

23.8 Characters and Orthogonality

Complete reducibility (Corollary 23.7.4) promises that every complex representation of a finite group breaks into irreducible pieces, but the promise is not constructive: it says nothing about which pieces, or how many of each. This section produces a small table of numbers that answers both questions. To each representation we attach one function $G \rightarrow \mathbb{C}$, its character, and we prove that the characters of the irreducible representations are an **orthonormal basis** of a space of functions we can compute the dimension of by counting. Everything a complex representation is, up to equivalence, is then read off from one row of numbers.

Throughout, G is a **finite** group and the field is $\mathbb{C}$. The hypothesis matters twice over: $\text{char } \mathbb{C} = 0$ never divides $|G|$, so Theorem 23.7.3 applies; and $\mathbb{C}$ is algebraically closed, so Schur's lemma (Theorem 23.4.1) gives that the only operators commuting with an irreducible representation are the scalars. The theory over a field whose characteristic divides $|G|$ — modular representation theory — is a different subject, and this book does not enter it.

23.8.1 Conjugacy classes

Before characters we need the sets they are constant on. The book has met them only in an exercise of Section 1 (Exercise (C1: The center of a group algebra)), so we build them here.

Definition 23.8.1. Conjugate elements, conjugacy class

Let G be a group. Elements $g, h \in G$ are **conjugate** if $h = xgx^{-1}$ for **some** $x \in G$. The **conjugacy class** of g is

$$C(g) := \{xgx^{-1} : x \in G\}.$$

Conjugacy is an equivalence relation (Definition 0.5.2): $g = 1g1^{-1}$ gives reflexivity; if $h = xgx^{-1}$, then $g = x^{-1}h(x^{-1})^{-1}$, which gives symmetry using Theorem 0.10.1; and if $h = xgx^{-1}$ and $k = yhy^{-1}$, then $k = (yx)g(yx)^{-1}$, which gives transitivity. Hence the conjugacy classes are the equivalence classes, and by Theorem 0.5.2 they **partition** G : every element lies in exactly one class. We write k for the number of conjugacy classes of G , a finite number since G is finite.

In an abelian group $xgx^{-1} = g$ for every x , so every class is a single element and $k = |G|$. At the other extreme, $C(1) = \{1\}$ always.

Example. The conjugacy classes of the symmetric group on three letters

Find the conjugacy classes of S_3 .

Solution. First a rule that does the work. For $\sigma \in S_n$ and a cycle $(i_1 i_2 \cdots i_r)$,

$$\sigma(i_1 i_2 \cdots i_r)\sigma^{-1} = (\sigma(i_1) \sigma(i_2) \cdots \sigma(i_r)).$$

Both sides are permutations, so it is enough to check they agree at every point. At $\sigma(i_j)$: the left side applies σ^{-1} to get i_j , then the cycle to get i_{j+1} (indices read cyclically), then σ to get $\sigma(i_{j+1})$; the right side does the same in one step. At a point $\sigma(m)$ with $m \notin \{i_1, \dots, i_r\}$: the left side gives $\sigma(m)$ back, and so does the right side, since $\sigma(m)$ is not among the $\sigma(i_j)$ by injectivity. Every point of $\{1, \dots, n\}$ is $\sigma(m)$ for exactly one m , so the two agree.

So conjugation relabels the entries of a cycle. In S_3 , conjugating $(1\ 2)$ by the six elements produces $(\sigma(1)\ \sigma(2))$, and as σ runs over S_3 the pair $\{\sigma(1), \sigma(2)\}$ runs over all three two-element subsets; so $C((1\ 2)) = \{(1\ 2), (1\ 3), (2\ 3)\}$. Conjugating $(1\ 2\ 3)$ gives $(\sigma(1)\ \sigma(2)\ \sigma(3))$, which is $(1\ 2\ 3)$ or $(1\ 3\ 2)$ according to the cyclic order; both occur, for $\sigma = \text{id}$ and $\sigma = (1\ 2)$, so $C((1\ 2\ 3)) = \{(1\ 2\ 3), (1\ 3\ 2)\}$. The three classes are

$$\{\text{id}\}, \quad \{(1\ 2), (1\ 3), (2\ 3)\}, \quad \{(1\ 2\ 3), (1\ 3\ 2)\},$$

of sizes 1, 3, 2, which add to $6 = |S_3|$ as a partition must. So $k = 3$ for S_3 — the same number as the three irreducible complex representations found in Example (The irreducible complex representations of the symmetric group on three letters). That is no coincidence, and the theorem explaining it is the main result of this section.

23.8.2 The character of a representation

A representation ρ on V is a whole family of matrices, and the matrices change when the basis does. To compare representations we want something attached to ρ that does not move. Chapter 2 already supplies the candidate: the trace of an operator does not depend on the basis (Definition 4.5.2), and similar operators have equal traces.

The character of a representation records one number per group element: the trace of the operator that element acts by.

Definition 23.8.2. Character

Let ρ be a representation of a finite group G on a finite-dimensional complex vector space V . The **character** of ρ is the function

$$\chi_\rho: G \rightarrow \mathbb{C}, \quad \chi_\rho(g) := \text{tr } \rho(g).$$

Its **degree** is $\chi_\rho(1) = \dim V$. A character is **irreducible** if ρ is irreducible (Definition 23.7.4).

In words: a character is a list of $|G|$ complex numbers, one per group element, obtained by computing traces. It throws away a great deal — the whole matrix becomes one number — and the content of this section is that it throws away exactly the information we did not want.

Examples. Let G be finite.

- The trivial representation has $\chi(g) = \text{tr}(\text{id}_{\mathbb{C}}) = 1$ for every g . We write χ_{triv} for the constant function 1.
- The sign representation of S_n has $\chi(\sigma) = \text{sgn}(\sigma)$.
- The permutation representation of S_n on $\mathbb{C}^n$ has $\chi(\sigma) = \#\{i : \sigma(i) = i\}$, the number of **fixed points**. Indeed the matrix of $\rho(\sigma)$ in the standard basis has (i, i) -entry equal to 1 exactly when $\sigma(i) = i$, and 0 otherwise.
- The regular representation on $\mathbb{C}[G]$ has $\chi_{\text{reg}}(1) = |G|$ and $\chi_{\text{reg}}(g) = 0$ for $g \neq 1$. The matrix of $\rho_{\text{reg}}(g)$ in the basis $\{\delta_h\}$ has (h, h) -entry equal to 1 exactly when $gh = h$, that is, when $g = 1$ by cancellation (Theorem 0.10.1).

One fact about finite groups is needed first, and Chapter 0 did not record it. It costs four lines, and Section 9 will use it constantly.

Lemma 23.8.1. Order of an Element

Let G be a finite group and $g \in G$. Then there is a least integer $m \geq 1$ with $g^m = 1$, called the **order** of g . Moreover $g^k = g^l$ if and only if m divides $k - l$, and

$$\langle g \rangle := \{g^k : k \in \mathbb{Z}\} = \{1, g, g^2, \dots, g^{m-1}\}$$

is a subgroup of G with exactly m elements.

Proof. The $|G| + 1$ elements $g^0, g^1, \dots, g^{|G|}$ cannot all be distinct, so $g^i = g^j$ for some $0 \leq i < j$. Canceling g^i (Theorem 0.10.1) gives $g^{j-i} = 1$ with $j - i \geq 1$. Hence the set of positive integers p with $g^p = 1$ is nonempty, and it has a least element m .

Suppose $g^k = g^l$, so $g^{k-l} = 1$. Divide: $k - l = qm + r$ with $0 \leq r < m$. Then

$$g^r = (g^m)^{-q} g^{k-l} = 1,$$

and minimality of m forces $r = 0$, that is $m \mid k - l$. Conversely $m \mid k - l$ gives $g^{k-l} = (g^m)^{(k-l)/m} = 1$, so $g^k = g^l$.

Consequently g^k depends only on k modulo m , so $\langle g \rangle = \{1, g, \dots, g^{m-1}\}$, and those m elements are distinct because $m \nmid k - l$ whenever $0 \leq l < k \leq m - 1$. Finally $1 \in \langle g \rangle$, $g^k g^l = g^{k+l} \in \langle g \rangle$, and $(g^k)^{-1} = g^{-k} \in \langle g \rangle$, so $\langle g \rangle$ is a subgroup (Definition 0.10.2). ■

The next proposition collects the properties we use constantly. Chapter 4's trace results do all the work.

Proposition 23.8.2. Basic properties of characters

Let ρ be a representation of a finite group G on a complex space V , with character $\chi = \chi_\rho$. Then:

- (a) $\chi(1) = \dim V$;
- (b) χ is a **class function**: $\chi(xgx^{-1}) = \chi(g)$ for all $g, x \in G$;
- (c) equivalent representations have equal characters;
- (d) if $V = U \oplus W$ with both summands G -invariant, then $\chi = \chi_{\rho|_U} + \chi_{\rho|_W}$;
- (e) $\chi(g^{-1}) = \overline{\chi(g)}$ for every $g \in G$, and $|\chi(g)| \leq \chi(1)$.

Idea. Parts (a)–(d) are the four standard facts about the trace read in this setting: the trace of the identity, invariance under conjugation, invariance under similarity, and additivity over a block decomposition. Part (e) is different: it needs to know that $\rho(g)$ is diagonalizable with eigenvalues on the unit circle, which comes from g having finite order.

Proof.

- (a) $\rho(1) = \text{id}_V$ (Theorem 0.10.3), and $\text{tr id}_V = (\dim V) \cdot 1 = \dim V$ in $\mathbb{C}$ (Theorem 4.5.3).

- (b) $\chi(xgx^{-1}) = \text{tr}(\rho(x)\rho(g)\rho(x)^{-1}) = \text{tr}(\rho(g)\rho(x)^{-1}\rho(x)) = \text{tr} \rho(g) = \chi(g)$, where the second equality is $\text{tr}(ST) = \text{tr}(TS)$ (Theorem 4.5.3) applied with $S = \rho(x)$ and $T = \rho(g)\rho(x)^{-1}$.
- (c) If $\rho'(g) = S\rho(g)S^{-1}$ for an isomorphism S (Definition 23.7.3), the same computation as in (b) gives $\text{tr} \rho'(g) = \text{tr} \rho(g)$.
- (d) Choose a basis of V formed by a basis of U followed by a basis of W . Since both are invariant, the matrix of $\rho(g)$ in this basis is block diagonal, with the matrices of $\rho(g)|_U$ and $\rho(g)|_W$ on the diagonal. The diagonal entries of the big matrix are those of the two blocks, so the traces add (Definition 4.5.2).
- (e) We may assume $V \neq \{0\}$, since otherwise χ is the zero function and both claims are clear.

Finite order. By Lemma 23.8.1 there is an integer $d \geq 1$ with $g^d = 1$, namely the order of g .

Diagonalizability. Then $\rho(g)^d = \rho(g^d) = \text{id}_V$, so the polynomial $x^d - 1$ annihilates $\rho(g)$. Over $\mathbb{C}$ it equals $\prod_{m=0}^{d-1} (x - \zeta^m)$ with $\zeta = e^{2\pi i/d}$, a product of **distinct** linear factors (Example (Roots of Unity)). By Theorem 9.8.2, (c) $\Rightarrow$ (a), $\rho(g)$ is diagonalizable, and every eigenvalue λ satisfies $\lambda^d = 1$, hence $|\lambda| = 1$.

Conclusion. Let $\lambda_1, \dots, \lambda_n$ be the diagonal entries in an eigenbasis, $n = \dim V$. Then $\chi(g) = \sum_j \lambda_j$. In the same basis $\rho(g)^{-1} = \rho(g^{-1})$ is diagonal with entries λ_j^{-1} , and $|\lambda_j| = 1$ gives $\lambda_j^{-1} = \overline{\lambda_j}$. Hence $\chi(g^{-1}) = \sum_j \overline{\lambda_j} = \overline{\chi(g)}$. Finally $|\chi(g)| \leq \sum_j |\lambda_j| = n = \chi(1)$ by the triangle inequality for complex numbers (Theorem 0.6.4), applied repeatedly. This proves the proposition. ■

Warning. A character is not a homomorphism. For the standard representation of S_3 (Example (The irreducible complex representations of the symmetric group on three letters)), $\chi((1\ 2)) = 0$ and $\chi((1\ 3)) = 0$, while $(1\ 2)(1\ 3) = (1\ 3\ 2)$ has $\chi((1\ 3\ 2)) = -1 \neq 0 \cdot 0$. Only characters of **degree** 1 are homomorphisms, because then $\chi(g)$ is the whole operator. Nor need the values of a character be real: for $\mathbb{Z}/4\mathbb{Z}$ they include $\pm i$.

Quick check. A representation of G has character χ with $\chi(1) = 3$ and $\chi(g) = 4$ for some g . What is wrong?

Solution. Nothing can have such a character: by Proposition 23.8.2 (e), $|\chi(g)| \leq \chi(1) = 3$ for every g , and $4 > 3$. The bound says a character is largest, in absolute value, at the identity.

23.8.3 Class functions and their inner product

Part (b) of Proposition 23.8.2 says every character lives in one particular space of functions, and that space is small enough to be a useful home.

Definition 23.8.3. Class function

Let G be a finite group. A function $f: G \rightarrow \mathbb{C}$ is a **class function** if $f(xgx^{-1}) = f(g)$ for all $g, x \in G$; equivalently, if f is constant on each conjugacy class. The set of class functions is written $\text{Cl}(G)$.

The functions $G \rightarrow \mathbb{C}$ form a complex vector space under the pointwise operations (Example (The Standard Vector Spaces) (d)). The zero function is a class function, and sums and scalar multiples of class functions are class functions, so $\text{Cl}(G)$ is a subspace of it (Theorem 1.2.1). Its dimension is exactly what we want to count.

Nonexample by minimal change. On S_3 , the function f that is 1 at $(1\ 2)$ and 0 elsewhere is **not** a class function: $(1\ 3) = \sigma(1\ 2)\sigma^{-1}$ for $\sigma = (2\ 3)$, so f would have to take the same value at $(1\ 3)$, and it takes 0 there. Changing f to be 1 on **all three** transpositions repairs it; that is the indicator function of a whole class, and those are the building blocks below.

Proposition 23.8.3. The class functions have dimension the number of classes

Let G be a finite group with conjugacy classes $C_1, \dots, C_k$. For each j let $1_{C_j}: G \rightarrow \mathbb{C}$ be the function that is 1 on C_j and 0 elsewhere. Then $(1_{C_1}, \dots, 1_{C_k})$ is a basis of $\text{Cl}(G)$, and $\dim \text{Cl}(G) = k$.

Proof. Each 1_{C_j} is constant on every class — it is 1 on C_j and 0 on the others — so it lies in $\text{Cl}(G)$.

Spanning. Let $f \in \text{Cl}(G)$ and pick $g_j \in C_j$ for each j . Every $g \in G$ lies in exactly one class C_j (the classes partition G), and then $f(g) = f(g_j)$ while $\sum_l f(g_l)1_{C_l}(g) = f(g_j)$. So $f = \sum_j f(g_j)1_{C_j}$.

Independence. Suppose $\sum_j c_j 1_{C_j} = 0$. Evaluating at $g_j \in C_j$ gives $c_j = 0$, since every other 1_{C_l} vanishes there. Hence the list is a basis and $\dim \text{Cl}(G) = k$. ■

Now the pairing. It is the standard inner product of $\mathbb{C}^{|G|}$, with the coordinates indexed by G and normalized by $|G|$ so that the trivial character has length 1.

Definition 23.8.4. The inner product of class functions

For functions $f, f': G \rightarrow \mathbb{C}$ on a finite group G , set

$$\langle f, f' \rangle := \frac{1}{|G|} \sum_{g \in G} f(g) \overline{f'(g)}.$$

This satisfies (IP1)–(IP3) of Definition 11.1.1 on the space of all functions $G \rightarrow \mathbb{C}$, hence on the subspace $\text{Cl}(G)$: linearity in the first slot and conjugate symmetry are immediate from the formula, and $\langle f, f \rangle = \frac{1}{|G|} \sum_g |f(g)|^2 > 0$ whenever $f \neq 0$, since some term is positive and none is negative. So $\text{Cl}(G)$ is a complex inner product space of dimension k , and all of Chapter 11 is available in it.

When f and f' are class functions the sum can be shortened to one term per class: with $c_j = |C_j|$ and $g_j \in C_j$,

$$\langle f, f' \rangle = \frac{1}{|G|} \sum_{j=1}^k c_j f(g_j) \overline{f'(g_j)}. \quad (23.8.1)$$

That is the formula used for hand computation, and it is the only place the class sizes enter.

23.8.4 Two averaging lemmas

Both proofs of orthogonality run through the same device as Maschke's theorem: average over G , and the average is G -invariant. Here we also need its trace.

Lemma 23.8.4. The average of a representation is the projection onto the fixed space

Let ρ be a representation of a finite group G on a complex space V , and set

$$V^G := \{v \in V : \rho(g)v = v \text{ for all } g \in G\}, \quad P := \frac{1}{|G|} \sum_{g \in G} \rho(g).$$

Then V^G is a subspace, P is a projection with $\text{im } P = V^G$, and

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \dim V^G.$$

Proof. $V^G = \bigcap_{g \in G} \ker(\rho(g) - \text{id}_V)$ is an intersection of subspaces, hence a subspace (Theorem 1.2.2).

Fix $h \in G$. Then $\rho(h)P = \frac{1}{|G|} \sum_g \rho(hg) = P$, because $g \mapsto hg$ is a bijection of G (Theorem 0.10.1) and so only reorders the terms. Hence $\rho(h)(Pv) = Pv$ for every v , giving $\text{im } P \subseteq V^G$. Conversely, if $v \in V^G$ then $Pv = \frac{1}{|G|} \sum_g \rho(g)v = v$. So P is the identity on V^G ; combined with $\text{im } P \subseteq V^G$ this gives $P^2 = P$ and $\text{im } P = V^G$.

Thus P is a projection (Definition 4.5.1) of rank $\dim V^G$, and $\text{tr } P = \dim V^G$ by Theorem 4.5.4, the field being $\mathbb{C}$. On the other hand the trace is linear (Theorem 4.5.3), so $\text{tr } P = \frac{1}{|G|} \sum_g \text{tr } \rho(g) = \frac{1}{|G|} \sum_g \chi_\rho(g)$. This proves the lemma. ■

Lemma 23.8.5. The trace of a two-sided multiplication

Let V and W be finite-dimensional vector spaces over a field F , let $A \in \mathcal{L}(V)$ and $B \in \mathcal{L}(W)$, and let

$$\Phi: \mathcal{L}(V, W) \rightarrow \mathcal{L}(V, W), \quad \Phi(f) = B \circ f \circ A.$$

Then Φ is linear and $\text{tr } \Phi = (\text{tr } A)(\text{tr } B)$.

Idea. Write everything in coordinates. In the basis of $\mathcal{L}(V, W)$ given by the matrix units $\mathbf{E}_{ij}$, the map $\mathbf{X} \mapsto \mathbf{B}\mathbf{X}\mathbf{A}$ sends $\mathbf{E}_{ij}$ to a matrix whose (i, j) -entry is $b_{ii}a_{jj}$; adding those diagonal coefficients factorizes as a product of two sums.

Proof. Linearity of Φ is immediate from bilinearity of composition. Fix bases of V and W and identify $\mathcal{L}(V, W)$ with $M_{m \times n}(F)$, where $n = \dim V$ and $m = \dim W$ (Theorem 4.1.4); let $\mathbf{A} = (a_{jl}) \in M_n(F)$ and $\mathbf{B} = (b_{ki}) \in M_m(F)$ be the matrices of A and B . Under the identification Φ becomes $\mathbf{X} \mapsto \mathbf{B}\mathbf{X}\mathbf{A}$.

Let $\mathbf{E}_{ij} \in M_{m \times n}(F)$ be the matrix unit with a 1 in position (i, j) and zeros elsewhere — the rectangular version of the matrix units of Lemma 23.2.1; these mn matrices form a basis. By Definition 0.9.9,

$$(\mathbf{B}\mathbf{E}_{ij}\mathbf{A})_{kl} = \sum_{p,q} b_{kp}(\mathbf{E}_{ij})_{pq}a_{ql} = b_{ki}a_{jl}.$$

In the basis $(\mathbf{E}_{ij})$, the coefficient of $\mathbf{E}_{ij}$ in $\Phi(\mathbf{E}_{ij})$ is the (i, j) -entry of $\mathbf{B}\mathbf{E}_{ij}\mathbf{A}$, namely $b_{ii}a_{jj}$. These coefficients are the diagonal entries of the matrix of Φ , so by Definition 4.5.2,

$$\mathrm{tr} \Phi = \sum_{i=1}^m \sum_{j=1}^n b_{ii}a_{jj} = \left(\sum_i b_{ii} \right) \left(\sum_j a_{jj} \right) = (\mathrm{tr} \mathbf{B})(\mathrm{tr} \mathbf{A}),$$

and $\mathrm{tr} \mathbf{A} = \mathrm{tr} A$, $\mathrm{tr} \mathbf{B} = \mathrm{tr} B$ by Definition 4.5.2. This proves the lemma. ■

Now put the two together. Given representations ρ on V and σ on W , make G act on the space $\mathcal{L}(V, W)$ of linear maps by

$$\pi(g)f := \sigma(g) \circ f \circ \rho(g)^{-1}.$$

Proposition 23.8.6. The representation on the space of linear maps

Let ρ, σ be representations of a finite group G on complex spaces V, W . Then π above is a representation of G on $\mathcal{L}(V, W)$, its character is $\chi_\pi(g) = \chi_\sigma(g)\overline{\chi_\rho(g)}$, and its fixed space is the space

$$\mathcal{L}_G(V, W) := \{f \in \mathcal{L}(V, W) : f\rho(g) = \sigma(g)f \text{ for all } g \in G\}$$

of maps intertwining ρ and σ (Definition 23.7.3); for $W = V$ and $\sigma = \rho$ this is the commutant $\mathrm{End}_{\mathbb{C}[G]}(V)$ of Definition 23.4.1. Consequently

$$\frac{1}{|G|} \sum_{g \in G} \chi_\sigma(g)\overline{\chi_\rho(g)} = \dim \mathcal{L}_G(V, W). \quad (23.8.2)$$

Proof. A representation. Each $\pi(g)$ is linear in f , and $\pi(g)\pi(h)f = \sigma(g)\sigma(h)f\rho(h)^{-1}\rho(g)^{-1} = \sigma(gh)f\rho(gh)^{-1} = \pi(gh)f$, using the homomorphism property of ρ and σ and Theorem 0.10.1 for the inverse of a product. Taking $h = g^{-1}$ shows $\pi(g)$ is invertible with inverse $\pi(g^{-1})$.

Its character. Apply Lemma 23.8.5 with $A = \rho(g)^{-1}$ and $B = \sigma(g)$:

$$\begin{aligned}\chi_\pi(g) &= \operatorname{tr} \pi(g) = \operatorname{tr}(\rho(g)^{-1}) \operatorname{tr}(\sigma(g)) \\ &= \chi_\rho(g^{-1}) \chi_\sigma(g) = \overline{\chi_\rho(g)} \chi_\sigma(g),\end{aligned}$$

the last step by Proposition 23.8.2 (e).

Its fixed space. $\pi(g)f = f$ means $\sigma(g)f\rho(g)^{-1} = f$, and composing with $\rho(g)$ on the right turns this into $\sigma(g)f = f\rho(g)$. So the fixed space is exactly $\mathcal{L}_G(V, W)$.

The count. Apply Lemma 23.8.4 to π and substitute the character just computed. This proves the proposition. ■

23.8.5 Orthogonality of the irreducible characters

Everything is now in place. Schur's lemma converts the right-hand side of Eq. 23.8.2 into a 0 or a 1.

Theorem 23.8.7. Orthogonality of Irreducible Characters

Let G be a finite group and let ρ on V and σ on W be **irreducible** complex representations of G . Then

$$\langle \chi_\sigma, \chi_\rho \rangle = \begin{cases} 1, & \text{if } \rho \cong \sigma, \\ 0, & \text{if } \rho \not\cong \sigma. \end{cases}$$

In particular the characters of pairwise inequivalent irreducible complex representations form an orthonormal list in $\operatorname{Cl}(G)$.

Idea. By Eq. 23.8.2, $\langle \chi_\sigma, \chi_\rho \rangle$ counts the dimension of the space of intertwining maps $V \rightarrow W$. Schur's lemma says that space is $\{0\}$ when the two are inequivalent, because a nonzero intertwiner would be an isomorphism. When they are equivalent, fix one isomorphism S_0 ; every other intertwiner differs from it by an intertwiner of V with itself, which over $\mathbb{C}$ is a scalar. So the space is the line $\mathbb{C}S_0$.

Proof. By Eq. 23.8.2, $\langle \chi_\sigma, \chi_\rho \rangle = \dim \mathcal{L}_G(V, W)$, which is a nonnegative integer. It remains to compute that dimension.

Case 1: $\rho \not\cong \sigma$. By Theorem 23.7.1, V and W are irreducible $\mathbb{C}[G]$ -spaces; and since the condition $f\rho(g) = \sigma(g)f$ for all $g \in G$ extends by linearity to all of $\mathbb{C}[G]$, the members of $\mathcal{L}_G(V, W)$ are exactly the $\mathbb{C}[G]$ -maps $V \rightarrow W$ (Definition 23.4.1). Let $f \in \mathcal{L}_G(V, W)$ be nonzero. By Theorem 23.4.1 (a) it is an isomorphism, making ρ and σ equivalent (Definition 23.7.3) — a contradiction. Hence $\mathcal{L}_G(V, W) = \{0\}$ and the inner product is 0.

Case 2: $\rho \cong \sigma$. Fix an isomorphism $S_0 \in \mathcal{L}_G(V, W)$, so $S_0\rho(g) = \sigma(g)S_0$ for all g . Applying S_0^{-1} on both sides gives $\rho(g)S_0^{-1} = S_0^{-1}\sigma(g)$, so S_0^{-1} intertwines σ with ρ . Let $f \in \mathcal{L}_G(V, W)$ and set $u := S_0^{-1}f \in \mathcal{L}(V)$. Then for every g ,

$$u\rho(g) = S_0^{-1}f\rho(g) = S_0^{-1}\sigma(g)f = \rho(g)S_0^{-1}f = \rho(g)u,$$

so u commutes with every $\rho(g)$. Since ρ is irreducible and the field is algebraically closed, Corollary 23.4.3 gives $u = \lambda \text{id}_V$ for some $\lambda \in \mathbb{C}$, and hence $f = S_0 u = \lambda S_0$. Therefore $\mathcal{L}_G(V, W) = \mathbb{C}S_0$, which has dimension 1 because $S_0 \neq 0$.

The last sentence of the statement follows: distinct members of a list of pairwise inequivalent irreducibles pair to 0, and each pairs with itself to 1. This proves the theorem. ■

Two consequences come for free, and both are used in every computation below.

Corollary 23.8.8. Irreducibility criterion

Let ρ be a complex representation of a finite group G on $V \neq \{0\}$, with character χ . Write $V = V_1 \oplus \cdots \oplus V_r$ with each V_i irreducible (Corollary 23.7.4), and let $\chi_1, \dots, \chi_s$ be the distinct characters occurring among the $\chi_{\rho|_{V_i}}$, with multiplicities $m_1, \dots, m_s \geq 1$. Then

$$\chi = \sum_{i=1}^s m_i \chi_i, \quad m_i = \langle \chi, \chi_i \rangle, \quad \langle \chi, \chi \rangle = \sum_{i=1}^s m_i^2.$$

In particular $\langle \chi, \chi \rangle = 1$ if and only if ρ is irreducible, and two complex representations of G with the same character are equivalent.

Proof. Repeated use of Proposition 23.8.2 (d) gives $\chi = \sum_{i=1}^r \chi_{\rho|_{V_i}}$, and grouping equal terms gives $\chi = \sum_i m_i \chi_i$. Distinct characters come from inequivalent representations, since equivalent ones have equal characters (Proposition 23.8.2 (c)); so by Theorem 23.8.7 the χ_i are orthonormal, and $\langle \chi, \chi_i \rangle = m_i$ and, expanding the pairing in the first slot and then in the second, $\langle \chi, \chi \rangle = \sum_i m_i^2$.

If ρ is irreducible then $s = 1$ and $m_1 = 1$, so $\langle \chi, \chi \rangle = 1$. Conversely, a sum of squares of positive integers equals 1 only when there is a single term equal to 1; then $r = 1$ and $V = V_1$ is irreducible.

For the last claim, suppose ρ and ρ' have the same character. Every irreducible character ψ of G then has $\langle \chi_\rho, \psi \rangle = \langle \chi_{\rho'}, \psi \rangle$, so each irreducible occurs with the same multiplicity in both decompositions. So the two decompositions into irreducibles use the same irreducibles with the same multiplicities, and the summands can be matched in pairs with equal characters. Two irreducible summands with equal characters are equivalent: were they inequivalent, Theorem 23.8.7 would make their characters orthogonal, hence unequal, since an irreducible character pairs with itself to 1 and not to 0. The direct sum of those equivalences is an isomorphism $V \rightarrow V'$ intertwining ρ and ρ' . This proves the corollary. ■

Example. Checking the standard representation by its character

Use Corollary 23.8.8 to confirm that the two-dimensional representation of S_3 built in Example (The irreducible complex representations of the symmetric group on three letters) is irreducible, and decompose the permutation representation of S_3 on $\mathbb{C}^3$.

Solution. From Example (The irreducible complex representations of the symmetric group on three letters) the matrices give $\chi_{\text{std}}(\text{id}) = 2$, $\chi_{\text{std}}((1\ 2)) = -1 + 1 = 0$, and $\chi_{\text{std}}((1\ 2\ 3)) = 0 + (-1) = -1$; by Proposition 23.8.2 (b) these three values determine χ_{std} on all of S_3 , the classes being those of Example (The conjugacy classes of the symmetric group on three letters). Using Eq. 23.8.1 with class sizes 1, 3, 2,

$$\langle \chi_{\text{std}}, \chi_{\text{std}} \rangle = \frac{1}{6}(1 \cdot 2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = \frac{4+0+2}{6} = 1,$$

so the representation is irreducible.

For the permutation representation, χ_{perm} counts fixed points: 3 at id, 1 at a transposition, 0 at a 3-cycle. Then

$$\begin{aligned} \langle \chi_{\text{perm}}, \chi_{\text{triv}} \rangle &= \frac{1}{6}(1 \cdot 3 + 3 \cdot 1 + 2 \cdot 0) = 1, \\ \langle \chi_{\text{perm}}, \chi_{\text{sgn}} \rangle &= \frac{1}{6}(1 \cdot 3 + 3 \cdot 1 \cdot (-1) + 2 \cdot 0) = 0, \\ \langle \chi_{\text{perm}}, \chi_{\text{std}} \rangle &= \frac{1}{6}(1 \cdot 3 \cdot 2 + 0 + 0) = 1. \end{aligned}$$

So $\chi_{\text{perm}} = \chi_{\text{triv}} + \chi_{\text{std}}$, and by Corollary 23.8.8 the permutation representation is the direct sum of the trivial and the standard one. That matches the splitting $\mathbb{C}^3 = \text{span}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \oplus W$ found by hand in Example (The irreducible complex representations of the symmetric group on three letters), and it was found here without looking at a single vector.

23.8.6 The characters span the class functions

Orthonormality already bounds the number of irreducible representations: an orthonormal list in $\text{Cl}(G)$ is independent (Theorem 11.1.4), so there are at most k of them up to equivalence. The next theorem shows there are exactly k . It is the deepest statement of the section, and its proof is the third and last averaging argument: this time we average the group elements themselves, weighted by a class function.

Theorem 23.8.9. The Irreducible Characters Are a Basis of the Class Functions

Let G be a finite group with k conjugacy classes. Then G has exactly k irreducible complex representations up to equivalence, and their characters $\chi_1, \dots, \chi_k$ form an orthonormal basis of $\text{Cl}(G)$.

Idea. Suppose a class function f is orthogonal to every irreducible character; we must show $f = 0$. Build from f the operator $T_f = \sum_g \overline{f(g)} \rho(g)$ on any representation ρ . ① Because f is a class function, T_f commutes with every $\rho(h)$ — conjugating by $\rho(h)$ permutes the terms and leaves the coefficients alone. ② So on an irreducible ρ it is a scalar, and its trace, which is $|G| \langle \chi_\rho, f \rangle = 0$, forces that scalar to be 0. ③ Complete reducibility (Corollary 23.7.4) spreads this to every representation. ④ Apply it to the regular representation and evaluate at δ_1 : the result is $\sum_g \overline{f(g)} \delta_g = 0$, and the δ_g are a basis.

Proof. Orthonormal lists are independent (Theorem 11.1.4) and $\dim \text{Cl}(G) = k$ (Proposition 23.8.3), so by Theorem 23.8.7 any list of pairwise inequivalent irreducible representations has at most k members. Choose such a list $\rho_1, \dots, \rho_s$ with s as large as possible; by maximal-

ity every irreducible complex representation of G is equivalent to some ρ_i . Write $\chi_i = \chi_{\rho_i}$, and let $U = \text{span}(\chi_1, \dots, \chi_s) \subseteq \text{Cl}(G)$, using [Proposition 23.8.2](#) (b).

Let $f \in U^\perp$, the orthogonal complement of U in $\text{Cl}(G)$ (Definition 11.3.1), so f is a class function with $\langle f, \chi_i \rangle = 0$, hence also $\langle \chi_i, f \rangle = \overline{\langle f, \chi_i \rangle} = 0$, for every i . For **any** representation ρ of G on V , define

$$T_f^\rho := \sum_{g \in G} \overline{f(g)} \rho(g) \in \mathcal{L}(V).$$

Claim. Claim 1. T_f^ρ commutes with $\rho(h)$ for every $h \in G$.

Proof. Fix h . Then

$$\rho(h) T_f^\rho \rho(h)^{-1} = \sum_{g \in G} \overline{f(g)} \rho(hgh^{-1}).$$

The map $g \mapsto hgh^{-1}$ is a bijection $G \rightarrow G$, with inverse $g \mapsto h^{-1}gh$. Substituting $g' = hgh^{-1}$, so that $g = h^{-1}g'h$, the sum becomes $\sum_{g'} \overline{f(h^{-1}g'h)} \rho(g')$. Since f is a class function, $f(h^{-1}g'h) = f(g')$, so the sum is T_f^ρ . Multiplying on the right by $\rho(h)$ gives the claim. ■

Claim. Claim 2. If ρ is irreducible, then $T_f^\rho = 0$.

Proof. By Claim 1 and Corollary 23.4.3, over the algebraically closed field $\mathbb{C}$ we get $T_f^\rho = \lambda \text{id}_V$ for some $\lambda \in \mathbb{C}$. Taking traces and using linearity of the trace (Theorem 4.5.3),

$$\lambda \dim V = \text{tr } T_f^\rho = \sum_g \overline{f(g)} \chi_\rho(g) = |G| \langle \chi_\rho, f \rangle.$$

Now ρ is equivalent to some ρ_i , so $\chi_\rho = \chi_i$ by [Proposition 23.8.2](#) (c), and $\langle \chi_i, f \rangle = 0$. Since $\dim V \geq 1$, $\lambda = 0$. ■

Claim. Claim 3. $T_f^\rho = 0$ for **every** representation ρ .

Proof. We may assume $V \neq \{0\}$. By Corollary 23.7.4, $V = V_1 \oplus \dots \oplus V_r$ with each V_j invariant and $\rho|_{V_j}$ irreducible. Each $\rho(g)$ maps V_j into V_j , hence so does T_f^ρ , and its restriction to V_j is $\sum_g \overline{f(g)} \rho|_{V_j}(g) = T_f^{\rho|_{V_j}}$, which is 0 by Claim 2. So T_f^ρ vanishes on each V_j , hence on their sum V . ■

Now apply Claim 3 to the regular representation ρ_{reg} on $\mathbb{C}[G]$, where $\rho_{\text{reg}}(g)\delta_h = \delta_{gh}$. Evaluating at the basis vector δ_1 ,

$$0 = T_f^{\rho_{\text{reg}}} \delta_1 = \sum_{g \in G} \overline{f(g)} \delta_g.$$

The δ_g are a basis of $\mathbb{C}[G]$, so every coefficient vanishes: $\overline{f(g)} = 0$, that is $f(g) = 0$, for all g . Hence $U^\perp = \{0\}$.

By Theorem 11.3.2, $\text{Cl}(G) = U \oplus U^\perp = U$. So $(\chi_1, \dots, \chi_s)$ is an orthonormal spanning list, hence a basis, and $s = \dim \text{Cl}(G) = k$ by [Proposition 23.8.3](#). Finally, inequivalent irreducibles have different characters, since orthonormality gives $\langle \chi_i, \chi_j \rangle = 0 \neq 1 = \langle \chi_i, \chi_i \rangle$ for $i \neq j$; so

the k characters correspond to the k equivalence classes of irreducible representations. This proves the theorem. ■

Corollary 23.8.10. The sum of the squares of the degrees

Let G be a finite group and let $\rho_1, \dots, \rho_k$ on $V_1, \dots, V_k$ be a complete list of pairwise inequivalent irreducible complex representations of G , one for each of the k conjugacy classes (Theorem 23.8.9). Then each ρ_i occurs in the regular representation with multiplicity exactly $\dim V_i$, and

$$|G| = \sum_{i=1}^k (\dim V_i)^2.$$

Proof. The regular character is $\chi_{\text{reg}}(1) = |G|$ and $\chi_{\text{reg}}(g) = 0$ for $g \neq 1$, as computed after Definition 23.8.2. Hence

$$\langle \chi_{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_g \chi_{\text{reg}}(g) \overline{\chi_i(g)} = \frac{1}{|G|} |G| \overline{\chi_i(1)} = \dim V_i,$$

the value $\chi_i(1) = \dim V_i$ being a real number (Proposition 23.8.2 (a)). By Corollary 23.8.8 this inner product is the multiplicity of ρ_i in ρ_{reg} , so $\chi_{\text{reg}} = \sum_i (\dim V_i) \chi_i$. Evaluating at $g = 1$ gives $|G| = \sum_i (\dim V_i) \chi_i(1) = \sum_i (\dim V_i)^2$. ■

This is Corollary 23.6.2, the count $\dim A = \sum n_i^2$ of the structure theorem, applied to $A = \mathbb{C}[G]$: the two proofs are different and the answer is the same.

Corollary 23.8.11. Orthogonality of the columns

Let G be a finite group with conjugacy classes $C_1, \dots, C_k$, representatives $g_j \in C_j$ and sizes $c_j = |C_j|$, and irreducible characters $\chi_1, \dots, \chi_k$. Then for all j, l ,

$$\sum_{i=1}^k \chi_i(g_j) \overline{\chi_i(g_l)} = \begin{cases} |G|/c_j, & j = l, \\ 0, & j \neq l. \end{cases}$$

Idea. Theorem 23.8.9 makes the table of character values a **square** array. Rescale it so that orthonormality of the rows says the matrix is unitary; then its columns are orthonormal too, because a one-sided inverse of a square matrix is two-sided.

Proof. Let $\mathbf{U} \in M_k(\mathbb{C})$ have entries $u_{ij} = \sqrt{c_j/|G|} \chi_i(g_j)$; the square root is a positive real.

By [Eq. 23.8.1](#) and [Theorem 23.8.7](#),

$$(\mathbf{U}\mathbf{U}^*)_{ii'} = \sum_{j=1}^k u_{ij} \overline{u_{i'j}} = \frac{1}{|G|} \sum_{j=1}^k c_j \chi_i(g_j) \overline{\chi_{i'}(g_j)} = \langle \chi_i, \chi_{i'} \rangle,$$

which is 1 for $i = i'$ and 0 otherwise. Hence $\mathbf{U}\mathbf{U}^* = \mathbf{I}_k$. By [Theorem 3.4.5](#), $\mathbf{U}^*\mathbf{U} = \mathbf{I}_k$ as well, so for all j, l

$$\sum_{i=1}^k \overline{u_{ij}} u_{il} = \frac{\sqrt{c_j c_l}}{|G|} \sum_{i=1}^k \overline{\chi_i(g_j)} \chi_i(g_l)$$

equals 1 when $j = l$ and 0 otherwise. For $j = l$ this gives $\sum_i |\chi_i(g_j)|^2 = |G|/c_j$, and for $j \neq l$ it gives $\sum_i \overline{\chi_i(g_j)} \chi_i(g_l) = 0$. Conjugating the second identity, which is a sum of complex numbers, gives the stated form. This proves the corollary. ■

23.8.7 Two character tables

The **character table** of G is the $k \times k$ array whose (i, j) -entry is $\chi_i(g_j)$: one row per irreducible character, one column per conjugacy class. By [Theorem 23.8.9](#) it is square, by [Theorem 23.8.7](#) its rows are orthonormal, and by [Corollary 23.8.11](#) its columns are orthogonal.

Example. The character table of the symmetric group on three letters

Write down the character table of S_3 and verify every orthogonality relation.

Solution. By [Example \(The conjugacy classes of the symmetric group on three letters\)](#) there are $k = 3$ classes, with representatives id , $(1\ 2)$, $(1\ 2\ 3)$ and sizes 1, 3, 2. So there are three irreducible complex representations ([Theorem 23.8.9](#)), and [Example \(The irreducible complex representations of the symmetric group on three letters\)](#) produced three: the trivial, the sign, and the standard one. Their characters, from the definitions and the traces computed in [Example \(Checking the standard representation by its character\)](#), are:

	id	(1 2)	(1 2 3)
class size	1	3	2
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{std}	2	0	-1

Degrees. $1^2 + 1^2 + 2^2 = 6 = |S_3|$, as [Corollary 23.8.10](#) requires.

Row orthogonality, from [Eq. 23.8.1](#) with weights 1, 3, 2 and divisor 6. All values are real, so conjugation does nothing:

$$\begin{aligned}\langle \chi_{\text{triv}}, \chi_{\text{triv}} \rangle &= \frac{1}{6}(1 + 3 + 2) = 1, \\ \langle \chi_{\text{sgn}}, \chi_{\text{sgn}} \rangle &= \frac{1}{6}(1 + 3 + 2) = 1, \\ \langle \chi_{\text{std}}, \chi_{\text{std}} \rangle &= \frac{1}{6}(4 + 0 + 2) = 1, \\ \langle \chi_{\text{triv}}, \chi_{\text{sgn}} \rangle &= \frac{1}{6}(1 - 3 + 2) = 0, \\ \langle \chi_{\text{triv}}, \chi_{\text{std}} \rangle &= \frac{1}{6}(2 + 0 - 2) = 0, \\ \langle \chi_{\text{sgn}}, \chi_{\text{std}} \rangle &= \frac{1}{6}(2 + 0 - 2) = 0.\end{aligned}$$

Column orthogonality, from [Corollary 23.8.11](#), with no weights this time:

$$\begin{aligned}\text{column id} : & 1 + 1 + 4 = 6 = 6/1, \\ \text{column (1 2)} : & 1 + 1 + 0 = 2 = 6/3, \\ \text{column (1 2 3)} : & 1 + 1 + 1 = 3 = 6/2, \\ \text{id against (1 2)} : & 1 \cdot 1 + 1 \cdot (-1) + 2 \cdot 0 = 0, \\ \text{id against (1 2 3)} : & 1 + 1 - 2 = 0, \\ \text{(1 2) against (1 2 3)} : & 1 - 1 + 0 = 0.\end{aligned}$$

Every relation holds.

Example. The character table of a cyclic group of order four

Write down the character table of $\mathbb{Z}/4\mathbb{Z}$ and verify the orthogonality relations.

Solution. The group is abelian, so every conjugacy class is a single element and $k = 4$. By [Theorem 23.8.9](#) there are four irreducible complex representations, and Example (The irreducible complex representations of a cyclic group) produced four, all of degree 1: $\chi_m([j]) = i^{jm}$, using $\omega = e^{2\pi i/4} = i$. Their degrees satisfy $1^2 + 1^2 + 1^2 + 1^2 = 4 = |G|$.

	[0]	[1]	[2]	[3]
χ_0	1	1	1	1
χ_1	1	i	-1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	-1	i

Row orthogonality. For characters of degree 1, $\overline{\chi_l([j])} = i^{-jl}$, so

$$\langle \chi_m, \chi_l \rangle = \frac{1}{4} \sum_{j=0}^3 i^{j(m-l)}.$$

If $m = l$ every term is 1 and the sum is $4/4 = 1$. If $m \neq l$, put $d = m - l$, so $i^d \neq 1$ and $(i^d)^4 = 1$; then $(i^d - 1) \sum_{j=0}^3 (i^d)^j = (i^d)^4 - 1 = 0$, and dividing by $i^d - 1 \neq 0$ gives $\sum_j i^{jd} = 0$. For instance $\langle \chi_1, \chi_0 \rangle = \frac{1}{4}(1 + i - 1 - i) = 0$ and $\langle \chi_3, \chi_1 \rangle = \frac{1}{4}(1 - 1 + 1 - 1) = 0$.

Column orthogonality. Every class has size 1, so [Corollary 23.8.11](#) predicts 4 down the diagonal and 0 off it. The same geometric sum settles every one of them: column $[j]$ against column $[l]$ gives $\sum_{m=0}^3 \chi_m([j]) \overline{\chi_m([l])} = \sum_{m=0}^3 i^{m(j-l)}$, which is 4 when $j = l$, and is 0 otherwise by the computation just made, read with $d = j - l$. Three of them written out. Column $[1]$ against itself: $1 + |i|^2 + 1 + |-i|^2 = 4$. Column $[0]$ against $[2]$: the entries of column $[0]$ are 1, 1, 1, 1 and those of column $[2]$ are 1, -1, 1, -1, so the sum is $1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 + 1 \cdot (-1) = 0$. Column $[1]$ against $[3]$: $1 + i \cdot (-i) + (-1)(-1) + (-i)i = 1 + i \cdot i + 1 + (-i)(-i) = 1 - 1 + 1 - 1 = 0$.

Read as a matrix, the table is $(i^{jm})_{m,j}$, which up to the factor $1/2$ is the 4×4 Fourier matrix of Example (The Fourier matrix of size four). That is not a coincidence, and Section 09 explains it.

Warning. The character table does not determine the group. Two nonisomorphic groups can have identical character tables. The standard witness is a pair of groups of order 8: the eight symmetries of a square (four rotations and four reflections), and the group $\{\pm \mathbf{I}_2, \pm i, \pm j, \pm k\}$ sitting inside the quaternions $\mathbb{H}$ of Definition 14.12.3 under multiplication. Both have five conjugacy classes and degrees 1, 1, 1, 1, 2, and their tables agree entry for entry, yet the groups are not isomorphic. **This is stated on credit and not proved here;** nothing below depends on it. What the character table *does* determine is every complex representation of the group, up to equivalence ([Corollary 23.8.8](#)).

23.8.8 Exercises

23.8.8.1 A. Check your understanding

Exercise. A1

- Define the character of a representation, and state why it does not depend on a choice of basis.
- State the orthogonality relation for irreducible complex characters.
- Determine whether the following is true: “if χ is a character of G then $\chi(gh) = \chi(g)\chi(h)$.” Justify your answer.
- A group of order 12 has exactly four conjugacy classes. Determine the degrees of its irreducible complex representations.
- Explain why the number of irreducible complex representations of an abelian group G is $|G|$.

Solution.

- (a) $\chi_\rho(g) = \text{tr } \rho(g)$ (Definition 23.8.2). The trace of an operator is computed from its matrix in any basis and the answer is the same for all of them (Definition 4.5.2), so χ_ρ depends only on ρ .
- (b) For irreducible complex representations ρ, σ of a finite group, $\langle \chi_\sigma, \chi_\rho \rangle$ is 1 if $\rho \cong \sigma$ and 0 otherwise (Theorem 23.8.7).
- (c) False. See the warning after Proposition 23.8.2: for the standard representation of S_3 , $\chi((1\ 2))\chi((1\ 3)) = 0$ but $\chi((1\ 3\ 2)) = -1$. It is true precisely for characters of degree 1.
- (d) Four classes means four irreducible representations (Theorem 23.8.9), with degrees $d_1, \dots, d_4 \geq 1$ satisfying $d_1^2 + d_2^2 + d_3^2 + d_4^2 = 12$ (Corollary 23.8.10). Every group has the trivial representation, so we may take $d_1 = 1$ and need $d_2^2 + d_3^2 + d_4^2 = 11$ with each $d_i \geq 1$. The squares available are 1, 4, 9, since $4^2 = 16 > 11$. A triple containing a 9 has its other two terms summing to 2, so it is $9 + 1 + 1 = 11$, which works; and a triple with no 9 is one of $4 + 4 + 4 = 12$, $4 + 4 + 1 = 9$, $4 + 1 + 1 = 6$, and $1 + 1 + 1 = 3$, none of them 11. So the degrees are 1, 1, 1, 3.
- (e) In an abelian group $gxg^{-1} = g$, so every conjugacy class is a singleton and $k = |G|$; now apply Theorem 23.8.9.

23.8.8.2 B. Practice

Exercise. B1: A character computed and decomposed

Let $\mathbf{P}_\tau$ be the permutation matrix of $\tau \in S_3$, so that $\mathbf{P}_\tau \mathbf{e}_i = \mathbf{e}_{\tau(i)}$ (Definition 3.4.2), and let π be the representation of S_3 on $\mathcal{L}(\mathbb{C}^3)$ given by $\pi(\tau)(f) = \mathbf{P}_\tau f \mathbf{P}_\tau^{-1}$.

- (a) Compute the character of π on the three conjugacy classes.
- (b) Decompose π into irreducibles.

Solution.

- (a) This is the construction of Proposition 23.8.6 with both representations equal to the permutation representation ρ_{perm} on $\mathbb{C}^3$. So its character is $\chi_\pi(\tau) = \chi_{\text{perm}}(\tau)\overline{\chi_{\text{perm}}(\tau)} = |\chi_{\text{perm}}(\tau)|^2$. From Example (Checking the standard representation by its character), $\chi_{\text{perm}} = (3, 1, 0)$ on the classes of id, $(1\ 2)$, $(1\ 2\ 3)$, so $\chi_\pi = (9, 1, 0)$.
- (b) With weights 1, 3, 2 and divisor 6:

$$\begin{aligned}\langle \chi_\pi, \chi_{\text{triv}} \rangle &= \frac{1}{6}(9 + 3 + 0) = 2, \\ \langle \chi_\pi, \chi_{\text{sgn}} \rangle &= \frac{1}{6}(9 - 3 + 0) = 1, \\ \langle \chi_\pi, \chi_{\text{std}} \rangle &= \frac{1}{6}(9 \cdot 2 + 0 + 0) = 3.\end{aligned}$$

So $\chi_\pi = 2\chi_{\text{triv}} + \chi_{\text{sgn}} + 3\chi_{\text{std}}$, and the degrees check: $2 + 1 + 6 = 9 = \dim \mathcal{L}(\mathbb{C}^3)$. As a check on the arithmetic, $\langle \chi_\pi, \chi_\pi \rangle = \frac{1}{6}(81 + 3 + 0) = 14 = 2^2 + 1^2 + 3^2$, as Corollary 23.8.8 requires. In particular $\dim \mathcal{L}_G(\mathbb{C}^3, \mathbb{C}^3) = 2$: the G -equivariant operators on $\mathbb{C}^3$ form a plane, spanned by the identity and the all-ones matrix.

Exercise. B2: Which of these are characters?

Let $G = S_3$, with classes ordered id , $(1\ 2)$, $(1\ 2\ 3)$ of sizes 1, 3, 2. Determine which of the following class functions are characters of complex representations of S_3 . Justify your answer.

- (a) $(4, 0, 1)$;
- (b) $(3, 1, 0)$;
- (c) $(2, 2, -1)$;
- (d) $(1, 1, -1)$.

Solution. A class function is a character exactly when it is a **nonnegative integer** combination of $\chi_{\text{triv}} = (1, 1, 1)$, $\chi_{\text{sgn}} = (1, -1, 1)$, and $\chi_{\text{std}} = (2, 0, -1)$, by Corollary 23.7.4 and Proposition 23.8.2 (d); the coefficients are the inner products $\langle \cdot, \chi_i \rangle$.

- (a) $\langle (4, 0, 1), \chi_{\text{triv}} \rangle = \frac{1}{6}(4 + 0 + 2) = 1$, $\langle \cdot, \chi_{\text{sgn}} \rangle = \frac{1}{6}(4 - 0 + 2) = 1$, $\langle \cdot, \chi_{\text{std}} \rangle = \frac{1}{6}(8 + 0 - 2) = 1$. All nonnegative integers, and $1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 = 4$ matches the degree. **Yes:** it is the character of the direct sum of the trivial, the sign, and the standard representation, equivalently of the direct sum of the permutation representation and the sign representation, since $(3, 1, 0) + (1, -1, 1) = (4, 0, 1)$.
- (b) Inner products 1, 0, 1 as computed in Example (Checking the standard representation by its character). **Yes:** the permutation representation.
- (c) $\langle \cdot, \chi_{\text{triv}} \rangle = \frac{1}{6}(2 + 6 - 2) = 1$, $\langle \cdot, \chi_{\text{sgn}} \rangle = \frac{1}{6}(2 - 6 - 2) = -1$. A negative multiplicity is impossible, so **no**. (It is a difference of characters, called a virtual character, but not a character.)
- (d) $\langle (1, 1, -1), \chi_{\text{triv}} \rangle = \frac{1}{6}(1 + 3 - 2) = \frac{1}{3}$, which is not an integer. So **no**: it is not even a combination of the three irreducible characters with integer coefficients, let alone nonnegative ones.

Exercise. B3: Degree-one characters of a cyclic group

Let $G = \mathbb{Z}/6\mathbb{Z}$.

- (a) Write down the character table.
- (b) Verify $\langle \chi_1, \chi_4 \rangle = 0$ and $\langle \chi_2, \chi_2 \rangle = 1$ by direct computation.

Solution.

- (a) The group is abelian of order 6, so there are six classes, six irreducible characters, all of degree 1, given by $\chi_m([j]) = \omega^{jm}$ with $\omega = e^{2\pi i/6}$ (Example (The irreducible complex representations of a cyclic group)). The table has (m, j) -entry ω^{jm} , with $\omega^6 = 1$.
- (b) Here

$$\begin{aligned} \langle \chi_1, \chi_4 \rangle &= \frac{1}{6} \sum_{j=0}^5 \omega^j \overline{\omega^{4j}} = \frac{1}{6} \sum_{j=0}^5 \omega^{-3j} \\ &= \frac{1}{6} \sum_{j=0}^5 (-1)^j = \frac{1}{6}(1 - 1 + 1 - 1 + 1 - 1) = 0, \end{aligned}$$

using $\omega^{-3} = \omega^3 = -1$. And $\langle \chi_2, \chi_2 \rangle = \frac{1}{6} \sum_j \omega^{2j} \overline{\omega^{2j}} = \frac{1}{6} \sum_j |\omega^{2j}|^2 = \frac{1}{6} \cdot 6 = 1$.

23.8.8.3 C. Going deeper

Exercise. C1: Counting fixed points

Let G be a finite group acting on $\{1, \dots, n\}$ through a homomorphism $\varphi: G \rightarrow S_n$, and let $\rho(g) = \mathbf{P}_{\varphi(g)}$ be the corresponding permutation representation on $\mathbb{C}^n$, where $\mathbf{P}_\sigma$ is the permutation matrix of Definition 3.4.2 (so $\mathbf{P}_\sigma \mathbf{P}_\tau = \mathbf{P}_{\sigma\tau}$ by Lemma 3.4.7 (a)).

- Prove that $\langle \chi_\rho, \chi_{\text{triv}} \rangle$ equals the average number of fixed points of the permutations $\varphi(g)$, and also equals $\dim(\mathbb{C}^n)^G$.
- Deduce that for every $n \geq 1$, the average number of fixed points of a permutation in S_n is exactly 1.
- For $G = S_3$ with $\varphi = \text{id}$, verify (a) numerically.

Solution.

- By the computation after Definition 23.8.2, $\chi_\rho(g)$ is the number of i with $\varphi(g)(i) = i$. Hence

$$\langle \chi_\rho, \chi_{\text{triv}} \rangle = \frac{1}{|G|} \sum_g \chi_\rho(g) \cdot \bar{1}$$

is exactly the average of those counts. By Lemma 23.8.4 applied to ρ , the same quantity is $\dim(\mathbb{C}^n)^G$.

- Take $G = S_n$ and $\varphi = \text{id}$. A vector $\mathbf{x} \in \mathbb{C}^n$ fixed by every $\rho(\tau)$ has $x_i = x_j$ for all $i \neq j$, by taking τ the transposition $(i \ j)$; conversely every vector with equal coordinates is fixed. So $(\mathbb{C}^n)^{S_n} = \text{span}(\mathbf{e}_1 + \dots + \mathbf{e}_n)$, of dimension 1 (for $n = 1$ this is $\mathbb{C}$ itself). By (a) the average number of fixed points equals that dimension, namely 1.
- For S_3 , $\chi_\rho = (3, 1, 0)$ on classes of sizes 1, 3, 2, so the average number of fixed points is $\frac{1}{6}(1 \cdot 3 + 3 \cdot 1 + 2 \cdot 0) = 1$. Directly: id fixes three points, each of the three transpositions fixes one, and each of the two 3-cycles fixes none, giving $(3 + 3 + 0)/6 = 1$. And $\dim(\mathbb{C}^3)^{S_3} = 1$ by (b). The three agree.

Exercise. C2: Degree-one characters form a group

Let G be a finite group and let $X(G)$ be the set of characters of degree 1 of complex representations of G . (For abelian G this group is the subject of Section 09.)

- Prove that the pointwise product of two degree-one characters is a degree-one character, and that $X(G)$ is a group under this product.
- Prove that $|X(G)| \leq k$, the number of conjugacy classes, with equality if and only if G is abelian. *Hint: use Corollary 23.8.10 in one direction and Corollary 23.4.4 in the other.*

Solution.

- A degree-one character is a homomorphism $\chi: G \rightarrow \mathbb{C} \setminus \{0\}$ (the representation is

the scalar it multiplies by). If χ, ψ are two of them then $(\chi\psi)(gh) = \chi(gh)\psi(gh) = \chi(g)\psi(g)\chi(h)\psi(h) = (\chi\psi)(g)(\chi\psi)(h)$, using commutativity of $\mathbb{C}$; so $\chi\psi$ is again a homomorphism into $\mathbb{C} \setminus \{0\}$, hence a degree-one character. The product is associative because multiplication in $\mathbb{C}$ is; χ_{triv} is an identity; and $g \mapsto \chi(g)^{-1}$ is a homomorphism and satisfies $\chi \cdot \chi^{-1} = \chi_{\text{triv}}$. So $X(G)$ is a group (Definition 0.10.1).

- (b) Degree-one characters are irreducible (a one-dimensional representation has no non-trivial invariant subspace) and pairwise distinct elements of $X(G)$ are inequivalent representations, since for degree one, equivalence means equality. So $X(G)$ is a subset of the k irreducible characters (Theorem 23.8.9) and $|X(G)| \leq k$.

Equality says every irreducible character has degree 1. If G is abelian then $\mathbb{C}[G]$ is a commutative algebra, so every irreducible $\mathbb{C}[G]$ -space is one-dimensional by Corollary 23.4.4; by Theorem 23.7.1 those are exactly the irreducible representations of G . Conversely, if every irreducible has degree 1 then $|G| = \sum_{i=1}^k 1 = k$ by Corollary 23.8.10, so every conjugacy class is a singleton; that is, $gxg^{-1} = g$ for all x, g , which says G is abelian.

Exercise. C3: The second orthogonality relation

Let G be a finite group with irreducible complex characters $\chi_1, \dots, \chi_k$.

- Prove that $\sum_{i=1}^k |\chi_i(g)|^2 = |G|/|C(g)|$ for every $g \in G$.
- Deduce that $\sum_i |\chi_i(g)|^2 = 1$ if and only if the class of g is all of G , and that this happens only for the trivial group. *Hint:* $C(1) = \{1\}$ always.
- For $G = S_3$ compute $\sum_i |\chi_i(g)|^2$ at each of the three classes and check (a).

Solution.

- This is the diagonal case $j = l$ of Corollary 23.8.11, with $g = g_j$ a representative of its own class and $c_j = |C(g)|$. The value is independent of the representative chosen, as it must be, since characters are class functions.
- By (a), the sum equals 1 exactly when $|C(g)| = |G|$, that is, when the class of g is all of G . But $C(1) = \{1\}$, and the classes partition G , so a class equal to G forces $G = \{1\}$. Conversely for the trivial group the only class is $\{1\} = G$ and the single character has $|\chi_1(1)|^2 = 1$.
- From Example (The character table of the symmetric group on three letters): at id, $1 + 1 + 4 = 6 = 6/1$; at $(1\ 2)$, $1 + 1 + 0 = 2 = 6/3$; at $(1\ 2\ 3)$, $1 + 1 + 1 = 3 = 6/2$. Each matches $|G|/|C(g)|$ with class sizes 1, 3, 2.