

4.5 Tautologies, Entailment and Equivalence

Section 4.4 gave every formula a value under every valuation, and collected the values in a truth table. One formula at a time, though, is not what Chapters 1 and 2 were about. They were about **sequents**: premises on the left, a conclusion on the right, and the claim that the conclusion can be derived. This section defines the counterpart of a sequent in terms of truth, $\Gamma \models \psi$: every valuation that makes the premises true makes the conclusion true. On the way it names the formulas that are true in every row, and the pairs of formulas that are true in the same rows. It then proves two lemmas about changing a formula, by replacing one part or by putting formulas in for its atoms, that later sections use, and ends by sorting the section's questions by whether one row or every row settles them, and by setting $\models$ beside $\vdash$.

4.5.1 Tautologies and contradictions

Some tables of Section 4.4 had a last column made entirely of T: the formula $((p \rightarrow q) \wedge p) \rightarrow q$ of Example (The truth table of modus ponens), and the biconditional of Example (An implication and its contrapositive). Their truth does not depend on the situation. One table in Exercise A1 there, that of $p \leftrightarrow \neg p$, was all F. These extremes deserve names.

A tautology is true for reasons of form alone: whatever the atoms mean, it comes out true.

Definition 4.5.1. Tautology and Contradiction

A formula φ is a **tautology** if $v(\varphi) = \text{T}$ for **every** valuation v . It is a **contradiction** if $v(\varphi) = \text{F}$ for **every** valuation v .

By Lemma 4.4.1, the value of φ under a valuation depends only on the row the valuation gives to the atoms of φ . So φ is a tautology exactly when its truth table has T in every row, and a contradiction exactly when it has F in every row. Infinitely many valuations, finitely many rows to check.

Examples, simplest first.

- $p \rightarrow p$ and $p \vee \neg p$ are tautologies: in the row $p = \text{T}$ both are true, and in the row $p = \text{F}$, $p \rightarrow p$ has a false hypothesis and $\neg p$ is true.
- $p \wedge \neg p$ and $p \leftrightarrow \neg p$ are contradictions: in each row one of p , $\neg p$ is false, and the two sides of the biconditional differ.
- *Nonexample.* $p \rightarrow q$ is neither: it is T in the row TT and F in the row TF. Most formulas are like this. Changing one symbol of the tautology $p \rightarrow p$, to get $p \rightarrow q$, loses the property: the rows in which the hypothesis and conclusion differ are no longer excluded by their being the same atom. A formula that is true in some rows and false in others, such as $p \rightarrow q$, is called **contingent**: every formula is a tautology, a contradiction, or contingent.

🕒 **Second reading** · Formulas with no atoms

A formula with no atoms, such as $\perp$ or $\neg\perp$, has the same value under every valuation (its table has one row). So it is either a contradiction, like $\perp$, or a tautology, like $\neg\perp$. This degenerate

case is the only one in which “not a tautology” forces “a contradiction”.

Lists of formulas need a word too, because a list of premises can fail to be true together even when each premise alone can be true.

Definition 4.5.2. Satisfiable

A formula φ is **satisfiable** if **there is** a valuation v with $v(\varphi) = \text{T}$. A list of formulas $\Gamma = \varphi_1, \dots, \varphi_n$, where $n \geq 0$, is **satisfiable** if **there is** a valuation that makes **every** φ_i true. A formula or list that is not satisfiable is **unsatisfiable**.

In words: a formula is satisfiable if some row makes it true, and a list is satisfiable if some **single** row makes all its members true at once. So a formula is unsatisfiable exactly when it is a contradiction. The **empty list** ($n = 0$) is satisfiable: every valuation makes all of its members true, since it has none to check.

The list $p, \neg p$ is unsatisfiable, although each of its members is satisfiable: a row making p true makes $\neg p$ false. Satisfiability of a list is about a common row, not about each formula separately. And the list $p \rightarrow q, q \rightarrow r, r \rightarrow \neg p$ is satisfiable, by the row with p false and q, r true (the first two have true conclusions, and the third has conclusion $\neg p$, which is true); in fact p is false in every row that makes all three true.

Tautologies and unsatisfiable formulas are two sides of one coin. For every formula ψ ,

$$\psi \text{ is a tautology} \iff \neg\psi \text{ is unsatisfiable.}$$

Indeed, by (V2) of Definition 4.4.2, $v(\neg\psi) = \text{T}$ exactly when $v(\psi) = \text{F}$. So “no valuation makes $\neg\psi$ true” says “no valuation makes ψ false”, which is “every valuation makes ψ true”. Sections 4.6 and 4.9 use it to pass between the two kinds of question.

Remark. Satisfiable and consistent. Section 3.2 called a list Γ *inconsistent* if $\Gamma \vdash \perp$ (Definition 3.2.1), and *consistent* otherwise: a word about derivations. *Satisfiable* is its twin about truth: some row makes every member of the list true. Nothing yet connects the two words. Section 4.7 proves that a satisfiable list is consistent, and Section 4.8 the converse; until then we keep them apart. The counterexample set lives in both worlds: $\Gamma \vdash \psi$ exactly when $\Gamma, \neg\psi$ is inconsistent (Proposition 3.2.2 (b)), and $\Gamma \models \psi$ exactly when $\Gamma, \neg\psi$ is unsatisfiable (below).

Warning. “Not a tautology” does not mean “a contradiction”. The formula $p \rightarrow q$ is neither: some rows make it true and some false. To show a formula is **not** a tautology, one row making it false is enough; to show it **is** a contradiction, every row must make it false.

4.5.2 Semantic entailment

Now the counterpart of a sequent. A derivation from premises Γ to a conclusion ψ is a promise that ψ follows from Γ . In terms of truth, “follows” should mean: no situation makes the premises true and the conclusion false. We want to prove, in Sections 4.7 and 4.8, that the rules

of Chapters 1 and 2 derive exactly the sequents that pass this test; for that we need a precise name for the test.

An argument is valid when its conclusion is true in every row in which all its premises are true.

Definition 4.5.3. Semantic Entailment

Let $\Gamma = \varphi_1, \dots, \varphi_n$ be a list of formulas, where $n \geq 0$, and let ψ be a formula. We say Γ **semantically entails** ψ , and write

$$\Gamma \models \psi,$$

if **every** valuation that makes **all** of $\varphi_1, \dots, \varphi_n$ true also makes ψ true. We then also call the argument from Γ to ψ **valid**. We write $\varphi_1, \dots, \varphi_n \models \psi$ for $\Gamma \models \psi$, and $\models \psi$ when Γ is empty. If $\Gamma \models \psi$ fails, we write $\Gamma \not\models \psi$, and a valuation that makes every formula of Γ true and ψ false is a **refuting valuation**, or **refuting row**.

Clause by clause. “Every valuation”: the condition is checked in all situations, not in the actual one; a valid argument cannot be spoiled by what the atoms happen to mean. “That makes all of $\varphi_1, \dots, \varphi_n$ true”: rows in which some premise is false are ignored, whatever the conclusion does there. “Also makes ψ true”: in the remaining rows, the conclusion must hold. In symbols, for every valuation v ,

$$(v(\varphi_1) = \text{T and } \dots \text{ and } v(\varphi_n) = \text{T}) \Rightarrow v(\psi) = \text{T}.$$

And $\Gamma \not\models \psi$ says: **there is** a valuation making every premise true and the conclusion false. One refuting row is enough.

Checking it. By Lemma 4.4.1, whether a valuation makes the premises and the conclusion true depends only on its row on the atoms occurring in Γ and ψ . So $\Gamma \models \psi$ can be decided by one truth table on those atoms (Definition 4.4.3 allows extra atoms in the list): find the rows in which every premise is T, and look at the conclusion there.

First examples. Row 1 of Example (The truth table of modus ponens) is the only row making $p \rightarrow q$ and p true, and q is true there, so $p \rightarrow q, p \models q$. By Example (An implication and its contrapositive), $p \rightarrow q \models \neg q \rightarrow \neg p$ and $\neg q \rightarrow \neg p \models p \rightarrow q$, since the two columns are equal.

The empty list. When Γ is empty, every valuation makes “all of Γ ” true, so $\models \psi$ says that every valuation makes ψ true: $\models \psi$ holds exactly when ψ is a tautology.

Unsatisfiable premises. If Γ is unsatisfiable, no valuation makes all of Γ true, so the condition has no row to check and holds: $\Gamma \models \psi$ for **every** formula ψ . For example $p, \neg p \models q$ and $\perp \models q$. This is the table’s version of Proposition 2.1.1, $P, \neg P \vdash Q$, and of $\perp E$.

More premises. If $\Gamma \models \psi$, and Δ is a list containing every formula of Γ (and perhaps others), then $\Delta \models \psi$: a valuation making all of Δ true makes all of Γ true, and so makes ψ true. Adding premises never destroys an entailment. In the same way, the order of the premises and repetitions among them do not matter: only which formulas occur in the list does.

☉ Second reading · The bookkeeping of entailment

Four facts about $\models$ need nothing but the definition. *Reflexivity*: $\Gamma \models \varphi$ whenever φ is a member of Γ , since a row making all of Γ true makes φ true. *Weakening*: adding premises keeps an entailment, as just shown. *Order and repetition*: rearranging Γ , or repeating a member, changes

nothing. *Cut*: if $\Gamma \models \varphi$ and $\Gamma, \varphi \models \psi$, then $\Gamma \models \psi$. Indeed, let a row make every member of Γ true. By the first entailment it makes φ true, so it makes every member of Γ, φ true, and by the second it makes ψ true. Section 2.3 proved weakening and cut for $\vdash$ (Lemma 2.3.3) by pasting derivations, and its weakening, which asks only that every member of Γ be a member of Δ , covers order and repetition; reflexivity for $\vdash$ is the derivation whose only line is the premise φ . Here each fact is one line about rows. Section 4.8 sets the two lists side by side.

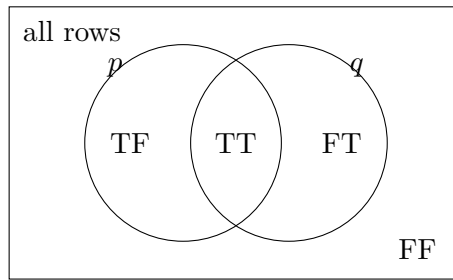
A test by satisfiability. Finally, for every list Γ and formula ψ ,

$$\Gamma \models \psi \iff \text{the list } \Gamma, \neg\psi \text{ is unsatisfiable.}$$

Both sides say that no valuation makes every formula of Γ true and ψ false, because $v(\neg\psi) = \text{T}$ exactly when $v(\psi) = \text{F}$. With Γ empty this is the equivalence between tautologies and unsatisfiable negations above.

The list $\Gamma, \neg\psi$ is the **counterexample set** of the claim (Definition 1.1.3): an argument is valid exactly when its counterexample set cannot be made true by any row.

Rows as regions. A table on p and q has four rows, and they can be drawn as the four regions of a picture.



Each row of the table on p, q is a region. A formula marks the rows where it is true: $p \wedge q$ the lens, $p \vee q$ both circles, $\neg p$ everything outside p 's circle. $\Gamma \models \psi$ says that the region where all of Γ is true lies inside the region of ψ : $p \wedge q \models p \vee q$.

Equivalent formulas mark the same region, a tautology marks the whole rectangle, and an unsatisfiable list marks nothing in common.

Quick check. Does $p \wedge q \models p \vee r$ hold? Does $p \vee r \models p \wedge q$?

Solution. The first holds: a row making $p \wedge q$ true makes p true, so $p \vee r$ is true. The second fails, for instance at $p = \text{F}, r = \text{T}$ (any q), which makes $p \vee r$ true and $p \wedge q$ false.

Example. Affirming the consequent is not valid

Show that $p \rightarrow q, q \not\models p$.

Solution. We need one row making both premises true and the conclusion false. The conclusion p must be F, and the premise q must be T. In the row $p = \text{F}$, $q = \text{T}$, the premise $p \rightarrow q$ is T, since its hypothesis is false. So this row makes $p \rightarrow q$ and q true and p false: it is a refuting row.

The pattern is the one of Section 1.3, “from $\varphi \rightarrow \psi$ and ψ , infer φ ”, which is not an instance of $\rightarrow\text{E}$. Section 1.1’s Quick check gave the row a face: with P : “ n is divisible by 4” and Q : “ n is even”, at $n = 2$ the premises $P \rightarrow Q$ and Q are true and P is false. It is also the step that would repair Argument B of Section 1.1: from “if n is even, then n^2 is even” and “ n^2 is even”, conclude “ n is even”. The conclusion of that instance happens to be true for every integer n , as Argument A proves; the refuting row shows that the **form** does not guarantee it, so the step proves nothing. Section 4.7 upgrades this row from “not valid” to “not derivable by any derivation”.

A valid argument with several premises is checked in the same way, and working backwards from F often settles it without a table.

Example. Two cases, two conclusions

Show that $p \vee q, p \rightarrow r, q \rightarrow s \models r \vee s$.

Solution. Suppose a valuation v makes all three premises true and the conclusion false; we derive a contradiction. Since $r \vee s$ is F, both r and s are F (V4). Since $p \rightarrow r$ is T and r is F, the hypothesis p is F (V5). In the same way $q \rightarrow s$ forces q to be F. But then $p \vee q$ is F, contradicting the first premise. So no refuting valuation exists, and the entailment holds.

The search replaced a table of $2^4 = 16$ rows by four forced steps. The entailment is the table’s picture of a proof by cases with $\vee\text{E}$: in the case p , the second premise gives r , and $\vee\text{I}$ gives $r \vee s$; in the case q , the third gives s , and $\vee\text{I}$ again.

Changing the grouping of a formula can change what it entails.

Example. One grouping entails the other, not conversely

Decide whether $(p \rightarrow q) \rightarrow r \models p \rightarrow (q \rightarrow r)$, and whether $p \rightarrow (q \rightarrow r) \models (p \rightarrow q) \rightarrow r$.

Solution. The first holds. Suppose a valuation makes $(p \rightarrow q) \rightarrow r$ true and $p \rightarrow (q \rightarrow r)$ false. The latter forces $p = \text{T}$ and $q \rightarrow r$ false, so $q = \text{T}$ and $r = \text{F}$. Then $p \rightarrow q$ is T, and $(p \rightarrow q) \rightarrow r$ has a true hypothesis and a false conclusion, so it is F, a contradiction.

The second fails. Take $p = \text{F}$ and $r = \text{F}$, with q either value. Then $p \rightarrow (q \rightarrow r)$ is T, since its hypothesis is false. But $p \rightarrow q$ is T for the same reason, so $(p \rightarrow q) \rightarrow r$ is $\text{T} \rightarrow \text{F}$, which is F. These two rows are refuting rows.

So the two groupings are not interchangeable, which is why Chapter 1 fixed a convention ($\rightarrow$ groups to the right) and why the convention matters.

Warning. $\models$ is not $\rightarrow$. The formula $\varphi \rightarrow \psi$ is a formula: it has a value in each row and can be a part of larger formulas. The statement $\varphi \models \psi$ is not a formula at all. It is a claim **about** two formulas, true or false once and for all, like $\varphi \vdash \psi$. They are related (Exercise B3 below asks you to prove that $\varphi \models \psi$ holds exactly when $\varphi \rightarrow \psi$ is a tautology), but writing $p \models q \models r$, or $\neg(p \models q)$ inside a formula, mixes the two levels and means nothing.

Quick check. Is $p, \neg p \models q$? Is $p \rightarrow q, \neg p \models \neg q$?

Solution. The first holds: the list $p, \neg p$ is unsatisfiable, so there is no row to check. The second fails: the row $p = \text{F}, q = \text{T}$ makes $p \rightarrow q$ true (false hypothesis) and $\neg p$ true, and $\neg q$ false. It is the pattern “denying the hypothesis”, the mirror image of affirming the consequent.

4.5.3 Semantic equivalence

Two formulas can be different strings and still say the same thing in every situation, as $p \rightarrow q$ and $\neg q \rightarrow \neg p$ did in Example (An implication and its contrapositive). Chapter 2 had a word for formulas that derive each other, provable equivalence (Definition 2.4.2). Here is the counterpart in terms of truth.

Two formulas are equivalent when they have the same truth table.

Definition 4.5.4. Semantic Equivalence

Formulas φ and ψ are **semantically equivalent**, written $\varphi \equiv \psi$, if $\varphi \models \psi$ **and** $\psi \models \varphi$.

In words, each entails the other. Equivalently, $v(\varphi) = v(\psi)$ for **every** valuation v . Indeed, if $\varphi \equiv \psi$ and $v(\varphi) = \text{T}$, then $v(\psi) = \text{T}$ by $\varphi \models \psi$; if $v(\varphi) = \text{F}$, then $v(\psi)$ cannot be T , since $\psi \models \varphi$ would make $v(\varphi) = \text{T}$. The converse is plain. By (V6) of Definition 4.4.2, “ $v(\varphi) = v(\psi)$ for every v ” says that $\varphi \leftrightarrow \psi$ is true under every v , so

$$\varphi \equiv \psi \iff \models \varphi \leftrightarrow \psi.$$

From “same value under every valuation” it follows at once that $\equiv$ is reflexive, symmetric, and transitive: $\varphi \equiv \varphi$; if $\varphi \equiv \psi$, then $\psi \equiv \varphi$; if $\varphi \equiv \psi$ and $\psi \equiv \chi$, then $\varphi \equiv \chi$.

Examples: $p \wedge q \equiv q \wedge p$ (the clause (V3) does not care about the order); $p \equiv \neg\neg p$ (two flips); $p \vee \neg p \equiv \neg\bot$ (both are true under every valuation). The degenerate case: any two tautologies are equivalent, and so are any two contradictions. *Nonexample:* $p \wedge q$ and p are not equivalent. One direction holds, $p \wedge q \models p$, but the other fails in the row $p = \text{T}, q = \text{F}$. Entailment in one direction is not equivalence. Nor is equivalence identity: $p \wedge q$ and $q \wedge p$ are different formulas, with different parse trees, and $\rightarrow\text{E}$ treats them as different (Section 1.3).

The equivalences used most often are collected below. They are stated for arbitrary formulas φ, ψ, χ , not just atoms, since Section 4.9 applies them to compound formulas.

Proposition 4.5.1. Standard Equivalences

For all formulas φ , ψ and χ :

- (a) (contrapositive) $\varphi \rightarrow \psi \equiv \neg\psi \rightarrow \neg\varphi$;
- (b) (De Morgan) $\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$ and $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$;
- (c) (implication as a disjunction) $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$;
- (d) (double negation) $\neg\neg\varphi \equiv \varphi$;
- (e) (biconditional) $\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$;
- (f) (commutativity) $\varphi \wedge \psi \equiv \psi \wedge \varphi$ and $\varphi \vee \psi \equiv \psi \vee \varphi$;
- (g) (distributivity) $\varphi \wedge (\psi \vee \chi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ and $\varphi \vee (\psi \wedge \chi) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \chi)$;
- (h) (associativity) $(\varphi \wedge \psi) \wedge \chi \equiv \varphi \wedge (\psi \wedge \chi)$ and $(\varphi \vee \psi) \vee \chi \equiv \varphi \vee (\psi \vee \chi)$.

Idea. For atoms these are truth-table checks. For arbitrary formulas the same tables work, with φ, ψ, χ in the place of atoms: under a fixed valuation, each of φ, ψ, χ has some value, and both sides are computed from those values by the clauses (V2)–(V6), exactly as in a table whose first columns are φ, ψ, χ .

Proof. Fix a valuation v , and let $a = v(\varphi)$, $b = v(\psi)$, $c = v(\chi)$. Each side of each equivalence is built from φ, ψ, χ by connectives, so by (V2)–(V6) of Definition 4.4.2 its value under v is computed from a, b, c by the clauses of those connectives. The combination (a, b) , or (a, b, c) , is one row of the tables below, whose first columns list all 4 (or 8) combinations. In each table the columns of the two sides of each equivalence agree in every row, as the reader can check column by column. Hence both sides have the same value under v , and since v was arbitrary, they are equivalent by the paraphrase of Definition 4.5.4.

Parts (a), (b), (c), and (e):

φ	ψ	$\varphi \rightarrow \psi$	$\neg\psi \rightarrow \neg\varphi$	$\neg\varphi \vee \psi$	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	T	T	F
F	F	T	T	T	T

The first three formula columns agree, which gives (a) and (c). The last agrees with the column of $\varphi \leftrightarrow \psi$ in the table of Definition 4.4.2 (T exactly when $a = b$), which gives (e).

φ	ψ	$\neg(\varphi \wedge \psi)$	$\neg\varphi \vee \neg\psi$	$\neg(\varphi \vee \psi)$	$\neg\varphi \wedge \neg\psi$
T	T	F	F	F	F
T	F	T	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

This gives (b). For (d): if $a = T$, then $v(\neg\varphi) = F$ and $v(\neg\neg\varphi) = T$; if $a = F$, then $v(\neg\varphi) = T$ and $v(\neg\neg\varphi) = F$; in both cases $v(\neg\neg\varphi) = a$. For (f): clause (V3) makes $v(\varphi \wedge \psi) = T$ exactly when $a = b = T$, a condition that does not change when a and b are swapped, so $v(\varphi \wedge \psi) = v(\psi \wedge \varphi)$; likewise (V4) for $\vee$.

Parts (g) and (h):

φ	ψ	χ	$\varphi \wedge (\psi \vee \chi)$	$(\varphi \wedge \psi) \vee (\varphi \wedge \chi)$	$\varphi \vee (\psi \wedge \chi)$	$(\varphi \vee \psi) \wedge (\varphi \vee \chi)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

φ	ψ	χ	$(\varphi \wedge \psi) \wedge \chi$	$\varphi \wedge (\psi \wedge \chi)$	$(\varphi \vee \psi) \vee \chi$	$\varphi \vee (\psi \vee \chi)$
T	T	T	T	T	T	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	T	T
F	F	T	F	F	T	T
F	F	F	F	F	F	F

In each of the two tables the first two formula columns agree, and so do the last two. This proves all parts. ■

Chapters 1 and 2 derived many of these laws as pairs of sequents, some directions with derivations that use RAA or LEM (Proposition 2.5.5 (d), Proposition 2.5.1, Proposition 2.5.6) and others without. That some of them cannot do without a classical rule, Chapter 2 stated without proof (for De Morgan (d), for instance); for $\neg\neg P \vdash P$, Section 4.7 proves it in an optional result. The tables do not see that difference: they see only that the two sides agree in every row. Section 4.8 shows that every such agreement can be turned into derivations, with RAA among the rules.

Long conjunctions and disjunctions. Associativity lets us stop worrying about grouping. We write $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ for the conjunction grouped to the right, $\varphi_1 \wedge (\varphi_2 \wedge (\cdots \wedge \varphi_n))$, and likewise for $\vee$; for $n = 1$ it is φ_1 itself. Then, for every valuation v and every $n \geq 1$,

$$\begin{aligned} v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \text{T} &\iff v(\varphi_i) = \text{T} \text{ for every } i \text{ with } 1 \leq i \leq n, \\ v(\varphi_1 \vee \cdots \vee \varphi_n) = \text{T} &\iff v(\varphi_i) = \text{T} \text{ for some } i \text{ with } 1 \leq i \leq n. \end{aligned}$$

🕒 **Second reading** · Why long conjunctions are evaluated term by term

For the conjunction, we prove by Axiom 4.2.1 on n , started at 1, that the first line holds for **every** list of n formulas. For $n = 1$ both sides say $v(\varphi_1) = \text{T}$. If it holds for every list of n formulas, take a list of $n + 1$. The conjunction is $\varphi_1 \wedge \theta$, where $\theta = \varphi_2 \wedge \cdots \wedge \varphi_{n+1}$ has n terms. By (V3), $v(\varphi_1 \wedge \theta) = \text{T}$ exactly when $v(\varphi_1) = \text{T}$ and $v(\theta) = \text{T}$, and by the hypothesis for n , $v(\theta) = \text{T}$ exactly when $v(\varphi_i) = \text{T}$ for $2 \leq i \leq n + 1$. Together: exactly when every $v(\varphi_i) = \text{T}$. The disjunction is the same with (V4) and “some”.

The same holds for any grouping. Call a formula built from $\varphi_1, \dots, \varphi_n$, in this order, using $\wedge$ alone, a *grouping* of the list. For $n \geq 2$ a grouping is $\theta_1 \wedge \theta_2$, where θ_1 is a grouping of $\varphi_1, \dots, \varphi_k$ and θ_2 a grouping of $\varphi_{k+1}, \dots, \varphi_n$, for some k with $1 \leq k < n$. Each part has fewer terms, so by (V3) and Theorem 4.2.2 on n , a grouping is true under v exactly when every φ_i is. Likewise for $\vee$ with (V4).

Two consequences follow at once, since the right-hand sides do not care about order, and by the fold every grouping has the same right-hand side. Reordering the terms of a long conjunction, or grouping them differently, gives an equivalent formula; likewise for disjunctions. And De Morgan's laws extend to many terms:

$$\neg(\varphi_1 \wedge \dots \wedge \varphi_n) \equiv \neg\varphi_1 \vee \dots \vee \neg\varphi_n, \quad \neg(\varphi_1 \vee \dots \vee \varphi_n) \equiv \neg\varphi_1 \wedge \dots \wedge \neg\varphi_n,$$

since, for instance, both sides of the first are true exactly when some φ_i is false.

Quick check. Is $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$? And is $p \rightarrow (q \rightarrow r) \equiv (p \rightarrow q) \rightarrow r$?

Solution. The first holds. Each side is false in exactly one row, $p = q = \text{T}, r = \text{F}$: the left needs p true and $q \rightarrow r$ false, the right needs $p \wedge q$ true and r false. (Chapter 1 derived the two from each other, Proposition 1.4.2.) The second fails: by [Example \(One grouping entails the other, not conversely\)](#), the row $p = r = \text{F}$ makes $p \rightarrow (q \rightarrow r)$ true and $(p \rightarrow q) \rightarrow r$ false.

Remark. Three ways to change a formula. The chapter has three operations that put one thing in place of another. Their names are close, and what each one keeps is different.

Operation	What is put in	Where	What it keeps	Example
replace a part	one occurrence of a subformula, by a formula equivalent to it	below, “Replacing a part by an equivalent part”	the truth table of the whole formula	$\neg\neg r$ by r inside $p \rightarrow (q \vee \neg\neg r)$
substitute for atoms	formulas for chosen atoms, at every occurrence, all at once	below, “Substituting formulas for atoms”	tautologies stay tautologies, valid arguments stay valid	$q \wedge r$ for p turns $p \vee \neg p$ into $(q \wedge r) \vee \neg(q \wedge r)$
substitute in a derivation	formulas for the letters of a derivation, at every occurrence	Section 4.1, last subsection	being a derivation	$p_1 \wedge p_2$ for P in the derivation of Proposition 1.4.1

Quick check. Which of the three operations turns $p \vee \neg p$ into $(q \rightarrow r) \vee \neg(q \rightarrow r)$, and what does it keep?

Solution. Substitution of $q \rightarrow r$ for the atom p , at both occurrences. It keeps tautologies, so the new formula is a tautology too. It is not a replacement of a part: $q \rightarrow r$ is not equivalent to p .

4.5.4 Relations between two formulas

Equivalent formulas agree in every row. Three other relations say how two formulas can fail to agree, and they have old names.

Definition 4.5.5. Contradictory, Contrary and Subcontrary Formulas

Formulas φ and ψ are **contradictory** if every row makes exactly one of them true; **contrary** if no row makes both true but some row makes both false; **subcontrary** if no row makes both false but some row makes both true.

In words: contradictory formulas are each other's negation in value, $\varphi \equiv \neg\psi$, since “exactly one true” says that $v(\varphi) = \text{T}$ iff $v(\psi) = \text{F}$, that is, iff $v(\neg\psi) = \text{T}$. Contrary formulas cannot both hold but can both fail; subcontrary formulas cannot both fail but can both hold. A row here is a valuation, or equally, by Lemma 4.4.1, a row of a table on the atoms of both formulas.

Pair	Relation	Why
$p \rightarrow q, p \wedge \neg q$	contradictory	each is F exactly where the other is T: $p \wedge \neg q$ is T only at TF, the one row where $p \rightarrow q$ is F
$p \wedge q, \neg p$	contrary	both false at $p = \text{T}, q = \text{F}$; both true would need p and $\neg p$ true
$p \vee q, \neg p \vee q$	subcontrary	both true at $p = \text{F}, q = \text{T}$; both false would need p and $\neg p$ false

Quick check. Classify the pair $p \leftrightarrow q, p \leftrightarrow \neg q$.

Solution. Contradictory. In each row the two sides of the first agree exactly when the two sides of the second disagree, since $\neg q$ has the other value from q . So exactly one of the two is true in each row.

4.5.5 Replacing a part by an equivalent part

In algebra, if $a = b$, then a may be replaced by b inside any expression. Semantic equivalence behaves in the same way, because the value of a formula depends only on the values of its parts. This is what makes chains of equivalences, one standard equivalence applied deep inside a formula at a time, a legitimate way of computing.

Recall from Section 4.1 that the subformulas of χ are the formulas at the nodes of its parse tree (Definition 4.1.3). An **occurrence** of φ in χ is a node whose subtree is a parse tree of φ ; **replacing** that occurrence by φ' means putting a parse tree of φ' in place of that subtree, which gives the parse tree of a new formula χ' .

Lemma 4.5.2. Replacement

Let $\varphi \equiv \varphi'$. If χ' is obtained from a formula χ by replacing one occurrence of φ in χ by φ' , then $\chi' \equiv \chi$.

Idea. Induct on the shape of χ . Either the occurrence is all of χ , and then χ is φ and χ' is φ' , equivalent by hypothesis; or it sits inside one immediate part of χ , the other part is untouched, and the value of χ is computed from the values of its parts, one of which has not changed and one of which has not changed its value.

The proof, by structural induction with exactly that case split, is in the fold below.

🕒 **Second reading** · *The proof of the replacement lemma*

Proof. We apply Theorem 4.3.1 to the property $C(\chi)$: *for every formula χ' obtained from χ by replacing one occurrence of φ by φ' , and every valuation v , $v(\chi') = v(\chi)$* . By the paraphrase of Definition 4.5.4, $C(\chi)$ says that each such χ' is equivalent to χ .

In every case, the replaced occurrence sits either at the top node of the tree of χ or at a node below it; in the second case the node lies in the subtree of exactly one immediate part of χ , since those subtrees are the parse trees of the immediate parts (Definition 4.1.2). If the occurrence is the root, then $\chi = \varphi$ and $\chi' = \varphi'$, and $v(\chi') = v(\chi)$ because $\varphi \equiv \varphi'$. It remains to treat an occurrence below the root.

Atoms and $\perp$. The parse tree is one node, so there is no node below the root.

Negation. Let $\chi = \neg\theta$ and assume $C(\theta)$. An occurrence below the root lies in θ , so $\chi' = \neg\theta'$, where θ' is obtained from θ by replacing that occurrence. By $C(\theta)$, $v(\theta') = v(\theta)$, and (V2) gives $v(\neg\theta') = v(\neg\theta)$.

Binary connectives. Let $\chi = \theta_1 \circ \theta_2$, with $\circ$ one of $\wedge, \vee, \rightarrow, \leftrightarrow$, and assume $C(\theta_1)$ and $C(\theta_2)$. If the occurrence lies in θ_1 , then $\chi' = \theta'_1 \circ \theta_2$, with θ'_1 obtained from θ_1 by the replacement, and $v(\theta'_1) = v(\theta_1)$ by $C(\theta_1)$. The clause of Definition 4.4.2 for $\circ$ computes $v(\chi')$ from $v(\theta'_1)$ and $v(\theta_2)$, and $v(\chi)$ from $v(\theta_1)$ and $v(\theta_2)$, by the same rule, so $v(\chi') = v(\chi)$. An occurrence in θ_2 is the same with the roles of the parts exchanged.

This covers every clause of Definition 4.1.1, so $C(\chi)$ holds for every χ , which proves the

lemma. ■

Replacing several occurrences, one after another, keeps equivalence at each step, and $\equiv$ is transitive, so the final formula is equivalent to the first. For instance, with (b), (d), and then (c) of Proposition 4.5.1, each applied inside the formula:

$$\begin{aligned} p \rightarrow \neg(q \wedge \neg r) &\equiv p \rightarrow (\neg q \vee \neg\neg r) \\ &\equiv p \rightarrow (\neg q \vee r) \\ &\equiv \neg p \vee (\neg q \vee r). \end{aligned}$$

The first step replaces the subformula $\neg(q \wedge \neg r)$ by an equivalent one (De Morgan with $\varphi = q$, $\psi = \neg r$); the second replaces $\neg\neg r$ by r ; the third replaces the whole formula, by (c) with $\varphi = p$ and $\psi = \neg q \vee r$. Section 4.9 computes normal forms this way.

Warning. Replacement needs **equivalence**; entailment in one direction is not enough. We have $p \wedge q \models p$, but replacing $p \wedge q$ by p inside $\neg(p \wedge q)$ gives $\neg p$, and the row $p = \text{T}, q = \text{F}$ makes $\neg(p \wedge q)$ true and $\neg p$ false. A negation reverses the direction of an entailment, so one-way information cannot pass through it.

4.5.6 Substituting formulas for atoms

The letters P, Q of Chapters 1 and 2 stood for arbitrary statements, and Section 4.1 turned that into a remark about derivations: a derivation written with letters is a pattern, and putting formulas in for the letters, uniformly, gives a derivation again. The truth table of Example (The truth table of modus ponens), though, was computed for the atoms p and q . Does it tell us anything about $(r \vee s) \rightarrow \neg t$ in place of p ? It should: the table only used the values of p and q , and a compound formula also has a value in each row. This subsection makes that precise.

Let $p_1, \dots, p_n$ be distinct atoms and $\varphi_1, \dots, \varphi_n$ formulas. The **substitution** σ of $\varphi_1, \dots, \varphi_n$ for $p_1, \dots, p_n$ is the function on formulas defined by recursion (Theorem 4.3.4):

$$\begin{aligned} \sigma(p_i) &= \varphi_i \quad (1 \leq i \leq n), & \sigma(q) &= q \text{ for every other atom } q, & \sigma(\perp) &= \perp, \\ \sigma(\neg\chi) &= \neg\sigma(\chi), & \sigma(\chi \circ \theta) &= \sigma(\chi) \circ \sigma(\theta) \text{ for each binary } \circ. \end{aligned}$$

In words: $\sigma(\chi)$ replaces **every** occurrence of each p_i in χ by φ_i , all at once, and changes nothing else. For a list $\Gamma = \gamma_1, \dots, \gamma_m$, we write $\sigma(\Gamma)$ for the list $\sigma(\gamma_1), \dots, \sigma(\gamma_m)$. For example, with σ putting $q \wedge r$ for p and p for q ,

$$\sigma((p \rightarrow q) \vee \neg p) = ((q \wedge r) \rightarrow p) \vee \neg(q \wedge r).$$

The p that σ puts in for q is not replaced again: the replacements happen simultaneously, in one pass over the parse tree.

Example. One row, two valuations

Let σ put $q \wedge r$ for p and p for q , as above, and let $v(p) = \text{F}$, $v(q) = \text{T}$, $v(r) = \text{T}$. Compute $v(\sigma(\chi))$ for $\chi = (p \rightarrow q) \vee \neg p$. Then define w by $w(s) := v(\sigma(s))$ for every atom s , and

compute $w(\chi)$.

Solution. By the display above, $\sigma(\chi) = ((q \wedge r) \rightarrow p) \vee \neg(q \wedge r)$. Under v , $q \wedge r$ is T and p is F, so $(q \wedge r) \rightarrow p$ is F; $\neg(q \wedge r)$ is F; so $v(\sigma(\chi)) = F$.

Next, $w(p) = v(q \wedge r) = T$, $w(q) = v(p) = F$, and $w(r) = v(r) = T$, since σ leaves r alone. Under w , $p \rightarrow q$ is F and $\neg p$ is F, so $w(\chi) = F$. The two values agree.

The formula $\sigma(\chi)$ under v behaves exactly as χ does under w . The lemma below says that this happens for every formula and every valuation, and draws the consequences.

Lemma 4.5.3. Substitution

Let σ be the substitution of $\varphi_1, \dots, \varphi_n$ for the distinct atoms $p_1, \dots, p_n$.

- (a) If $\models \chi$, then $\models \sigma(\chi)$.
- (b) If $\Gamma \models \psi$, then $\sigma(\Gamma) \models \sigma(\psi)$.

Idea. Under a valuation v , the formula $\sigma(\chi)$ behaves exactly like χ itself under a different valuation w : the one that gives each atom p_i the value that v gives to φ_i . So every row for $\sigma(\chi)$ is, in disguise, a row for χ , and whatever holds in all rows of χ holds in all rows of $\sigma(\chi)$. The disguise is proved by induction on the shape of χ .

Proof. Let v be a valuation, and let w be the valuation with $w(p) := v(\sigma(p))$ for every atom p ; so $w(p_i) = v(\varphi_i)$ and $w(q) = v(q)$ for every other atom q .

Claim. $v(\sigma(\chi)) = w(\chi)$ for every formula χ .

Proof of Claim. By Theorem 4.3.1. For an atom p , $v(\sigma(p)) = w(p)$ is the definition of w . For $\perp$, $v(\sigma(\perp)) = v(\perp) = F = w(\perp)$ by (V1). For a negation, assume $v(\sigma(\chi)) = w(\chi)$; then $v(\sigma(\neg\chi)) = v(\neg\sigma(\chi))$, which by (V2) is the flip of $v(\sigma(\chi)) = w(\chi)$, and so equals $w(\neg\chi)$. For a binary connective $\circ$, assume the claim for χ and θ ; then $v(\sigma(\chi \circ \theta)) = v(\sigma(\chi) \circ \sigma(\theta))$ is computed by the clause for $\circ$ from $v(\sigma(\chi)) = w(\chi)$ and $v(\sigma(\theta)) = w(\theta)$, and so equals $w(\chi \circ \theta)$, which the same clause computes from the same two values. ■

(a) Let $\models \chi$, and let v be any valuation. By the claim, $v(\sigma(\chi)) = w(\chi) = T$, since χ is a tautology. So $\sigma(\chi)$ is true under every valuation.

(b) Let $\Gamma \models \psi$, and let v make every formula of $\sigma(\Gamma)$ true. For each γ in Γ , the claim gives $w(\gamma) = v(\sigma(\gamma)) = T$, so w makes every formula of Γ true. By $\Gamma \models \psi$, $w(\psi) = T$, and by the claim again $v(\sigma(\psi)) = w(\psi) = T$. This proves $\sigma(\Gamma) \models \sigma(\psi)$. ■

So a table computed once for atoms settles infinitely many instances. From $\models p \vee \neg p$ we get $\models (q \rightarrow r) \vee \neg(q \rightarrow r)$, and every other instance of excluded middle. From $p \rightarrow q$, $p \models q$

(Example (The truth table of modus ponens)) we get $\varphi \rightarrow \psi$, $\varphi \models \psi$ for all formulas φ, ψ , which is the table's version of $\rightarrow E$ as a schema. And from [Example \(Affirming the consequent is not valid\)](#)'s refuting row we learn only that the pattern is not valid, not that its instances fail; the lemma runs one way only, as the next warning says.

Warning. Substitution must be **uniform**: every occurrence of p_i is replaced by the same φ_i . Replacing only the first p of the tautology $p \rightarrow p$ by q gives $q \rightarrow p$, which is false in the row $q = T, p = F$. And the lemma runs one way only. An instance can be a tautology when the pattern is not: $p \vee q$ is not a tautology, but its instance with $\neg p$ put in for q is $p \vee \neg p$, which is. So a refuting row for a pattern with atoms shows that the **pattern** is not valid, not that every instance fails.

4.5.7 One row or every row

Every question of this section is about rows, and each comes in two halves: one half is settled by a single row, the other needs them all.

Question	A yes is shown by	A no is shown by	The same as: is this list unsatisfiable?
φ satisfiable?	one row making φ true	every row	no: a yes is one row, so this is the negation of “is the list φ unsatisfiable?”
φ a tautology?	every row	one row making φ false	the list $\neg\varphi$
$\Gamma \models \psi$?	every row (or working backwards from F)	one refuting row	the counterexample set $\Gamma, \neg\psi$
$\varphi \equiv \psi$?	every row	one row where they differ	the list $\neg(\varphi \leftrightarrow \psi)$, since $\varphi \equiv \psi$ exactly when $\models \varphi \leftrightarrow \psi$ (after Definition 4.5.4)
Γ satisfiable?	one row	every row	no: it is the negation of “is Γ unsatisfiable?”

So every question is one question in disguise: is some list unsatisfiable? A yes needs every row, a no needs one. Section 4.6 searches the rows of a list without writing them all down.

4.5.8 Two questions, side by side

We now have two ways of saying that ψ follows from Γ . They were built from different materials, and it is worth setting them next to each other before Sections 4.7 and 4.8 connect them.

	$\Gamma \vdash \psi$	$\Gamma \models \psi$
What it says	there is a derivation from premises in Γ to ψ	every valuation making all of Γ true makes ψ true
What it is about	syntax: lines, rules, boxes	meaning: truth values

	$\Gamma \vdash \psi$	$\Gamma \models \psi$
How to show it holds	exhibit one derivation	check every row (or work backwards from F)
How to show it fails	no method yet: a failed search proves nothing	exhibit one refuting row
Defined in	Definition 1.2.3 (Section 1.2)	Definition 4.5.3 (this section)

Look at the fourth row. To show that something is **not** derivable, we would have to survey infinitely many derivations; to show that something is **not** valid, one row suffices. And to show that something **is** valid takes a finite, mechanical check, while showing it derivable may need an idea. Each column is easy exactly where the other is hard.

Sections 4.7 and 4.8 prove that the two columns always give the same answer. Section 4.7 proves **soundness**: if $\Gamma \vdash \psi$, then $\Gamma \models \psi$; so a refuting row proves that no derivation exists, which answers the question Chapter 3 ended on. Section 4.8 proves **completeness**: if $\Gamma \models \psi$, then $\Gamma \vdash \psi$; so a successful table check guarantees that a derivation exists, even before anyone has found it.

4.5.9 Exercises

4.5.9.1 A. Check your understanding

Exercise. A1

- Classify each formula as a tautology, a contradiction, or satisfiable but not a tautology: $p \rightarrow (q \rightarrow p)$; $p \wedge \neg(p \vee q)$; $(p \rightarrow q) \wedge (q \rightarrow p)$; $\neg(p \rightarrow p)$; $(p \wedge q) \rightarrow (p \vee r)$.
- For each, decide whether it holds, and give a refuting row if it does not: $p \wedge q \models p \leftrightarrow q$; $p \leftrightarrow q \models p \wedge q$; $\neg p \rightarrow p \models p$.
- True or false, with a reason: if $\Gamma \models \psi$ and $\Gamma \models \neg\psi$, then Γ is unsatisfiable.
- True or false, with a reason: $p \vee q \models p$ or $p \vee q \models q$.

Solution. (a) $p \rightarrow (q \rightarrow p)$: a tautology (if p is T, the conclusion $q \rightarrow p$ is T; if p is F, the hypothesis is false). $p \wedge \neg(p \vee q)$: a contradiction (if p is T, so is $p \vee q$, and $\neg(p \vee q)$ is F). $(p \rightarrow q) \wedge (q \rightarrow p)$: satisfiable, not a tautology (T in the row TT, F in TF). $\neg(p \rightarrow p)$: a contradiction, since $p \rightarrow p$ is a tautology. $(p \wedge q) \rightarrow (p \vee r)$: a tautology (if the hypothesis is T, then p is T, so is $p \vee r$).

(b) $p \wedge q \models p \leftrightarrow q$ holds: the only row making $p \wedge q$ true has $p = q = \text{T}$, where $p \leftrightarrow q$ is true. $p \leftrightarrow q \models p \wedge q$ fails: the row $p = q = \text{F}$ makes $p \leftrightarrow q$ true and $p \wedge q$ false. $\neg p \rightarrow p \models p$ holds: in the row $p = \text{F}$, the premise $\neg p \rightarrow p$ has a true hypothesis and a false conclusion, so it is false; the only row making the premise true has $p = \text{T}$.

(c) True. If a valuation made all of Γ true, it would make both ψ and $\neg\psi$ true, which (V2) forbids. So no valuation makes all of Γ true.

(d) False. The row $p = \text{F}$, $q = \text{T}$ refutes $p \vee q \models p$, and the row $p = \text{T}$, $q = \text{F}$ refutes $p \vee q \models q$. Each disjunct fails in some row where the disjunction holds, although in every such row one of them holds: $\models$ must work row by row with a single conclusion.

Exercise. A2: Drill

The parts run from classifying one formula, through lists and entailments, to pairs of formulas and the reduction to unsatisfiability.

- (a) Tautology, contradiction or contingent: $(p \rightarrow q) \vee p$?
- (b) The same for $(p \vee q) \wedge \neg p \wedge \neg q$.
- (c) The same for $(p \wedge q) \rightarrow \neg p$.
- (d) Is the list $p \vee q, \neg p, \neg q$ satisfiable?
- (e) Does $p \rightarrow q \models \neg p \vee q$?
- (f) Give a refuting row for $p \vee q \models p \wedge q$.
- (g) Does $\perp \models p$?
- (h) Does $p \models q \rightarrow p$?
- (i) Is $\neg(p \rightarrow q) \equiv p \wedge \neg q$?
- (j) Classify the pair $p \wedge q, \neg p \vee \neg q$: contradictory, contrary, subcontrary, or none.
- (k) Classify the pair $p \wedge \neg q, q$.
- (l) Which list must be unsatisfiable for $p \rightarrow q \equiv \neg q \rightarrow \neg p$ to hold?

Solution.

- (a) A tautology. If p is F, the left disjunct is T; if p is T, the right one is.
- (b) A contradiction: $\neg p$ and $\neg q$ true leave $p \vee q$ false.
- (c) Contingent: F at $p = q = T$, and T whenever p is F.
- (d) No: a row making $\neg p$ and $\neg q$ true makes $p \vee q$ false.
- (e) Yes. The two are equivalent, by [Proposition 4.5.1](#) (c).
- (f) $p = T, q = F$.
- (g) Yes: no row makes $\perp$ true.
- (h) Yes: in every row with p true, $q \rightarrow p$ has a true conclusion.
- (i) Yes: both are T in the row TF only.
- (j) Contradictory: by [Proposition 4.5.1](#) (b) the second is equivalent to $\neg(p \wedge q)$.
- (k) Contrary: never both true, since the first needs q false; both false at $p = q = F$.
- (l) The one-formula list $\neg((p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p))$.

4.5.9.2 B. Practice**Exercise. B1: Valid or not**

Decide whether each entailment holds, by a table or by working backwards from F. If it holds, also give a derivation of the corresponding sequent with $\vdash$ in place of $\models$, using the rules of Chapters 1 and 2; if it fails, give a refuting row.

- (a) $p \rightarrow q, \neg q \models \neg p$.
- (b) $p \rightarrow (q \vee r), \neg q \models p \rightarrow r$.
- (c) $p \rightarrow q \models \neg p \rightarrow \neg q$.

(d) $p \vee q, p \rightarrow r, q \rightarrow r \models r$.

Solution. (a) Holds. Backwards from F: $\neg p$ false means $p = T$; $\neg q$ true means $q = F$; then $p \rightarrow q$ is F, so no row makes the premises true and the conclusion false. A derivation, with modus tollens (Proposition 2.3.1) as a derived rule:

1		$p \rightarrow q$	
2		$\neg q$	
3		$\neg p$	MT 1, 2

(b) Holds. Backwards from F: $p \rightarrow r$ false means $p = T, r = F$; $\neg q$ true means $q = F$; then $q \vee r$ is F, and $p \rightarrow (q \vee r)$ is F, contradicting the first premise. A derivation, with disjunctive syllogism (Proposition 2.3.4), which refutes the left disjunct:

1		$p \rightarrow (q \vee r)$	
2		$\neg q$	
3		p	asm, for $\rightarrow I$
4		$q \vee r$	$\rightarrow E$ 1, 3
5		r	DS 4, 2
6		$p \rightarrow r$	$\rightarrow I$ 3–5

(c) Fails. The row $p = F, q = T$ makes $p \rightarrow q$ true (false hypothesis) and $\neg p \rightarrow \neg q$ false ($\neg p$ true, $\neg q$ false). It is the pattern “denying the hypothesis” of the Quick check $p \rightarrow q, \neg p \models \neg q$, now in the form of an implication.

(d) Holds. Backwards from F: $r = F$; then $p \rightarrow r$ true forces $p = F$, and $q \rightarrow r$ true forces $q = F$; then $p \vee q$ is F, a contradiction. A derivation by cases:

1		$p \vee q$	
2		$p \rightarrow r$	
3		$q \rightarrow r$	
4		p	asm, for $\vee E$
5		r	$\rightarrow E$ 2, 4
6		q	asm, for $\vee E$
7		r	$\rightarrow E$ 3, 6
8		r	$\vee E$ 1, 4–5, 6–7

Exercise. B2: A biconditional as a disjunction

Show by a truth table that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$. Which theorem of this chapter will guarantee, without any search, that each side can be derived from the other?

Solution.

p	q	$p \leftrightarrow q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	T	T	F	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T

The columns of $p \leftrightarrow q$ and of the disjunction agree in every row, so the two formulas are equivalent: each side entails the other. The theorem that turns each entailment into a derivation is completeness, in Section 4.8: if $\Gamma \models \psi$, then $\Gamma \vdash \psi$. Once it is proved, it will give $p \leftrightarrow q \vdash (p \wedge q) \vee (\neg p \wedge \neg q)$ and its converse, before any derivation has been written; until then the table shows only that the two sequents are valid.

Exercise. B3: Entailment and implication

Let Γ be a list of formulas and φ, ψ formulas. Prove:

- (a) $\varphi \models \psi$ if and only if $\models \varphi \rightarrow \psi$;
- (b) $\Gamma, \varphi \models \psi$ if and only if $\Gamma \models \varphi \rightarrow \psi$.

Solution. We prove (b); then (a) is (b) with Γ empty, since Γ, φ is then the one-formula list φ .

($\Rightarrow$) Suppose $\Gamma, \varphi \models \psi$, and let v make every formula of Γ true. If $v(\varphi) = F$, then $v(\varphi \rightarrow \psi) = T$ by (V5). If $v(\varphi) = T$, then v makes every formula of the list Γ, φ true, so $v(\psi) = T$ by hypothesis, and again $v(\varphi \rightarrow \psi) = T$. So $\Gamma \models \varphi \rightarrow \psi$.

($\Leftarrow$) Suppose $\Gamma \models \varphi \rightarrow \psi$, and let v make every formula of Γ, φ true. Then v makes every formula of Γ true, so $v(\varphi \rightarrow \psi) = T$, and $v(\varphi) = T$. By (V5), an implication with a true hypothesis is true only if its conclusion is, so $v(\psi) = T$. So $\Gamma, \varphi \models \psi$.

Part (b) is the table's version of $\rightarrow I$ (the direction $\Rightarrow$) and of $\rightarrow E$ (the direction $\Leftarrow$).

Exercise. B4: Contradictory, contrary, subcontrary, or none

Classify each pair, and give the rows that show it.

- (a) $p \rightarrow q, p \wedge \neg q$.
- (b) $p \wedge q, \neg p \wedge q$.
- (c) $p \vee q, \neg p \vee q$.
- (d) $p \rightarrow q, q \rightarrow p$.
- (e) $p \leftrightarrow q, p \wedge \neg q$.
- (f) $p, p \vee q$.

Solution. (a) Contradictory: the first is F only at TF, where the second is T, and the second is T only there.

(b) Contrary: both true would need p and $\neg p$ true; both are false at $q = F$.

(c) Subcontrary: both true at $p = F, q = T$; both false would need q false and both p and $\neg p$ false.

(d) Subcontrary: both true at TT ; $p \rightarrow q$ is F only at TF and $q \rightarrow p$ only at FT , so they are never both false.

(e) Contrary: both true would need p and q equal (for $p \leftrightarrow q$) and unequal (for $p \wedge \neg q$); both false at $p = F, q = T$.

(f) None of the three: both true at $p = T$, both false at $p = q = F$. (In fact $p \models p \vee q$.)

The pair (d) is the surprise: an implication and its converse can never both fail.

Marking scheme. (6 points)

- 1 per part: the class, with a witness for each of the two questions (both true? both false?): a row where it happens, or the reason it cannot.

Fatal error (0 for the part): a class claimed with one of the two questions unanswered.

4.5.9.3 C. Going deeper

Exercise. C1: Substitution with a new connective, the last half blank, used later

Extend the formulas by a new binary connective $|$ (“not both”, often called NAND): add to Definition 4.1.1 the clause “if φ and ψ are formulas, so is $(\varphi | \psi)$ ”, and to Definition 4.4.2 the clause

$$(V7) \ v(\varphi | \psi) := F \text{ if } v(\varphi) = v(\psi) = T; \text{ otherwise } T.$$

Assume that structural induction and recursion work for the extended formulas as before, with one more clause. A substitution σ of $\varphi_1, \dots, \varphi_n$ for $p_1, \dots, p_n$ is defined by the clauses in the text, with $\sigma(\chi | \theta) := \sigma(\chi) | \sigma(\theta)$ added. Prove that if χ is a tautology of the extended language, so is $\sigma(\chi)$.

Here is the first half of a proof. *Let v be a valuation, and let w be the valuation with $w(p) := v(\sigma(p))$ for every atom p . We show $v(\sigma(\chi)) = w(\chi)$ for every formula χ of the extended language, by structural induction. Atoms: $v(\sigma(p)) = w(p)$ is the definition of w .*

Name the move, then complete it.

Solution. The move is **induction on the shape of a formula**, now with eight cases: atoms, $\perp$, $\neg$, and five binary connectives. The property is $C(\chi)$: $v(\sigma(\chi)) = w(\chi)$.

$\perp$. $\sigma(\perp) = \perp$, so $v(\sigma(\perp)) = F = w(\perp)$ by (V1).

Negation. Assume $C(\chi)$. Then $v(\sigma(\neg\chi)) = v(\neg\sigma(\chi))$, which by (V2) is T exactly when $v(\sigma(\chi)) = F$, that is, when $w(\chi) = F$, that is, when $w(\neg\chi) = T$.

The four old binary connectives. Assume $C(\chi)$ and $C(\theta)$, and let $\circ$ be one of $\wedge, \vee, \rightarrow, \leftrightarrow$. Then $v(\sigma(\chi \circ \theta)) = v(\sigma(\chi) \circ \sigma(\theta))$ is computed by the clause for $\circ$ from the two values $v(\sigma(\chi)) = w(\chi)$ and $v(\sigma(\theta)) = w(\theta)$; the same clause computes $w(\chi \circ \theta)$ from the same two values. So they are equal.

The new connective. Assume $C(\chi)$ and $C(\theta)$. By (V7), $v(\sigma(\chi|\theta)) = v(\sigma(\chi)|\sigma(\theta))$ is F exactly when $v(\sigma(\chi)) = v(\sigma(\theta)) = T$, that is, when $w(\chi) = w(\theta) = T$, that is, when $w(\chi|\theta) = F$.

This covers every clause, so $C(\chi)$ holds for every formula. Now let χ be a tautology and v any valuation: $v(\sigma(\chi)) = w(\chi) = T$. So $\sigma(\chi)$ is a tautology.

Only one case is new. The proof of [Lemma 4.5.3](#) never looked at what the clauses say, only that each computes the value of a compound from the values of its parts; any connective defined by a table passes.

Marking scheme. (10 points)

- 1: names the move and states the property $v(\sigma(\chi)) = w(\chi)$ for all χ .
- 1: the $\perp$ case.
- 2: the $\neg$ case, from the hypothesis for χ .
- 2: the old binary cases, using the hypotheses for both parts.
- 3: the $|$ case, with (V7) and both hypotheses.
- 1: the conclusion for tautologies.

Fatal errors (0 for the step): omitting the $|$ case; using w where v is meant in the conclusion.

Exercise. C2: Is equivalence preserved by substitution?

Let σ be a substitution and suppose $\varphi \equiv \psi$. Is $\sigma(\varphi) \equiv \sigma(\psi)$? Prove it, or give a counterexample. Then suppose that De Morgan's law (b) of [Proposition 4.5.1](#) had been checked only for atoms, $\neg(p \wedge q) \equiv \neg p \vee \neg q$, by its four-row table. Use your answer to deduce from this, in one line, that $\neg((r \rightarrow s) \wedge \neg t) \equiv \neg(r \rightarrow s) \vee \neg\neg t$.

Solution. Yes. By the paraphrase of [Definition 4.5.4](#), $\varphi \equiv \psi$ means $\models \varphi \leftrightarrow \psi$. By [Lemma 4.5.3](#) (a), $\models \sigma(\varphi \leftrightarrow \psi)$, and by the definition of σ , $\sigma(\varphi \leftrightarrow \psi) = \sigma(\varphi) \leftrightarrow \sigma(\psi)$. So $\models \sigma(\varphi) \leftrightarrow \sigma(\psi)$, that is, $\sigma(\varphi) \equiv \sigma(\psi)$. (Alternatively, apply [Lemma 4.5.3](#) (b) to $\varphi \models \psi$ and to $\psi \models \varphi$.)

For the instance: the substitution of $r \rightarrow s$ for p and $\neg t$ for q turns the two sides of $\neg(p \wedge q) \equiv \neg p \vee \neg q$ into $\neg((r \rightarrow s) \wedge \neg t)$ and $\neg(r \rightarrow s) \vee \neg\neg t$, so these are equivalent by the first part. Checking a law for atoms, by one small table, is therefore enough.

A tempting counterexample fails for a reason worth seeing. With σ putting $\neg p$ for q , the nonequivalent formulas p and $\neg q$ become the equivalent p and $\neg\neg p$: substitution can create equivalences. It cannot destroy one, which is what the question asks.

Marking scheme. (10 points)

- 3: reduces $\equiv$ to a tautology $\models \varphi \leftrightarrow \psi$, or to two entailments, citing the definition.
- 3: applies [Lemma 4.5.3](#) correctly.
- 2: in the $\leftrightarrow$ route, computes $\sigma(\varphi \leftrightarrow \psi) = \sigma(\varphi) \leftrightarrow \sigma(\psi)$ from the definition of σ ; in the entailment route, applies (b) in both directions.
- 2: the instance, with the substitution named.

Fatal errors (0 for the step): answering “no” by a pair that is not equivalent before the substitution; using a nonuniform replacement.

Exercise. C3: The rows of implication are forced

Pretend we do not know the table of $\rightarrow$. Say only that there is **some** table: a rule t giving a truth value $t(a, b)$ for each pair of truth values a, b , with

$$v(\varphi \rightarrow \psi) = t(v(\varphi), v(\psi))$$

for every valuation v and all formulas φ, ψ . (The other connectives keep their clauses.) Suppose the two rules for implication preserve truth in the following sense.

- (E) For every valuation v and all formulas φ, ψ : if $v(\varphi \rightarrow \psi) = \text{T}$ and $v(\varphi) = \text{T}$, then $v(\psi) = \text{T}$.
- (I) For every list of formulas Γ and all formulas φ, ψ : if $\Gamma, \varphi \models \psi$, then $\Gamma \models \varphi \rightarrow \psi$, where $\models$ is read with the table t for $\rightarrow$.

Prove that t is the table of (V5): $t(\text{T}, \text{F}) = \text{F}$, and $t(a, b) = \text{T}$ for the other three pairs.

Solution (Hint 1). Each of the four values of t is one instance of (E) or of (I), with atoms for φ and ψ and a well-chosen valuation.

Solution (Hint 2). For $t(\text{T}, \text{T})$ and $t(\text{F}, \text{F})$, try (I) with Γ empty and $\varphi = \psi = p$.

Solution. Let p and q be distinct atoms.

$t(\text{T}, \text{F}) = \text{F}$. Take a valuation v with $v(p) = \text{T}$, $v(q) = \text{F}$. If $t(\text{T}, \text{F})$ were T , then $v(p \rightarrow q) = t(\text{T}, \text{F}) = \text{T}$ and $v(p) = \text{T}$, so (E) would give $v(q) = \text{T}$, which is false. So $t(\text{T}, \text{F}) = \text{F}$.

$t(\text{T}, \text{T}) = t(\text{F}, \text{F}) = \text{T}$. Apply (I) with Γ empty and $\varphi = \psi = p$. Its hypothesis $p \models p$ holds. So every valuation makes $p \rightarrow p$ true. A valuation with $v(p) = \text{T}$ gives $t(\text{T}, \text{T}) = v(p \rightarrow p) = \text{T}$, and one with $v(p) = \text{F}$ gives $t(\text{F}, \text{F}) = \text{T}$.

$t(\text{F}, \text{T}) = \text{T}$. Apply (I) with Γ the list $q, \varphi = p$ and $\psi = q$. $q, p \models q$ holds, so $q \models p \rightarrow q$. A valuation with $v(p) = \text{F}$, $v(q) = \text{T}$ makes q true, so $t(\text{F}, \text{T}) = v(p \rightarrow q) = \text{T}$.

No step used a value of t that had not yet been found, since the formulas in Γ , φ and ψ contain no $\rightarrow$. So the only table of $\rightarrow$ under which both rules preserve truth is the table of (V5). Conditions (E) and (I) are what Section 4.7 proves about the rules $\rightarrow\text{E}$ and $\rightarrow\text{I}$; the exercise shows that nothing but the standard table could have passed.

Marking scheme. (10 points)

- 3: the row (T, F) from (E), with a valuation making p true and q false, and the argument by contradiction.
- 3: the rows (T, T) and (F, F) from (I), with the hypothesis of (I) checked, and a valuation for each value of p .
- 3: the row (F, T) from (I) with a non-empty Γ (or any other correct instance), hypothesis checked.
- 1: says why the argument is not circular (no $\rightarrow$ in the formulas used).

Fatal errors (0 for the step): using the standard table of $\rightarrow$ to justify a value of t ; an instance

of (I) whose hypothesis is not checked.

Exercise. C4: One question for all

Prove:

- (a) $\varphi \equiv \psi$ if and only if the list $\neg(\varphi \leftrightarrow \psi)$ is unsatisfiable;
- (b) for a list Γ : Γ is unsatisfiable if and only if $\Gamma \models \chi$ for every formula χ , if and only if $\Gamma \models \perp$;
- (c) φ and ψ are contrary if and only if $\varphi \models \neg\psi$ and the list $\neg\varphi, \neg\psi$ is satisfiable;
- (d) φ and ψ are contradictory if and only if $\varphi \equiv \neg\psi$.

Solution (Hint 1). Write out what each side says about rows: “every row ...” or “some row ...”. Then compare the two sentences.

Solution. (a) $\varphi \equiv \psi$ iff $\models \varphi \leftrightarrow \psi$, by the display after [Definition 4.5.4](#), iff $\neg(\varphi \leftrightarrow \psi)$ is unsatisfiable, by the equivalence between tautologies and unsatisfiable negations, applied to $\varphi \leftrightarrow \psi$.

(b) Three implications in a cycle. If Γ is unsatisfiable, then $\Gamma \models \chi$ for every χ , since there is no row to check. If $\Gamma \models \chi$ for every χ , then in particular $\Gamma \models \perp$. If $\Gamma \models \perp$, then a row making every member of Γ true would make $\perp$ true, which (V1) forbids; so no row makes Γ true, and Γ is unsatisfiable.

(c) “No row makes both true” says that every row making φ true makes ψ false, that is, makes $\neg\psi$ true: $\varphi \models \neg\psi$. “Some row makes both false” says that some row makes $\neg\varphi$ and $\neg\psi$ true: the list $\neg\varphi, \neg\psi$ is satisfiable.

(d) “Exactly one is true in every row” says that for every valuation v , $v(\varphi) = \text{T}$ iff $v(\psi) = \text{F}$, iff $v(\neg\psi) = \text{T}$ by (V2). That is $v(\varphi) = v(\neg\psi)$ for every v , which is $\varphi \equiv \neg\psi$ by the paraphrase of [Definition 4.5.4](#).

Marking scheme. (10 points)

- 2: (a), naming both facts used.
- 3: (b), the three implications in a cycle.
- 3: (c), both halves.
- 2: (d).

Fatal error (0 for the step): in (b), “ $\Gamma \models \perp$, so Γ is false” (a list is not true or false; rows make its members true or false).