

1.3 Using an Implication: Modus Ponens

Section 1.2 gave the two rules for “and”. This section gives one rule for “implies”: the rule that **uses** an implication already in hand. Proving one needs a new device, the subject of Section 1.4; using one is what a proof does every time it applies a theorem. The work is to see exactly what the rule allows, and to find chains of implications forwards and backwards.

1.3.1 Four cards

Before the rule, a puzzle about what “if ... then” says.

Example. Four cards

Each of four cards has a whole number on one side and a color on the other. They lie on a table showing

6, 7, red, green.

Someone claims: *if a card shows an even number on one side, then its other side is red.* Which cards must be turned over to find out whether the claim is true of these four cards? Decide before reading the solution.

Solution. For each card, try to break the claim: can its hidden side make a card that is even and not red? (By the promise reading of Section 1.1, that is the only way to break it.)

The card showing 6. Its number is even, so the hypothesis of the implication holds, and the claim says its other side is red. If that side is green, the claim is false. This card must be turned.

The card showing 7. Its number is odd, so the hypothesis fails, and the claim says nothing about this card. Whatever its color, the card cannot contradict the claim. Leave it.

The red card. Its color is the **conclusion** of the implication. If its number is even, the card fits the claim; if its number is odd, the card still fits, since the claim says only what even cards look like and nothing about which cards are red. No hidden number can contradict the claim. Leave it.

The green card. If its number is even, it is an even card that is not red, and the claim is false. This card must be turned.

So the answer is the 6 and the green card. Many people choose the 6 and the **red** card. The red card is the tell: choosing it means reading the claim as if it also said “if a card is red, then its number is even”, which is the **converse**. An implication and its converse are different claims, and the claim here is only the first.

Two of the cards show the two ways an implication meets a case: the 6 and the red card (not the pair that must be turned). The 6 shows the **hypothesis**, and an implication together with its hypothesis gives its conclusion: that is the rule of this section. The red card shows the **conclusion**, and an implication together with its conclusion gives nothing: that is the most important thing the rule does *not* allow. The green card shows that the conclusion fails; saying exactly what follows from that needs the rules for “not”.

1.3.2 The rule

An implication fires on its hypothesis, and gives its conclusion.

Definition 1.3.1. Implication Elimination

i	$\varphi \rightarrow \psi$	
j	φ	
k	ψ	$\rightarrow E\ i, j$

This rule is called $\rightarrow E$ (**implication-elimination**, or **modus ponens**).

We read it in four bullets.

- **Two given lines.** One line is an implication $\varphi \rightarrow \psi$, the other is φ . Both stand earlier in the derivation, anywhere above line k and in either order on the page.
- **Read φ and ψ off the main connective.** To see a line as $\varphi \rightarrow \psi$, its **main connective** (Definition 1.1.2) must be $\rightarrow$. Then φ , the **hypothesis**, is everything that this $\rightarrow$ joins on its left, and ψ , the **conclusion**, is everything on its right. In $(P \wedge Q) \rightarrow R$, the hypothesis is $P \wedge Q$ and the conclusion is R . In $P \rightarrow (Q \rightarrow R)$, the hypothesis is P and the conclusion is $Q \rightarrow R$, itself an implication.
- **The hypothesis must match exactly.** Line j must be the statement φ , symbol for symbol: the letter φ occurs in both given lines, and a use of the rule replaces it by the same statement in both (Definition 1.2.1). A part of φ will not do, and neither will a statement that “says the same thing”. From $(P \wedge Q) \rightarrow R$ and $Q \wedge P$, the rule gives nothing, although $Q \wedge P$ and $P \wedge Q$ are each derivable from the other (Proposition 1.2.1); the hypothesis must first be rebuilt as $P \wedge Q$, by $\wedge E$ and $\wedge I$.
- **The new line is the whole conclusion.** The rule writes ψ , the whole of what stands right of the main $\rightarrow$, and nothing else. If ψ is $Q \wedge R$, the rule writes $Q \wedge R$, and getting Q takes one more line, by $\wedge E$.

The citation lists the implication first and its hypothesis second, always: “ $\rightarrow E\ i, j$ ”, as in the schema. Only ψ can come out, so the order changes nothing; it makes every $\rightarrow E$ line read the same way.

🕒 **Second reading** · *A legal, useless line*

The statements φ and ψ may be the same. From $P \rightarrow P$ and P , the rule writes P , which was already there. The line is legal and useless.

Why this rule. It is what “by Theorem so-and-so” means. For integers d and n , write $d \mid n$, read “ d divides n ”, when $n = dk$ for some integer k . A theorem of school arithmetic says: if an integer is divisible by 2 and by 3, then it is divisible by 6. For $n = 42$ it says

$$((2 \mid 42) \wedge (3 \mid 42)) \rightarrow (6 \mid 42).$$

We know $2 \mid 42$ ($42 = 2 \cdot 21$) and $3 \mid 42$ ($42 = 3 \cdot 14$). $\wedge I$ builds the hypothesis, and $\rightarrow E$ fires: $6 \mid 42$, which is right, as $42 = 6 \cdot 7$. As a derivation, with the theorem’s instance and the two facts as premises:

1		$((2 \mid 42) \wedge (3 \mid 42)) \rightarrow (6 \mid 42)$	
2		$2 \mid 42$	
3		$3 \mid 42$	
4		$(2 \mid 42) \wedge (3 \mid 42)$	$\wedge I$ 2, 3
5		$6 \mid 42$	$\rightarrow E$ 1, 4

Every application of a theorem has this shape: check the theorem’s hypotheses, assemble them into exactly the theorem’s hypothesis, and fire.

Quick check. To get “36 divides 72” from the theorem “if 4 and 9 divide n , then 36 divides n ” at $n = 72$, which line, besides the theorem’s instance itself, must stand on the page before $\rightarrow E$ fires?

Solution. The conjunction $(4 \mid 72) \wedge (9 \mid 72)$, exactly: the theorem’s hypothesis at $n = 72$. It is built by $\wedge I$ from the lines $4 \mid 72$ and $9 \mid 72$, which hold because $72 = 4 \cdot 18$ and $72 = 9 \cdot 8$. Then $\rightarrow E$ gives $36 \mid 72$, and indeed $72 = 36 \cdot 2$.

Why it is safe. If $\varphi \rightarrow \psi$ and φ are true, then ψ is true: that is what the promise says.

Why nothing more. From $\varphi \rightarrow \psi$ alone, $\rightarrow E$ cannot fire: an implication by itself asserts neither its hypothesis nor its conclusion. And $\rightarrow E$ does not run backwards: from $\varphi \rightarrow \psi$ together with its conclusion ψ , it gives nothing about φ . That pattern,

$$\varphi \rightarrow \psi, \psi \text{ therefore } \varphi,$$

is called **affirming the consequent**, and it is the red card of the puzzle. It is not an instance of $\rightarrow E$, because the second given line of $\rightarrow E$ must be the hypothesis φ , and here it is the conclusion ψ . It is a bad form: Section 1.1 broke it at $n = 2$. It is also the step that would repair Argument B.

Warning. Affirming the consequent looks like modus ponens and is not. Let x be a real number. “If $x > 2$, then $x^2 > 4$ ” is true for every x , and at $x = -3$ we have $x^2 = 9 > 4$; but $x > 2$ is false there. From an implication and its **conclusion**, nothing follows about the hypothesis.

Quick check. Line 1 is $P \rightarrow (Q \rightarrow R)$. What does $\rightarrow E$ give from line 1 and a line Q ? And from line 1 and a line P ?

Solution. From line 1 and Q : nothing. The main connective of line 1 is the first $\rightarrow$, so its hypothesis is P , and Q does not match it. From line 1 and P : the conclusion $Q \rightarrow R$. A second application, to $Q \rightarrow R$ and the line Q , would then give R .

The hypothesis can itself be an implication, and then the brackets put it on the left.

Quick check. Line 1 is $(P \rightarrow Q) \rightarrow (R \wedge P)$ and line 2 is $P \rightarrow Q$. What does $\rightarrow E$ 1, 2 give? And what does $\rightarrow E$ give from line 1 and a line P ?

Solution. $R \wedge P$. The main connective of line 1 is the $\rightarrow$ outside the brackets, so its hypothesis is $P \rightarrow Q$, and line 2 is that whole hypothesis. From line 1 and P : nothing. P is only a part of the hypothesis $P \rightarrow Q$.

Remark. At this point an implication can be **used** but not **proved**. A derivation can write an implication only by taking it out of a line that already contains it: $\wedge E$ applied to $(P \rightarrow Q) \wedge R$, or $\rightarrow E$ applied to $P \rightarrow (Q \rightarrow R)$ and P . No rule so far builds a new $\rightarrow$. Section 1.4 gives the rule that does.

🕒 **Second reading** · A rule is not one more premise

Picture a reader who grants both “if $x > 2$, then $x^2 > 4$ ” and “ $x > 2$ ”, yet will not write “ $x^2 > 4$ ”. We might try to persuade this reader by adding the step itself as a third premise. With P : “ $x > 2$ ” and Q : “ $x^2 > 4$ ”, the new premise is

$$((P \rightarrow Q) \wedge P) \rightarrow Q,$$

“if (if $x > 2$, then $x^2 > 4$) and $x > 2$, then $x^2 > 4$ ”. Nothing is gained. Getting from the three premises to Q still takes two steps of the rules: $\wedge I$ to join the first two premises into one line, and $\rightarrow E$ to fire the new premise on that line. A reader who would not take one step may refuse these two in the same way. A fourth premise that states this passage would call for a fifth, and so on without end: a step that is written down as a premise never gets taken.

So $\rightarrow E$ is not a statement standing among the premises. It is a **permission** to write a line, and that is why a derivation names its rules apart from its premises (Definition 1.2.2). The puzzle is an old one: it was first told, as a dialogue, in 1895.

1.3.3 Chains of implications

With $\rightarrow E$, a derivation can pass along a chain: from P and $P \rightarrow Q$ to Q , from Q and $Q \rightarrow R$ to R , and so on. Such a chain can be found from either end.

Forwards, from the premises. An available implication (for now, one on an earlier line) whose hypothesis is also available can **fire**: $\rightarrow E$ writes its conclusion, which becomes available in turn and may be the hypothesis of another implication. Keep firing until the goal appears. This is premise-directed reasoning, as in Section 1.2.

Backwards, from the goal. Suppose the goal is ψ . Look among the available lines for an implication whose **conclusion** is ψ , say $\varphi \rightarrow \psi$. If there is one, it is enough to reach φ : then $\rightarrow E$ writes ψ . So φ becomes the new goal, and we repeat. This is goal-directed reasoning of a new kind: in Section 1.2 the goal’s main connective chose an introduction rule, but here the goal may be a letter, and it still tells us which implication to look for.

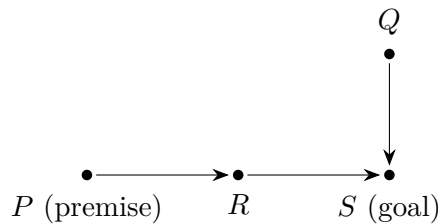
The two directions meet in the middle, when a line reached forwards is a goal reached backwards. In written mathematics the backward step has a standard phrase, “to show ψ , it suffices to show φ ”.

In the Givens/Goal table, $\rightarrow E$ fired forwards adds ψ to the Givens; worked backwards, when $\varphi \rightarrow \psi$ is among the Givens, it replaces the goal ψ by φ .

Quick check. The premises are P , $P \rightarrow R$, $Q \rightarrow S$, and $R \rightarrow S$, and the goal is S . Search backwards: which implications could give the goal, and which of them leads to a derivation?

Solution. Two premises have conclusion S : $Q \rightarrow S$ and $R \rightarrow S$. The first makes Q the new goal, and no line is Q or has conclusion Q , so that branch is a dead end. The second makes R the new goal; $P \rightarrow R$ has conclusion R , and its hypothesis P is a premise. So the derivation is R by $\rightarrow E$ from $P \rightarrow R$ and P , then S by $\rightarrow E$ from $R \rightarrow S$ and R . A backward search may branch, and a branch that stalls is abandoned for the next one.

A picture shows the search at a glance. Draw each statement as a dot and each implication as an arrow from its hypothesis to its conclusion. Forwards, follow the arrows out of what you have; backwards, follow the arrows into the goal.



Backwards from S , two arrows arrive. One comes from Q , which no arrow reaches and no premise gives: a dead end. The other comes from R , which P reaches.

Example. Chains of implications

Give a derivation of each sequent.

- (a) $P \rightarrow Q, P \vdash Q$.
- (b) $P, P \rightarrow Q, Q \rightarrow R \vdash R$.
- (c) $P \wedge Q, P \rightarrow (Q \rightarrow R) \vdash R$.
- (d) $(P \wedge Q) \rightarrow R, P, Q \vdash R$.
- (e) $(P \rightarrow Q) \wedge (R \rightarrow S), P \wedge R \vdash Q \wedge S$.

Solution.

- (a) *Idea.* There is nothing to choose: the premises are an implication and its hypothesis, and the rule fires once.

1	$P \rightarrow Q$	
2	P	
3	Q	$\rightarrow E$ 1, 2

This is the form of the two short arguments at the start of “The form of an argument” in Section 1.1 (n odd, so n^2 odd; $x > 2$, so $x^2 > 4$), now shown as a sequent: a derivation with these premises and this last line exists.

- (b) *Idea, forwards.* The premise $P \rightarrow Q$ has its hypothesis P available, so it fires and gives Q . Now $Q \rightarrow R$ has its hypothesis available, so it fires and gives R , the goal.

1	P	
2	$P \rightarrow Q$	
3	$Q \rightarrow R$	
4	Q	$\rightarrow E$ 2, 1
5	R	$\rightarrow E$ 3, 4

Idea, backwards. The goal R is a letter. Which available implication has conclusion R ? Only $Q \rightarrow R$, so it suffices to reach Q (left picture: the goal at the bottom, its new subgoal Q above it). Which implication has conclusion Q ? Only $P \rightarrow Q$, so it suffices to reach P ; and P is a premise, so the gap closes (right picture).

1	P	
2	$P \rightarrow Q$	
3	$Q \rightarrow R$	
	$\vdots$	
4	Q	
5	R	$\rightarrow E$ 3, 4

1	P	
2	$P \rightarrow Q$	
3	$Q \rightarrow R$	
4	Q	$\rightarrow E$ 2, 1
5	R	$\rightarrow E$ 3, 4

Both searches write the same five lines, the forward one from the top and the backward one from the last line up. Exercise B3 has a case where the backward search is much shorter.

- (c) *Idea.* The goal R is a letter. From the goal: the only line that mentions R is $P \rightarrow (Q \rightarrow R)$, but its conclusion is $Q \rightarrow R$, not R ; the R sits one implication deeper. From the premises: $P \wedge Q$ is a conjunction, so take it apart by $\wedge E$, giving P and Q . Now P is exactly the hypothesis of $P \rightarrow (Q \rightarrow R)$, which fires and gives $Q \rightarrow R$. This new line is an implication whose hypothesis Q is available, so it fires too. In the pictures, the goal stands alone at first; then $\wedge E$ writes lines 3 and 4, and only line 5 is missing; line 6 stays unjustified until line 5 is there for $\rightarrow E$ to cite.

1	$P \wedge Q$	
2	$P \rightarrow (Q \rightarrow R)$	
	$\vdots$	
6	R	

1	$P \wedge Q$	
2	$P \rightarrow (Q \rightarrow R)$	
3	P	$\wedge E$ 1
4	Q	$\wedge E$ 1
	$\vdots$	
6	R	

1	$P \wedge Q$	
2	$P \rightarrow (Q \rightarrow R)$	
3	P	$\wedge E 1$
4	Q	$\wedge E 1$
5	$Q \rightarrow R$	$\rightarrow E 2, 3$
6	R	$\rightarrow E 5, 4$

Line 6 is the first line in this chapter where $\rightarrow E$ fires on an implication that was itself **written by** $\rightarrow E$, not given as a premise.

- (d) *Idea.* This time the goal leads. The goal R is the conclusion of $(P \wedge Q) \rightarrow R$, so it suffices to reach its hypothesis $P \wedge Q$. That subgoal is a conjunction, so, goal-directed as in Section 1.2, it comes from $\wedge I$ with subgoals P and Q , and both are premises. In the pictures, the goal stands alone at first; then the subgoal $P \wedge Q$, read off line 1, sits above it as line 4, still unjustified.

1	$(P \wedge Q) \rightarrow R$	1	$(P \wedge Q) \rightarrow R$
2	P	2	P
3	Q	3	Q
	$\vdots$		$\vdots$
5	R	4	$P \wedge Q$
		5	R
			$\rightarrow E 1, 4$

1	$(P \wedge Q) \rightarrow R$	
2	P	
3	Q	
4	$P \wedge Q$	$\wedge I 2, 3$
5	R	$\rightarrow E 1, 4$

Compare (c). There the implication took its two hypotheses one at a time, $P \rightarrow (Q \rightarrow R)$, and the conjunction was a premise to be taken apart. Here the implication takes both hypotheses at once, $(P \wedge Q) \rightarrow R$, and the conjunction is a subgoal to be built. The content is the same; the brackets decide which rules the derivation uses. (Section 1.4 shows that each of the two implications is derivable from the other.)

- (e) *Idea.* Both ends at once. From the goal: $Q \wedge S$ is a conjunction, so its last line is $\wedge I$, with subgoals Q and S . From the premises: both are conjunctions, so $\wedge E$ takes each apart, giving $P \rightarrow Q$, $R \rightarrow S$, P , and R . The two ends meet through $\rightarrow E$: $P \rightarrow Q$ fires on P and gives the subgoal Q , and $R \rightarrow S$ fires on R and gives the subgoal S . In the second picture lines 3–6 came from the premises and line 9 from the goal; lines 7–8, still unjustified, are where the two searches meet.

		1	$(P \rightarrow Q) \wedge (R \rightarrow S)$	
		2	$P \wedge R$	
		3	$P \rightarrow Q$	$\wedge E 1$
1	$(P \rightarrow Q) \wedge (R \rightarrow S)$	4	$R \rightarrow S$	$\wedge E 1$
2	$P \wedge R$	5	P	$\wedge E 2$
	$\vdots$	6	R	$\wedge E 2$
9	$Q \wedge S$		$\vdots$	
		7	Q	
		8	S	
		9	$Q \wedge S$	$\wedge I 7, 8$

		1	$(P \rightarrow Q) \wedge (R \rightarrow S)$	
		2	$P \wedge R$	
		3	$P \rightarrow Q$	$\wedge E 1$
		4	$R \rightarrow S$	$\wedge E 1$
		5	P	$\wedge E 2$
		6	R	$\wedge E 2$
		7	Q	$\rightarrow E 3, 5$
		8	S	$\rightarrow E 4, 6$
		9	$Q \wedge S$	$\wedge I 7, 8$

Each derivation in the example is a box-free derivation (Definition 1.2.2) with $\wedge I$, $\wedge E$, and $\rightarrow E$ as its rules, so by Definition 1.2.3 each shows its sequent.

1.3.4 Checking a line of modus ponens

Finding a chain can take some searching; checking one never does. For a line justified “ $\rightarrow E i, j$ ”, the questions of Section 1.2 (which rule? is it an instance? are the cited lines earlier?) become four small ones:

1. Are lines i and j earlier than this line?
2. Is the main connective of line i a $\rightarrow$? If so, read off φ (its left side) and ψ (its right side).
3. Is line j exactly φ , symbol for symbol?
4. Is this line exactly ψ ?

Here is the check run on lines 5 and 6 of (c), and on a tempting wrong line that might replace them, “ $R \rightarrow \rightarrow E 2, 4$ ”.

Line	Cites	φ, ψ read off line i	Line j is φ ?	Line is ψ ?
5: $Q \rightarrow R$	2, 3	$P, Q \rightarrow R$	yes (P)	yes
6: R	5, 4	Q, R	yes (Q)	yes
wrong: R	2, 4	$P, Q \rightarrow R$	no (Q is not P)	no (R is not $Q \rightarrow R$)

The wrong line fails twice: it feeds line 2 the wrong hypothesis and takes out a part of the conclusion. The main connective settles it: the first $\rightarrow$ is the one applied last, so everything after it, $Q \rightarrow R$, is the conclusion.

1.3.5 Two chains about numbers

The chains so far were made of letters. Here are two made of facts about numbers, one found forwards and one backwards. In both, the premises are true school facts, and the work is to see which implication fires on what.

Example. Two chains about numbers

(a) Let n be an integer. From the five premises

$$\begin{aligned} 6 \mid n, \quad (6 \mid n) \rightarrow ((2 \mid n) \wedge (3 \mid n)), \quad (2 \mid n) \rightarrow (4 \mid n^2), \\ (3 \mid n) \rightarrow (9 \mid n^2), \quad ((4 \mid n^2) \wedge (9 \mid n^2)) \rightarrow (36 \mid n^2), \end{aligned}$$

derive $36 \mid n^2$.

(b) Let x and y be real numbers. From the six premises

$$\begin{aligned} x > 1, \quad y > 1, \quad (x > 1) \rightarrow (x^2 > 1), \quad (y > 1) \rightarrow (y^2 > 1), \\ ((x^2 > 1) \wedge (y^2 > 1)) \rightarrow (x^2 + y^2 > 2), \quad (x > 2) \rightarrow (x^2 + y^2 > 2), \end{aligned}$$

derive $x^2 + y^2 > 2$.

Solution.

(a) *The premises.* Each implication is a fact about every integer. If $n = 6k$, then $n = 2(3k)$ and $n = 3(2k)$. If $n = 2k$, then $n^2 = 4k^2$; if $n = 3k$, then $n^2 = 9k^2$. And a number divisible by 4 and by 9 is divisible by 36, the theorem of the Quick check on 72.

Idea, forwards. Fire every implication whose hypothesis is in hand. Only the second premise can fire at first, on $6 \mid n$. Its conclusion is a conjunction, which $\wedge E$ splits into $2 \mid n$ and $3 \mid n$. Now the third and fourth premises fire. The last premise wants its whole hypothesis on one line, so $\wedge I$ assembles it exactly, and the last premise fires. In the pictures, the goal stands alone at first; then lines 6–8 are written forwards, and line 11, still unjustified, is the hypothesis the last premise asks for.

1	$6 \mid n$
2	$(6 \mid n) \rightarrow ((2 \mid n) \wedge (3 \mid n))$
3	$(2 \mid n) \rightarrow (4 \mid n^2)$
4	$(3 \mid n) \rightarrow (9 \mid n^2)$
5	$((4 \mid n^2) \wedge (9 \mid n^2)) \rightarrow (36 \mid n^2)$
	$\vdots$
12	$36 \mid n^2$

Then lines 6–8, written forwards:

1	$6 \mid n$	
2	$(6 \mid n) \rightarrow ((2 \mid n) \wedge (3 \mid n))$	
3	$(2 \mid n) \rightarrow (4 \mid n^2)$	
4	$(3 \mid n) \rightarrow (9 \mid n^2)$	
5	$((4 \mid n^2) \wedge (9 \mid n^2)) \rightarrow (36 \mid n^2)$	
6	$(2 \mid n) \wedge (3 \mid n)$	$\rightarrow E \ 2, 1$
7	$2 \mid n$	$\wedge E \ 6$
8	$3 \mid n$	$\wedge E \ 6$
	$\vdots$	
11	$(4 \mid n^2) \wedge (9 \mid n^2)$	
12	$36 \mid n^2$	$\rightarrow E \ 5, 11$

The finished derivation:

1	$6 \mid n$	
2	$(6 \mid n) \rightarrow ((2 \mid n) \wedge (3 \mid n))$	
3	$(2 \mid n) \rightarrow (4 \mid n^2)$	
4	$(3 \mid n) \rightarrow (9 \mid n^2)$	
5	$((4 \mid n^2) \wedge (9 \mid n^2)) \rightarrow (36 \mid n^2)$	
6	$(2 \mid n) \wedge (3 \mid n)$	$\rightarrow E \ 2, 1$
7	$2 \mid n$	$\wedge E \ 6$
8	$3 \mid n$	$\wedge E \ 6$
9	$4 \mid n^2$	$\rightarrow E \ 3, 7$
10	$9 \mid n^2$	$\rightarrow E \ 4, 8$
11	$(4 \mid n^2) \wedge (9 \mid n^2)$	$\wedge I \ 9, 10$
12	$36 \mid n^2$	$\rightarrow E \ 5, 11$

Read back, the twelve lines say: if 6 divides n , then 36 divides n^2 .

(b) *The premises.* Each is true for all real x and y . If $x > 1$, then $x^2 = x \cdot x > x > 1$, and

the same for y . If $x^2 > 1$ and $y^2 > 1$, adding gives $x^2 + y^2 > 2$. If $x > 2$, then $x^2 > 4$, and $y^2 \geq 0$, so $x^2 + y^2 > 4 > 2$.

Idea, backwards. Two premises end in the goal. The last one makes $x > 2$ the subgoal, but no line is $x > 2$ and no implication ends in it: a dead end, abandoned. The fifth premise makes its hypothesis, a conjunction, the subgoal. $\wedge I$ splits it into $x^2 > 1$ and $y^2 > 1$, and each of these is the conclusion of an implication whose hypothesis is a premise. In the pictures, the goal stands alone at first; then the subgoal read off premise 5 sits above it as line 9, still unjustified.

1	$x > 1$
2	$y > 1$
3	$(x > 1) \rightarrow (x^2 > 1)$
4	$(y > 1) \rightarrow (y^2 > 1)$
5	$((x^2 > 1) \wedge (y^2 > 1)) \rightarrow (x^2 + y^2 > 2)$
6	$(x > 2) \rightarrow (x^2 + y^2 > 2)$
	$\vdots$
10	$x^2 + y^2 > 2$

Then the subgoal read off premise 5:

1	$x > 1$	
2	$y > 1$	
3	$(x > 1) \rightarrow (x^2 > 1)$	
4	$(y > 1) \rightarrow (y^2 > 1)$	
5	$((x^2 > 1) \wedge (y^2 > 1)) \rightarrow (x^2 + y^2 > 2)$	
6	$(x > 2) \rightarrow (x^2 + y^2 > 2)$	
	$\vdots$	
9	$(x^2 > 1) \wedge (y^2 > 1)$	
10	$x^2 + y^2 > 2$	$\rightarrow E\ 5, 9$

The finished derivation:

1	$x > 1$	
2	$y > 1$	
3	$(x > 1) \rightarrow (x^2 > 1)$	
4	$(y > 1) \rightarrow (y^2 > 1)$	
5	$((x^2 > 1) \wedge (y^2 > 1)) \rightarrow (x^2 + y^2 > 2)$	
6	$(x > 2) \rightarrow (x^2 + y^2 > 2)$	
7	$x^2 > 1$	$\rightarrow E$ 3, 1
8	$y^2 > 1$	$\rightarrow E$ 4, 2
9	$(x^2 > 1) \wedge (y^2 > 1)$	$\wedge I$ 7, 8
10	$x^2 + y^2 > 2$	$\rightarrow E$ 5, 9

Premise 6 is true and never cited. A true premise can still be a dead end: what decides a backward search is whether the subgoal can be reached, not whether the implication is true.

Quick check. In (b), why is premise 6 never fired?

Solution. Its hypothesis $x > 2$ stands on no line, and no line has it as its conclusion, so $\rightarrow E$ has nothing to fire it on.

1.3.6 How this looks in a mathematical proof

Let x be a real number with $x > 2$. Suppose we know two facts of school algebra:

- if $x > 2$, then $x^2 > 4$ (since $x > 2 > 0$, multiplying $x > 2$ by x gives $x^2 > 2x$, and $2x > 4$);
- if $x^2 > 4$, then $x^2 + 1 > 5$ (add 1 to both sides).

A written proof that $x^2 + 1 > 5$ takes one sentence: *Since $x > 2$, we have $x^2 > 4$, and so $x^2 + 1 > 5$.* As a derivation, with the two facts and the hypothesis as premises, it reads as follows.

1	$(x > 2) \rightarrow (x^2 > 4)$	
2	$(x^2 > 4) \rightarrow (x^2 + 1 > 5)$	
3	$x > 2$	
4	$x^2 > 4$	$\rightarrow E$ 1, 3
5	$x^2 + 1 > 5$	$\rightarrow E$ 2, 4

Each piece of the sentence is one line.

Line	Sentence
3	“Since $x > 2$, ...” (the hypothesis)
4	“... we have $x^2 > 4$, ...” ($\rightarrow$ E, firing the first fact)
5	“... and so $x^2 + 1 > 5$.” ($\rightarrow$ E, firing the second fact)

The sentence never mentions lines 1 and 2: writers leave a known implication unstated. Each “since φ , ψ ” in a proof is one application of $\rightarrow$ E to an implication the reader is trusted to supply. When a proof is hard to follow, writing those implications down is often how one finds the gap: an implication is false, or its hypothesis was never established.

1.3.7 Common errors

The first warning of this section was about the wrong given line: the conclusion in place of the hypothesis. Another common error is a hypothesis that is only partly there.

Warning. $\rightarrow$ E needs the **whole** hypothesis. From $(P \wedge Q) \rightarrow R$ and P alone, it gives nothing. Take the true theorem “if $x > 0$ and $y > 0$, then $xy > 0$ ” and the line $x > 0$: at $x = 1$, $y = -1$ the theorem and $x > 0$ are true, and $xy > 0$ is false.

○ **Second reading** · *The same slip in analysis*

This is the commonest slip in analysis: “ f is continuous, so by the theorem f attains a maximum”. The theorem meant has a second hypothesis, that f is defined on a closed bounded interval, and without it the conclusion fails: $f(x) = x$ on the open interval $(0, 1)$ is continuous and has no largest value, since $\frac{a+1}{2}$ is a larger value than a for every a in $(0, 1)$.

A near relative is a hypothesis in the wrong order, as in the third bullet of the rule’s reading: $Q \wedge P$ is not the hypothesis of $(P \wedge Q) \rightarrow R$. The repair is two $\wedge$ E lines and one $\wedge$ I line, and Exercise B1 asks for it.

Warning. A third error: firing an arrow that is not the whole line. $\rightarrow$ E fires only on a **whole line** whose main connective is $\rightarrow$. From $P \wedge (Q \rightarrow R)$ and Q , first take out $Q \rightarrow R$ by $\wedge$ E; the step straight to R is not an instance of the rule, although it never leads from truth to falsehood. From $(P \rightarrow Q) \vee R$ and P , $\rightarrow$ E cannot fire at all, and the line Q does not follow. Let x be a real number, with P : “ $x > 2$ ”, Q : “ $x > 4$ ”, R : “ $x \leq 4$ ”. The first line holds for every x : if $x \leq 4$, then R is true, and if $x > 4$, then Q is true, so $P \rightarrow Q$ is. At $x = 3$ the line P is true and Q is false.

1.3.8 The rule card

Before the card, the rule at work on large statements. Each row is one use of $\rightarrow$ E: the two given lines, and the line it writes.

Example. Instances of $\rightarrow E$

φ	ψ	The implication	The hypothesis line	New line
$P \wedge (Q \vee R)$	$S \rightarrow P$	$(P \wedge (Q \vee R)) \rightarrow (S \rightarrow P)$	$P \wedge (Q \vee R)$	$S \rightarrow P$
$P \rightarrow Q$	$R \wedge (S \vee P)$	$(P \rightarrow Q) \rightarrow (R \wedge (S \vee P))$	$P \rightarrow Q$	$R \wedge (S \vee P)$
$(P \vee Q) \wedge \neg R$	$(P \wedge S) \vee Q$	$((P \vee Q) \wedge \neg R) \rightarrow ((P \wedge S) \vee Q)$	$(P \vee Q) \wedge \neg R$	$(P \wedge S) \vee Q$

Not an instance. From $(P \rightarrow Q) \rightarrow R$ and P , $\rightarrow E$ writes neither Q nor R : the hypothesis is $P \rightarrow Q$, and P is only a part of it.

The card of Section 1.2 grows by one row. *Shape* shows the given lines, then the new line; *You write* is the phrase a written proof uses for the rule.

Rule	Shape	You write	Lean
$\wedge I$	$\varphi, \psi \Longrightarrow \varphi \wedge \psi$	“We show P , and we show Q .”	constructor
$\wedge E$	$\varphi \wedge \psi \Longrightarrow \varphi$ (or ψ)	“In particular, P .”	h.1 (and h.2)
$\rightarrow E$	$\varphi \rightarrow \psi, \varphi \Longrightarrow \psi$	“Since P implies Q , and P holds, Q .”	exact h hp

In the Lean column, **h** is the implication and **hp** the proof of its hypothesis; **h hp** applies the one to the other, which is modus ponens.

Warning (Not allowed: $\rightarrow E$). Each step below is wrong, for the reason after the colon; the reason names the bullet of the rule’s reading that it breaks.

- “ R from $P \rightarrow (Q \rightarrow R)$ and Q ”: Q is not the hypothesis, which is P (*the hypothesis must match exactly*).
- “ P from $P \rightarrow Q$ and Q ”: the second given line must be the hypothesis, not the conclusion; this is affirming the consequent (*two given lines*).
- “ R from $(P \wedge Q) \rightarrow R$ and P ”: the whole hypothesis must stand on a line (*the hypothesis must match exactly*).
- “ Q from $P \rightarrow (Q \wedge R)$ and P ”: the new line is the whole conclusion, $Q \wedge R$ (*the new line is the whole conclusion*).
- “ R from $P \wedge (Q \rightarrow R)$ and Q ”: the cited line’s main connective is $\wedge$; take out $Q \rightarrow R$ first (*read φ and ψ off the main connective*).

Remark (Looking ahead). Chapter 4 proves more than “affirming the consequent is a bad form”: no derivation, however long, works for every choice of P and Q and leads from $P \rightarrow Q$ and Q to P .

1.3.9 Exercises

1.3.9.1 A. Check your understanding

Exercise. A1

- (a) Write the schema of $\rightarrow E$ from memory.
- (b) Lines 1 and 2 of a derivation are $P \rightarrow (Q \wedge R)$ and P . Which of the following can line 3 be, by one application of $\rightarrow E$? $Q \wedge R$; Q ; $R \wedge Q$; $P \rightarrow Q$.
- (c) In [Example \(Four cards\)](#), why is the red card not worth turning?
- (d) True or false, with a reason: from lines $P \rightarrow Q$ and $Q \rightarrow R$, $\rightarrow E$ gives $P \rightarrow R$.

Solution.

- (a) Given lines i : $\varphi \rightarrow \psi$ and j : φ ; new line k : ψ , justified “ $\rightarrow E$ i, j ” ([Definition 1.3.1](#)).
- (b) Only $Q \wedge R$, the conclusion of line 1. The line Q needs a further $\wedge E$, and $R \wedge Q$ two further $\wedge E$ lines and a $\wedge I$ line. The line $P \rightarrow Q$ is not the conclusion of line 1, so $\rightarrow E$ cannot write it.
- (c) The claim is “even $\rightarrow$ red”, and the red card shows the conclusion. An odd number behind it would not contradict the claim, which says nothing about which cards are red; an even number would fit it. Inferring anything about its number from its color would be affirming the consequent.
- (d) False. $\rightarrow E$ needs an implication and a line that is its hypothesis. The hypothesis of $Q \rightarrow R$ is Q , and $P \rightarrow Q$ is not Q ; the hypothesis of $P \rightarrow Q$ is P , which is not a line. So $\rightarrow E$ does not apply at all. (The sequent $P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R$ does hold, but its derivation needs the rule of Section 1.4.)

Exercise. A2: Drill

Each part takes a line or two. Lines are numbered as in a derivation. The parts run from one step, through a missing line, a justification column and steps to judge, to short derivations.

- (a) Lines 1 $(P \vee Q) \rightarrow R$, 2 $P \vee Q$. What does “ $\rightarrow E$ 1, 2” give?
- (b) Lines 1 $P \rightarrow Q \wedge R$, 2 P . What does “ $\rightarrow E$ 1, 2” give?
- (c) Lines 1 $((P \vee Q) \wedge R) \rightarrow (S \rightarrow P)$, 2 $(P \vee Q) \wedge R$. What does “ $\rightarrow E$ 1, 2” give?
- (d) Line 7 is $R \wedge S$, justified “ $\rightarrow E$ 4, 6”, and line 6 is $P \rightarrow Q$. What is line 4?
- (e) Line 5 is Q , justified “ $\rightarrow E$ 2, 3”, and line 2 is $(R \wedge S) \rightarrow Q$. What is line 3?
- (f) Justify each line, the first two being premises: 1 $(P \rightarrow Q) \wedge R$, 2 $R \rightarrow P$, 3 $P \rightarrow Q$, 4 R , 5 P , 6 Q .
- (g) Lines 1 $P \rightarrow (Q \rightarrow R)$, 2 Q . Is “ $\rightarrow E$ 1, 2” legal?
- (h) Lines 1 $(P \wedge Q) \rightarrow R$, 2 $Q \wedge P$. Is “ R — $\rightarrow E$ 1, 2” legal?
- (i) Lines 1 $P \rightarrow Q$, 2 R , 3 $R \wedge P$. Is “4. Q — $\rightarrow E$ 1, 3” legal?
- (j) Lines 1 $(P \rightarrow Q) \wedge R$, 2 P . Is “3. Q — $\rightarrow E$ 1, 2” legal? If not, repair it.
- (k) Break the form of “if n is divisible by 6, then by 3; n is divisible by 3; so n is divisible by 6”.

- (l) Premises $P \rightarrow Q, R \rightarrow Q, R$; goal Q . Searching backwards, which implication leads to a derivation?
- (m) Derive $P \wedge (P \rightarrow Q) \vdash Q$.
- (n) Derive $P \rightarrow (Q \rightarrow R), P, Q \vdash R$.
- (o) Derive $P \rightarrow (Q \wedge R), (Q \wedge R) \rightarrow S, P \vdash S \wedge Q$.

Solution.

- (a) R , the conclusion of line 1, whose hypothesis $P \vee Q$ is line 2 exactly.
- (b) $Q \wedge R$: since $\wedge$ binds more tightly, line 1 is $P \rightarrow (Q \wedge R)$. Getting Q needs one more $\wedge E$ line.
- (c) $S \rightarrow P$: the hypothesis of line 1 is $(P \vee Q) \wedge R$, which is line 2 exactly, and its conclusion is the implication $S \rightarrow P$.
- (d) $(P \rightarrow Q) \rightarrow (R \wedge S)$. $\rightarrow E$ cites the implication first, so line 4 is an implication whose hypothesis is line 6 and whose conclusion is line 7.
- (e) $R \wedge S$, the hypothesis of line 2, exactly.
- (f) Lines 1 and 2 are premises; 3 is $\wedge E$ 1; 4 is $\wedge E$ 1; 5 is $\rightarrow E$ 2, 4; 6 is $\rightarrow E$ 3, 5. An implication cut out of a conjunction fires on a line that $\rightarrow E$ wrote.
- (g) No: the hypothesis of line 1 is P , and line 2 is Q .
- (h) No: the hypothesis must be $P \wedge Q$ exactly, and $Q \wedge P$ is a different statement.
- (i) No: line 3 is $R \wedge P$, not the hypothesis P . First write P by $\wedge E$ 3 as line 4, then Q by $\rightarrow E$ 1, 4.
- (j) No: the main connective of line 1 is $\wedge$, so line 1 is not an implication, and $\rightarrow E$ cannot fire on it. Repair: 3 $P \rightarrow Q, \wedge E$ 1; 4 $Q, \rightarrow E$ 3, 2.
- (k) Take $n = 3$. The first premise is true for every integer, since $6k = 3 \cdot 2k$; the second is true, since $3 = 3 \cdot 1$; and 3 is not divisible by 6. Both premises true, conclusion false: the form is affirming the consequent.
- (l) $R \rightarrow Q$: its hypothesis R is a premise, so $\rightarrow E$ fires at once. The other, $P \rightarrow Q$, makes P the new goal, and nothing gives P : a dead end.
- (m) 2 $P, \wedge E$ 1; 3 $P \rightarrow Q, \wedge E$ 1; 4 $Q, \rightarrow E$ 3, 2.
- (n) 4 $Q \rightarrow R, \rightarrow E$ 1, 2; 5 $R, \rightarrow E$ 4, 3.
- (o) 4 $Q \wedge R, \rightarrow E$ 1, 3; 5 $S, \rightarrow E$ 2, 4; 6 $Q, \wedge E$ 4; 7 $S \wedge Q, \wedge I$ 5, 6.

1.3.9.2 B. Practice

Exercise. B1: Two chains

Give a derivation of each sequent. For each, say which step you found first, and whether it came from the goal or from a premise.

- (a) $P, P \rightarrow Q, Q \rightarrow (R \wedge S) \vdash S \wedge Q$.
- (b) $(P \wedge Q) \rightarrow R, Q \wedge P \vdash R$.

Solution.

- (a) The first step found was from the goal: $S \wedge Q$ is a conjunction, so the last line is $\wedge I$ with subgoals S and Q . Then from the premises: $P \rightarrow Q$ fires on P and gives Q , and $Q \rightarrow (R \wedge S)$ fires on Q and gives $R \wedge S$, which $\wedge E$ takes apart.

1	P	
2	$P \rightarrow Q$	
3	$Q \rightarrow (R \wedge S)$	
4	Q	$\rightarrow E$ 2, 1
5	$R \wedge S$	$\rightarrow E$ 3, 4
6	S	$\wedge E$ 5
7	$S \wedge Q$	$\wedge I$ 6, 4

Line 4 is cited twice, by line 5 and by line 7.

- (b) The first step found was from the goal: R is the conclusion of line 1, so it suffices to reach $P \wedge Q$ exactly. The premise $Q \wedge P$ is not it. So take the premise apart and rebuild in the right order.

1	$(P \wedge Q) \rightarrow R$	
2	$Q \wedge P$	
3	Q	$\wedge E$ 2
4	P	$\wedge E$ 2
5	$P \wedge Q$	$\wedge I$ 4, 3
6	R	$\rightarrow E$ 1, 5

Exercise. B2: Two facts and a number

Let x be a real number. Take as premises the two facts “if $0 < x < 1$, then $x^2 < x$ ” and “if $x^2 < x$, then $x^3 < x^2$ ”, and the hypothesis $0 < x < 1$. Derive $x^3 < x^2$, naming the rule on each line. Then write the argument as one or two sentences of prose.

Solution. The hypothesis $0 < x < 1$ is the conjunction $(0 < x) \wedge (x < 1)$, which is exactly the hypothesis of the first fact, so the first fact fires on it at once.

1	$((0 < x) \wedge (x < 1)) \rightarrow (x^2 < x)$	
2	$(x^2 < x) \rightarrow (x^3 < x^2)$	
3	$(0 < x) \wedge (x < 1)$	
4	$x^2 < x$	$\rightarrow E$ 1, 3
5	$x^3 < x^2$	$\rightarrow E$ 2, 4

In prose: *Since* $0 < x < 1$, *we have* $x^2 < x$, *and therefore* $x^3 < x^2$. Both facts are true for every real x : if $0 < x < 1$, multiplying $x < 1$ by $x > 0$ gives $x^2 < x$; and if $x^2 < x$, then

$x(x - 1) < 0$, so $x > 0$, and multiplying $x^2 < x$ by x gives $x^3 < x^2$.

Exercise. B3: Which premises are needed?

Find a derivation of

$$P, P \rightarrow Q, P \rightarrow R, R \rightarrow S, S \rightarrow U, Q \rightarrow T \vdash T,$$

and list the premises it uses. Explain why a backward search finds the needed premises faster than a forward one here.

Solution. Search backwards. The goal T is the conclusion of $Q \rightarrow T$ only, so it suffices to reach Q . The goal Q is the conclusion of $P \rightarrow Q$ only, so it suffices to reach P , a premise.

1	P	
2	$P \rightarrow Q$	
3	$P \rightarrow R$	
4	$R \rightarrow S$	
5	$S \rightarrow U$	
6	$Q \rightarrow T$	
7	Q	$\rightarrow E$ 2, 1
8	T	$\rightarrow E$ 6, 7

It uses premises 1, 2, and 6. The others are never cited, which is allowed: a derivation need not use every premise (Definition 1.2.3). Deleting lines 3–5 and renumbering gives a derivation of $P, P \rightarrow Q, Q \rightarrow T \vdash T$.

A forward search fires every implication whose hypothesis is available, relevant or not: from P it writes Q and R , then S from R , then U from S , and T from Q — five new lines, three of them useless. The backward search looks only at implications whose conclusion is the current goal, and at each stage exactly one qualifies, so it writes only the two lines the derivation needs.

Exercise. B4: Assemble the hypothesis

A search for $P \rightarrow (Q \wedge R)$, $(R \wedge P) \rightarrow S$, $P \vdash S$ has reached the stage below. Name the move, then complete it.

1	$P \rightarrow (Q \wedge R)$	
2	$(R \wedge P) \rightarrow S$	
3	P	
	$\vdots$	
7	S	?

Solution. The move is to work **backwards from the goal**. The goal S is the conclusion of line 2, so it suffices to reach line 2's hypothesis $R \wedge P$, exactly, and then fire. That subgoal is a conjunction: P is line 3, and R comes from line 1, which fires on P and gives $Q \wedge R$, whose right half is R .

1	$P \rightarrow (Q \wedge R)$	
2	$(R \wedge P) \rightarrow S$	
3	P	
4	$Q \wedge R$	$\rightarrow E$ 1, 3
5	R	$\wedge E$ 4
6	$R \wedge P$	$\wedge I$ 5, 3
7	S	$\rightarrow E$ 2, 6

Line 6 writes $R \wedge P$, with R on the left, because that is how the hypothesis of line 2 is written; $P \wedge R$ would not match it. Line 3 is cited twice, by lines 4 and 6.

Exercise. B5: Supply the missing premise

In (a)–(c), add **one** premise, not the conclusion itself, so that the sequent can be derived with $\wedge I$, $\wedge E$ and $\rightarrow E$, and give the derivation. In (d), each sentence about an integer n leaves an implication unsaid: write it, and decide whether it is true for every integer n .

- (a) $P \rightarrow Q, Q \rightarrow R \vdash R$.
- (b) $P, (P \wedge Q) \rightarrow R \vdash R$.
- (c) $P, Q \vdash R \wedge P$.
- (d) (i) “Since 4 divides n , n is even.” (ii) “Since n is even, n^2 is divisible by 4.” (iii) “Since n^2 is even, n is even.” (iv) “Since n^2 is divisible by 4, n is divisible by 4.”

Solution.

- (a) Several premises work. Add P :

1	$P \rightarrow Q$	
2	$Q \rightarrow R$	
3	P	
4	Q	$\rightarrow E$ 1, 3
5	R	$\rightarrow E$ 2, 4

Or add Q : then 4 R , $\rightarrow E$ 2, 3. Any premise from which these rules reach P or Q (for instance $P \wedge Q$) is right too.

(b) Several premises work. Add Q :

1	P	
2	$(P \wedge Q) \rightarrow R$	
3	Q	
4	$P \wedge Q$	$\wedge I$ 1, 3
5	R	$\rightarrow E$ 2, 4

Or add $P \wedge Q$ itself (then one $\rightarrow E$ line), or $Q \wedge P$ (then two $\wedge E$ lines and one $\wedge I$ line first). Any premise from which these rules reach Q or $P \wedge Q$ is right.

(c) Several premises work: any premise from which, together with P and Q , these rules reach R . For instance $P \rightarrow R$:

1	P	
2	Q	
3	$P \rightarrow R$	
4	R	$\rightarrow E$ 3, 1
5	$R \wedge P$	$\wedge I$ 4, 1

$Q \rightarrow R$ works the same way, and so does $(P \wedge Q) \rightarrow R$, with an $\wedge I$ line first. The premise R alone also works, and it says the most: it hands over the new part of the conclusion outright.

- (d) (i) “If $4 \mid n$, then n is even”: true, since $4k = 2(2k)$. (ii) “If n is even, then $4 \mid n^2$ ”: true, since $n = 2k$ gives $n^2 = 4k^2$. (iii) “If n^2 is even, then n is even”: true, by Argument A of Section 1.1. (iv) “If $4 \mid n^2$, then $4 \mid n$ ”: false at $n = 2$, where $4 \mid 4$ and 4 does not divide 2. The sentence in (iv) looks just like the other three. A missing premise is where a hidden error hides.

1.3.9.3 C. Going deeper

Exercise. C1: Find the error

Here is an argument, followed by the derivation its author offers for it.

If 10 is divisible by 6, then 10 is even. And 10 is even. So 10 is divisible by 6.

With D : “10 is divisible by 6” and E : “10 is even”:

1	$D \rightarrow E$	
2	E	
3	D	$\rightarrow E$ 1, 2

- (a) Name the line that fails, and say why it is not an instance of the schema of $\rightarrow E$.
- (b) Show that the step can lead from true premises to a false conclusion.
- (c) What new statement about divisibility can be correctly derived from the two premises with the rules so far?

Solution.

- (a) Line 3. The schema needs, besides the implication $\varphi \rightarrow \psi$, a line that is its hypothesis φ . Matching line 1 gives $\varphi := D$ and $\psi := E$, so the second cited line would have to be D ; line 2 is E , which is ψ . The line affirms the consequent.
- (b) Try to break the step: this instance already does. The first premise is true, because every multiple of 6 is even ($6k = 2(3k)$); the second is true, since $10 = 2 \cdot 5$; and the conclusion is false: $6k$ is at most 6 when $k \leq 1$ and at least 12 when $k \geq 2$, so $10 = 6k$ for no integer k . So the step at line 3 leads from two true premises to a false line, and no safe rule can allow it.
- (c) None. Go down any derivation from these premises line by line, as in Exercise (C4: Challenge: What conjunction cannot do): every line is $D \rightarrow E$, E , or a conjunction built from them by $\wedge$ alone. A premise is of this kind. $\wedge I$ joins two such lines into another one. $\wedge E$ gives back a conjunct of such a line, which is again of this kind. $\rightarrow E$ would need, besides the implication $D \rightarrow E$, a line D , and D is not of this kind; so $\rightarrow E$ never fires, and no line is D . What we can reach are the premises and conjunctions of them, such as $(D \rightarrow E) \wedge E$, which say nothing the premises did not. The premises tell us nothing about whether 10 is divisible by 6, which is why the conclusion had to be smuggled in.

Exercise. C2: Both ends

Consider the sequent

$$P \rightarrow (Q \rightarrow R), S \rightarrow P, S \wedge Q \vdash R \wedge S.$$

- (a) Describe a search for a derivation in words, in three parts: what the goal asks for; what the premises give; where the two meet.
- (b) Write the derivation, naming the rule on every line.

Solution (Hint 1). Two premises have main connective $\rightarrow$ and one has $\wedge$; the goal has $\wedge$. Write down what each main connective lets you do.

Solution (Hint 2). The goal splits into R and S . Which available implication could end in R , and after how many firings?

Solution.

- (a) *From the goal:* $R \wedge S$ is a conjunction, so the last line is $\wedge I$ with subgoals R and S . The only line mentioning R is $P \rightarrow (Q \rightarrow R)$, whose conclusion is $Q \rightarrow R$; so R needs $Q \rightarrow R$ and Q , and $Q \rightarrow R$ needs P . *From the premises:* $S \wedge Q$ splits by $\wedge E$ into S and Q ; then $S \rightarrow P$ fires on S and gives P . *Where they meet:* S is at once a subgoal and a piece of the premise; P , reached forwards, is the hypothesis the backward search asked for; and Q , from $\wedge E$, is the last hypothesis R needs.
- (b) The derivation, with the rule on every line:

1	$P \rightarrow (Q \rightarrow R)$	
2	$S \rightarrow P$	
3	$S \wedge Q$	
4	S	$\wedge E$ 3
5	Q	$\wedge E$ 3
6	P	$\rightarrow E$ 2, 4
7	$Q \rightarrow R$	$\rightarrow E$ 1, 6
8	R	$\rightarrow E$ 7, 5
9	$R \wedge S$	$\wedge I$ 8, 4

Line 4 is cited twice, by line 6 and by line 9.

Marking scheme. (10 points)

- 2: from the goal, splits $R \wedge S$ into the subgoals R and S for a final $\wedge I$.
- 2: takes $S \wedge Q$ apart by $\wedge E$.
- 3: fires $S \rightarrow P$, then $P \rightarrow (Q \rightarrow R)$, then $Q \rightarrow R$, each $\rightarrow E$ citing an implication and exactly its hypothesis.
- 2: the final $\wedge I$, whose line is $R \wedge S$.
- 1: the three-part description names where the searches meet.

Any other correct derivation earns the same points step for step (its lines may come in another order, as long as each cites only earlier lines). Fatal errors (0 for the step): $\rightarrow E$ on $P \rightarrow (Q \rightarrow R)$ with Q (the wrong hypothesis); affirming the consequent anywhere.