, x)$, **irreflexive** if $\forall x \neg R(x, x)$, **symmetric** if $\forall x \forall y (R(x, y) \rightarrow R(y, x))$, **asymmetric** if $\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$, and **transitive** if $\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$. Read each claim with these formulas as its premises and conclusion, and derive it, or break it with a counter-reading. Then say which two claims are converses, and which claim repairs which.

- (a) asymmetric $\vdash$ irreflexive.
- (b) irreflexive $\vdash$ asymmetric.
- (c) irreflexive, transitive $\vdash$ asymmetric.
- (d) symmetric, transitive $\vdash$ reflexive.
- (e) symmetric, transitive, $\forall x R(x, f(x)) \vdash$ reflexive.
- (f) reflexive $\vdash$ symmetric.

Solution.

- (a) Derive. The goal is universal, so a flagged box; its goal $\neg R(x_0, x_0)$ is a negation, so a $\neg$ I box; the premise at x_0 , twice, gives the clash.

1	$\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$			
2	x_0	$R(x_0, x_0)$		asm, for $\neg$ I
3		$\forall y (R(x_0, y) \rightarrow \neg R(y, x_0))$		$\forall$ E 1
4		$R(x_0, x_0) \rightarrow \neg R(x_0, x_0)$		$\forall$ E 3
5		$\neg R(x_0, x_0)$		$\rightarrow$ E 4, 2
6		$\perp$		$\neg$ E 2, 5
7		$\neg R(x_0, x_0)$		$\neg$ I 2–6
8	$\forall x \neg R(x, x)$			$\forall$ I 2–7

At line 8, x_0 is not in the premise, no box is open, and x_0 is not in $\forall x \neg R(x, x)$.

- (b) Break. The domain $\{0, 1\}$, R true of $(0, 1)$ and $(1, 0)$ only. No element is related to itself; $R(0, 1)$ and $R(1, 0)$ both hold.

- (c) Derive: two flags, then an $\rightarrow$ I box, then a $\neg$ I box. Transitivity at x_0, y_0, x_0 turns the two assumptions into $R(x_0, x_0)$, which irreflexivity refutes.

1	$\forall x \neg R(x, x)$			
2	$\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$			
3	x_0	y_0	$R(x_0, y_0)$	asm, for $\rightarrow$ I
4			$R(y_0, x_0)$	asm, for $\neg$ I
5			$\forall y \forall z ((R(x_0, y) \wedge R(y, z)) \rightarrow R(x_0, z))$	$\forall$ E 2
6			$\forall z ((R(x_0, y_0) \wedge R(y_0, z)) \rightarrow R(x_0, z))$	$\forall$ E 5
7			$(R(x_0, y_0) \wedge R(y_0, x_0)) \rightarrow R(x_0, x_0)$	$\forall$ E 6
8			$R(x_0, y_0) \wedge R(y_0, x_0)$	$\wedge$ I 3, 4
9			$R(x_0, x_0)$	$\rightarrow$ E 7, 8
10			$\neg R(x_0, x_0)$	$\forall$ E 1
11			$\perp$	$\neg$ E 9, 10
12			$\neg R(y_0, x_0)$	$\neg$ I 4–11
13			$R(x_0, y_0) \rightarrow \neg R(y_0, x_0)$	$\rightarrow$ I 3–12
14			$\forall y (R(x_0, y) \rightarrow \neg R(y, x_0))$	$\forall$ I 3–13
15			$\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$	$\forall$ I 3–14

At line 14, y_0 is in no premise, the only open box (flagged x_0) has no assumption, and y_0 is not in the conclusion; at line 15 the same holds for x_0 , with no box open.

- (d) Break. The domain $\{0, 1\}$, R true of $(0, 0)$ only. R is symmetric and transitive (the only pair is $(0, 0)$), and $R(1, 1)$ fails.
- (e) Derive. Inside the box, the partner $f(x_0)$ plays the part that b played in Exercise C1 of Section 5.3.

1	$\forall x \forall y (R(x, y) \rightarrow R(y, x))$	
2	$\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$	
3	$\forall x R(x, f(x))$	
4	$x_0 \mid R(x_0, f(x_0))$	$\forall E 3$
5	$\forall y (R(x_0, y) \rightarrow R(y, x_0))$	$\forall E 1$
6	$R(x_0, f(x_0)) \rightarrow R(f(x_0), x_0)$	$\forall E 5$
7	$R(f(x_0), x_0)$	$\rightarrow E 6, 4$
8	$\forall y \forall z ((R(x_0, y) \wedge R(y, z)) \rightarrow R(x_0, z))$	$\forall E 2$
9	$\forall z ((R(x_0, f(x_0)) \wedge R(f(x_0), z)) \rightarrow R(x_0, z))$	$\forall E 8$
10	$(R(x_0, f(x_0)) \wedge R(f(x_0), x_0)) \rightarrow R(x_0, x_0)$	$\forall E 9$
11	$R(x_0, f(x_0)) \wedge R(f(x_0), x_0)$	$\wedge I 4, 7$
12	$R(x_0, x_0)$	$\rightarrow E 10, 11$
13	$\forall x R(x, x)$	$\forall I 4-12$

At line 13, x_0 is in no premise, no box is open, and x_0 is not in the conclusion.

- (f) Break. The domain $\{0, 1\}$, R true of $(0, 0)$, $(1, 1)$ and $(0, 1)$. R is reflexive; $R(0, 1)$ holds and $R(1, 0)$ does not.
- (g) and (b) are converses, and only (a) derives. (d) is the trap “symmetric and transitive give reflexive”, which breaks at an element related to nothing, and (e) repairs it by giving every element a partner $f(x)$ (Exercise C1 of Section 5.3 shows the same for one element that has a partner).

Marking scheme. (10 points) - 2 each for (a), (c), (e): flags opened in the order read off the goal, every $\forall E$ at the flag (or at $f(x_0)$) the box needs, the side condition checked once at the last $\forall I$. - 1 each for (b), (d), (f): a counter-reading with the domain and every pair of R listed, each premise and the conclusion checked. - 1: the closing sentence. Fatal errors (0 for the claim): a $\forall I$ over a flag that an open assumption mentions; a counter-reading that is not symmetric (in (d)) or not reflexive (in (f)).

5.4.8.3 C. Going deeper

Exercise. C1: How fresh is fresh?

- (a) Give a derivation of $\forall x \forall y R(x, y) \vdash R(x_0, x_0) \wedge \forall x R(x, x)$ in which the line $R(x_0, x_0)$ is written **before** a flagged box that uses the same name x_0 . Check every clause of the side condition at the $\forall I$ line.
- (b) A stronger condition is sometimes proposed: “ x_0 occurs nowhere in the derivation outside its box”. Does your derivation meet it? Explain why the conclusion of your derivation is nonetheless correct, so that the stronger condition forbids more than it needs to.
- (c) Could the last line of the flagged box in (a) be obtained by R, repeating the earlier line $R(x_0, x_0)$? Is the result a derivation?

Solution (Hint 1). The goal is a conjunction. Derive the left conjunct first, by two $\forall E$ lines outside any box.

Solution (Hint 2). For the right conjunct, open a box flagged x_0 and instantiate the premise at x_0 again inside it. At the $\forall I$ line, list the premises, the open assumptions and the conclusion, and look for x_0 in each.

Solution.

1		$\forall x \forall y R(x, y)$	
2		$\forall y R(x_0, y)$	$\forall E$ 1
3		$R(x_0, x_0)$	$\forall E$ 2
(a) 4	x_0	$\forall y R(x_0, y)$	$\forall E$ 1
5		$R(x_0, x_0)$	$\forall E$ 4
6		$\forall x R(x, x)$	$\forall I$ 4–5
7		$R(x_0, x_0) \wedge \forall x R(x, x)$	$\wedge I$ 3, 6

The side condition at line 6. *Premises:* the only premise is line 1, $\forall x \forall y R(x, y)$, and x_0 does not appear in it. *Open assumptions:* no box is open at line 6, so there are none. *Conclusion:* $\forall x R(x, x)$ does not contain x_0 . The last line of the box, line 5, is $R(x, x)[x_0/x]$ and is a line of the box itself. So line 6 is legal, and the derivation has 7 lines. Lines 2, 3 and 7 mention x_0 outside the box, and the side condition does not ask about them.

- (b) No: lines 2, 3 and 7 mention x_0 outside its box. Yet the conclusion is correct. The reason the side condition looks only at premises and open assumptions is that these are the only lines taken without proof, so they are the only lines that can single out what x_0 names; every other available line is derived from them. Lines 2 and 3 were derived from a premise that does not mention x_0 ; they hold whatever x_0 names. So the argument in the box really is about an arbitrary object, and $\forall x R(x, x)$ follows. The stronger condition would reject this derivation, although renaming the box’s flag to a new letter z_0 (lines 4–5 becoming $\forall y R(z_0, y)$ and $R(z_0, z_0)$) gives a derivation that meets it and has the same last line. A condition that rejects a correct derivation for the spelling of a name forbids more than safety needs.
- (c) Yes. R may copy into the box any line available there, and line 3 lies in no box, so it is available at every later line. With line 5 justified “R 3” (and line 4 deleted), the box is a single line $R(x_0, x_0)$, and the side condition at the $\forall I$ line is checked exactly as in (a): nothing in it looks at where the box’s last line came from. The derivation is legal, and it is correct for the reason in (b): line 3 was derived from a premise that says nothing about x_0 .

Marking scheme. (10 points) - 2: the left conjunct by two $\forall E$ lines at x_0 , outside any box. - 2: a flagged box with $\forall E$ at the flag and $\forall I$ citing it. - 2: the side condition checked clause by clause: premise, open assumptions, conclusion. - 2: (b) explains that lines outside the box that do not rest on premises or assumptions about x_0 cannot make x_0 special. Alternatively, 2 for showing that renaming the box’s flag to a new letter gives a derivation that meets the

stronger condition and has the same last line. - 2: (c) correct answer with the availability check for the repeated line. Fatal errors (0 for the step): claiming that line 3 makes line 6 illegal; checking the side condition against lines that are not premises, open assumptions or the conclusion.

Exercise. C2: Find the error

A student offers this proof.

Claim. For every real number x , $x^2 - 2x + 1 \geq 0$.

Proof. (1) Let x be a real number. (2) At $x = 0$: $0 - 0 + 1 = 1 \geq 0$. (3) At $x = 1$: $1 - 2 + 1 = 0 \geq 0$. (4) At $x = 2$: $4 - 4 + 1 = 1 \geq 0$. (5) So $x^2 - 2x + 1 \geq 0$, and since x was arbitrary, the claim holds for every real x .

- Is the claim true?
- Does the argument prove it? Name the step that fails and the rule it misuses.
- Show that the step can lead to a false conclusion: give a polynomial g for which the same checks at $x = 0, 1, 2$ succeed and $g(x) \geq 0$ fails for some real x .
- Point to a correct proof.

Solution (Hint 1). Write lines (2)–(4) as formulas. Which terms do they mention?

Solution (Hint 2). Write the student’s proof as a derivation: open a box flagged x_0 for line (1), and write lines (2)–(4) as lines inside it. What would line (5) have to cite?

Solution.

- Yes. For every real x , $x^2 - 2x + 1 = (x - 1)^2 \geq 0$, since a square is never negative.
- No. Lines (2)–(4) are true, and line (1) opens a flagged box, “let x be arbitrary”. The error is in line (5). With φ the formula $x^2 - 2 \cdot x + 1 \geq 0$, lines (2)–(4) establish $\varphi[0/x]$, $\varphi[1/x]$ and $\varphi[2/x]$: statements about the constants 0, 1, 2. None of them is $\varphi[x_0/x]$ for the flag. Inside the box, nothing about the arbitrary x was ever proved. The first sentence of (5), “so $x^2 - 2x + 1 \geq 0$ ”, has the right form: it is $\varphi[x_0/x]$, with the student’s x as the flag. But nothing the student wrote yields it: lines (2)–(4) are the only lines it rests on, and no rule takes $\varphi[0/x]$, $\varphi[1/x]$ and $\varphi[2/x]$ to $\varphi[x_0/x]$. So the closing “since x was arbitrary” applies $\forall I$ to a box whose last line was never derived: lines (2)–(4) are all instances at constants, not the instance at the flag. Line (1) was idle: the argument never used the x it introduced.
- Take $g(x) = x(x - 1)(x - 2)$. The same checks succeed: $g(0) = 0 \geq 0$, $g(1) = 0 \geq 0$, $g(2) = 0 \geq 0$. But $g(-1) = (-1)(-2)(-3) = -6 < 0$, so “ $g(x) \geq 0$ for every real x ” is false. So the pattern of (5), concluding “for every x ” from checks at a few constants, can lead from true checks to a false conclusion. It is not an instance of $\forall I$, as (b) showed. (A legal step never leads from true premises to a false conclusion; Section 7.2 states this.) The claim in (a) is true for a different reason, one that the argument never gave.

- (d) The derivation under “A first flagged box” in this section: inside the box flagged x_0 , $\forall E$ applies “squares are not negative” at $x_0 - 1$, an arith line rewrites it as $x_0^2 - 2x_0 + 1 \geq 0$, and that line **is** $\varphi[x_0/x]$, so $\forall I$ applies.

Marking scheme. (10 points) - 2: (a) the claim is true, with a reason (it is a square). - 3: (b) names step (5) and $\forall I$; says that lines (2)–(4) are instances at constants, not the instance at the flag. - 3: (c) a polynomial that passes the three checks and fails somewhere, with both computations shown. - 2: (d) a correct proof, generalizing over a statement proved about the arbitrary x . Fatal errors (0 for the step): answering (b) with “the claim is false”; a “counterexample” in (c) that fails one of the three checks or is non-negative everywhere.