

22.8 Duality

Two facts about the projective plane sit side by side. Two distinct points lie on exactly one line; two distinct lines meet in exactly one point. Read either sentence aloud with “point” and “line” exchanged and you get the other. This is not a coincidence and it is not a philosophy: Chapter 5 already built the machine that produces it. A hyperplane of $\mathbb{P}(V)$ is the set of points killed by a linear functional, the functional is determined by the hyperplane up to a scalar, and so the hyperplanes of $\mathbb{P}(V)$ are themselves the points of a projective space, namely $\mathbb{P}(V^*)$. This section turns that observation into an inclusion-reversing dictionary between the projective subspaces of $\mathbb{P}(V)$ and those of $\mathbb{P}(V^*)$, gets the dimension bookkeeping right at both ends, and says exactly how much of the “duality principle” we are entitled to claim.

Throughout, F is an arbitrary field and V is a finite-dimensional vector space over F with $\dim V = n + 1 \geq 1$, so that $\mathbb{P}(V)$ has dimension n (Definition 22.6.1). No hypothesis on the characteristic is needed anywhere in this section, and no analysis is used. We keep the two extreme projective subspaces in play: $\mathbb{P}(\{0\}) = \emptyset$, of dimension -1 , and $\mathbb{P}(V)$ itself, of dimension n . The correspondence below exchanges them, and it is precisely at those two ends that a duality argument is easiest to get wrong.

22.8.1 Projective subspaces remember their subspaces

Everything rests on being able to move between a projective subspace $\mathbb{P}(U)$ and the subspace U it came from. That is the content of a short lemma, proved in Section 6 and recalled here because almost every proof below uses it.

Lemma 22.8.1. Passing Between U and $\mathbb{P}(U)$

Let U, U_1, U_2 be subspaces of V .

- (a) $\mathbb{P}(U_1) \subseteq \mathbb{P}(U_2)$ if and only if $U_1 \subseteq U_2$. In particular $U \mapsto \mathbb{P}(U)$ is injective, so a projective subspace determines its subspace.
- (b) $\mathbb{P}(U_1 \cap U_2) = \mathbb{P}(U_1) \cap \mathbb{P}(U_2)$.
- (c) $\mathbb{P}(U_1) \vee \mathbb{P}(U_2) = \mathbb{P}(U_1 + U_2)$, where $\vee$ is the join, the smallest projective subspace containing both (Proposition 22.6.5).

Proof. Part (a) is Lemma 22.6.4, and parts (b) and (c) are Proposition 22.6.5 (a) and (b), the latter being where $\vee$ was defined. ■

One consequence will be used repeatedly: if $S \subseteq T$ are projective subspaces of the same finite dimension, then $S = T$. Indeed $U_S \subseteq U_T$ by (a) and $\dim U_S = \dim U_T$, so $U_S = U_T$ by Theorem 1.6.8.

22.8.2 The points of the dual space are the hyperplanes

A **hyperplane** of $\mathbb{P}(V)$ is a projective subspace of dimension $n-1$, that is, $\mathbb{P}(H)$ for a hyperplane H of V in the sense of Definition 5.5.2. In $\mathbb{P}^2$ these are the lines; in $\mathbb{P}^3$, the planes.

Now the hook. A hyperplane of V is the kernel of a nonzero functional (Theorem 5.5.3), and the functional is determined by its kernel up to a scalar, which is the injectivity half of the next proposition. “A nonzero vector of V^* , up to a nonzero scalar” is exactly the description of a point of $\mathbb{P}(V^*)$. So the hyperplanes of $\mathbb{P}(V)$ are not merely a set: they are the points of a projective space of their own, of the same dimension.

The dual projective space is the space whose points are the hyperplanes of the original.

Definition 22.8.1. Dual Projective Space

Let V be a finite-dimensional vector space over F with $\dim V = n + 1 \geq 1$. The **dual projective space** of $\mathbb{P}(V)$ is the projective space

$$\mathbb{P}(V^*)$$

of the dual space V^* (Definition 5.1.2). A point of $\mathbb{P}(V^*)$ is a class $[\varphi]$ with $\varphi \in V^*$ **nonzero**, and $[\varphi] = [\psi]$ exactly when $\psi = \lambda\varphi$ for some $\lambda \in F \setminus \{0\}$.

Its dimension is $\dim V^* - 1 = n$, by Corollary 5.1.4: *the dual projective space has the same dimension as the original*. That is worth pausing on, because the correspondence we are about to build reverses dimensions inside a space of unchanged size.

Nonexample by minimal change. Take $\varphi = 0$. It is a perfectly good element of V^* , but $[0]$ is not a point of $\mathbb{P}(V^*)$ — the definition excludes the zero vector, as Definition 22.6.1 does. And indeed $\ker 0 = V$, which is not a hyperplane: it has codimension 0, not 1 (Definition 5.5.2). The clause “nonzero” is exactly what keeps the following bijection honest.

Proposition 22.8.2. Points of the Dual Are Hyperplanes

Let $\dim V = n + 1 \geq 1$. The map

$$[\varphi] \mapsto \mathbb{P}(\ker \varphi)$$

is a well-defined bijection from $\mathbb{P}(V^*)$ onto the set of hyperplanes of $\mathbb{P}(V)$.

Idea. Three checks. Well defined: proportional functionals have the same kernel. Surjective: a hyperplane $\mathbb{P}(H)$ has $\dim H^0 = 1$ by the annihilator count, so it is cut out by some nonzero functional. Injective: two functionals with the same kernel H both lie in the one-dimensional space H^0 , so they are proportional. The counting does all the work.

Proof. Well defined. If $\varphi \neq 0$ then $\ker \varphi$ is a hyperplane of V by Theorem 5.5.3, so $\mathbb{P}(\ker \varphi)$ is a projective subspace of dimension $\dim \ker \varphi - 1 = n - 1$, a hyperplane of $\mathbb{P}(V)$. If $[\psi] = [\varphi]$, then $\psi = \lambda\varphi$ with $\lambda \neq 0$, and $\psi(v) = 0$ if and only if $\varphi(v) = 0$, so $\ker \psi = \ker \varphi$.

Surjective. Let $\mathbb{P}(H)$ be a hyperplane of $\mathbb{P}(V)$, so $\dim H = n$. By Theorem 5.2.2, $\dim H^0 = (n+1) - n = 1$, so there is a nonzero $\varphi \in H^0$. Then $H \subseteq \ker \varphi$ by Definition 5.2.1, and both have dimension n — the second by Theorem 5.5.3 and Theorem 4.4.5 — so $H = \ker \varphi$ by Theorem 1.6.8.

Injective. Suppose $\ker \varphi = \ker \psi = H$ with $\varphi, \psi \neq 0$. Then $\varphi, \psi \in H^0$, which has dimension 1 as above. Two nonzero vectors of a one-dimensional space are proportional, so $\psi = \lambda \varphi$ with $\lambda \neq 0$, that is, $[\psi] = [\varphi]$. ■

In coordinates this is completely concrete. Take $V = F^{n+1}$ with coordinates $x_0, \dots, x_n$. By Theorem 5.1.1 every functional is $\varphi_{\mathbf{a}}(\mathbf{x}) = a_0x_0 + \dots + a_nx_n$ for a unique $\mathbf{a} \in F^{n+1}$, and this identification is a linear isomorphism $F^{n+1} \rightarrow (F^{n+1})^*$. So a point of the dual plane is a class $[a_0 : \dots : a_n]$, and the hyperplane it names is

$$\{[\mathbf{x}] \in \mathbb{P}^n(F) : a_0x_0 + \dots + a_nx_n = 0\}.$$

The equation is homogeneous, so it does not matter which representative $\mathbf{x}$ we substitute; and scaling $\mathbf{a}$ does not change the solution set. Both scalings are exactly the ones the projective spaces divide out.

Example. A line and its point

In $\mathbb{P}^2(\mathbb{Q})$, find the point of the dual plane naming the line through $[1 : 1 : 0]$ and $[0 : 1 : 1]$. Then find the point where the lines $x_0 - x_1 + x_2 = 0$ and $x_0 + x_2 = 0$ meet.

Solution. The line. Its subspace is $U = \text{span}((1, 1, 0), (0, 1, 1))$, of dimension 2. By Theorem 5.2.2, $\dim U^0 = 3 - 2 = 1$, so one nonzero functional does it. We need $a_0 + a_1 = 0$ and $a_1 + a_2 = 0$, whose solutions are the multiples of $\mathbf{a} = (1, -1, 1)$. Check: $1 - 1 + 0 = 0$ and $0 - 1 + 1 = 0$. So the line is $x_0 - x_1 + x_2 = 0$, named by the dual point $[1 : -1 : 1]$.

The meet. Dually, we need $\mathbf{x}$ with $x_0 - x_1 + x_2 = 0$ and $x_0 + x_2 = 0$. Subtracting, $x_1 = 0$; then $x_0 = -x_2$. The solution space is spanned by $(-1, 0, 1)$, so the two lines meet in the single point $[-1 : 0 : 1]$. Check: $-1 - 0 + 1 = 0$ and $-1 + 1 = 0$.

The two computations are the same computation, run in V and in V^* . That is the whole of duality in one example.

Warning. There is no canonical duality between $\mathbb{P}(V)$ and itself. The rule “the point $[a_0 : a_1 : a_2]$ names the line $a_0x_0 + a_1x_1 + a_2x_2 = 0$ ” looks like a self-duality of $\mathbb{P}^2(F)$, but it depends on the coordinates. In $\mathbb{P}^2(\mathbb{Q})$ the rule assigns to $[1 : 0 : 1]$ the line $x_0 + x_2 = 0$. Change coordinates by $y_0 = x_0$, $y_1 = x_1$, $y_2 = 2x_2$. The same point is $[1 : 0 : 2]$ in the new coordinates, and the same rule now assigns to it the line $y_0 + 2y_2 = 0$, that is, $x_0 + 4x_2 = 0$. These are different lines: the point $[1 : 0 : -1]$ lies on the first, since $1 - 1 = 0$, and not on the second, since $1 - 4 = -3 \neq 0$. What is canonical is the correspondence between $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, not one between $\mathbb{P}(V)$ and itself. Identifying the two requires a choice — a basis, or a nondegenerate form, which is how the pole and polar of a conic will be built in Section 10 of this chapter.

22.8.3 The duality correspondence

We now extend $[\varphi] \mapsto \mathbb{P}(\ker \varphi)$ from hyperplanes to all projective subspaces. The right tool is the annihilator, and we keep Chapter 5's notation: for a subspace $U \subseteq V$, $U^0 = \{\varphi \in V^* : \varphi(u) = 0 \text{ for all } u \in U\}$ (Definition 5.2.1).

For a projective subspace $S = \mathbb{P}(U)$ of $\mathbb{P}(V)$, define

$$S^0 := \mathbb{P}(U^0) \subseteq \mathbb{P}(V^*).$$

This is well defined because S determines U , by Lemma 22.8.1 (a).

Before the theorem, the dimension bookkeeping, because it is the step where such a correspondence is usually stated wrongly. Projective dimension is one less than linear dimension. So from $\dim U + \dim U^0 = n + 1$ we get

$$(\dim S + 1) + (\dim S^0 + 1) = n + 1, \quad \text{that is} \quad \dim S^0 = n - 1 - \dim S.$$

Test it at both ends. If $S = \emptyset$, so $U = \{0\}$ and $\dim S = -1$, the formula gives $\dim S^0 = n$, and indeed $\{0\}^0 = V^*$, so $S^0 = \mathbb{P}(V^*)$. If $S = \mathbb{P}(V)$, with $\dim S = n$, the formula gives -1 , and indeed $V^0 = \{0\}$, so $S^0 = \emptyset$. A point ($\dim S = 0$) goes to a hyperplane ($\dim S^0 = n - 1$), and a hyperplane goes to a point, agreeing with Proposition 22.8.2. The formula is symmetric in the sense that $d \mapsto n - 1 - d$ is its own inverse on $\{-1, 0, \dots, n\}$.

Theorem 22.8.3. The Duality Correspondence

Let $\dim V = n + 1 \geq 1$, and let S, T be projective subspaces of $\mathbb{P}(V)$, the empty set and $\mathbb{P}(V)$ included. Then:

- (a) $\dim S^0 = n - 1 - \dim S$;
- (b) $S \subseteq T$ if and only if $T^0 \subseteq S^0$;
- (c) $(S \vee T)^0 = S^0 \cap T^0$ and $(S \cap T)^0 = S^0 \vee T^0$;
- (d) $S \mapsto S^0$ is a bijection from the set of projective subspaces of $\mathbb{P}(V)$ onto the set of projective subspaces of $\mathbb{P}(V^*)$.

Idea. Every part is a Chapter 5 statement on annihilators, translated through Lemma 22.8.1, with the dimension shifted by one at each end. The only part that needs thought is the reverse implication in (b) and the surjectivity in (d): both go up a level to V^{**} and come back with Corollary 5.3.4, which is available because V is finite-dimensional.

Proof. Write $S = \mathbb{P}(U)$ and $T = \mathbb{P}(U')$, which is legitimate and unambiguous by Lemma 22.8.1 (a).

- (a) By Theorem 5.2.2, $\dim U^0 = (n + 1) - \dim U$. Hence

$$\dim S^0 = \dim U^0 - 1 = n - \dim U = n - (\dim S + 1) = n - 1 - \dim S.$$

- (b) $(\Rightarrow)$ If $S \subseteq T$, then $U \subseteq U'$ by Lemma 22.8.1 (a), so $(U')^0 \subseteq U^0$ by Theorem 5.2.3 (a), so $T^0 \subseteq S^0$ by the same lemma. $(\Leftarrow)$ Suppose $T^0 \subseteq S^0$, so $(U')^0 \subseteq U^0$. Applying Theorem 5.2.3 (a) inside V^* gives $U^{00} \subseteq (U')^{00}$. By Corollary 5.3.4, $U^{00} = \text{ev}_V(U)$ and $(U')^{00} = \text{ev}_V(U')$, so $\text{ev}_V(U) \subseteq \text{ev}_V(U')$. Since ev_V is injective (Theorem 5.3.2), $U \subseteq U'$, and $S \subseteq T$.

- (c) By [Lemma 22.8.1](#) (c), $S \vee T = \mathbb{P}(U + U')$, and $(U + U')^0 = U^0 \cap (U')^0$ by Theorem 5.2.3 (c). Applying $\mathbb{P}(\cdot)$ and [Lemma 22.8.1](#) (b),

$$(S \vee T)^0 = \mathbb{P}(U^0 \cap (U')^0) = \mathbb{P}(U^0) \cap \mathbb{P}((U')^0) = S^0 \cap T^0.$$

For the second identity, $S \cap T = \mathbb{P}(U \cap U')$ by [Lemma 22.8.1](#) (b), and $(U \cap U')^0 = U^0 + (U')^0$ by Theorem 5.2.3 (d), which needs V finite-dimensional. Applying $\mathbb{P}(\cdot)$ and part (c) of the same lemma gives $(S \cap T)^0 = S^0 \vee T^0$.

- (d) *Injective.* If $S^0 = T^0$ then $U^0 = (U')^0$ by [Lemma 22.8.1](#) (a), so $U = U'$ by Corollary 5.3.5, so $S = T$.

Surjective. Let $R = \mathbb{P}(W)$ be a projective subspace of $\mathbb{P}(V^*)$, where W is a subspace of V^* . Put

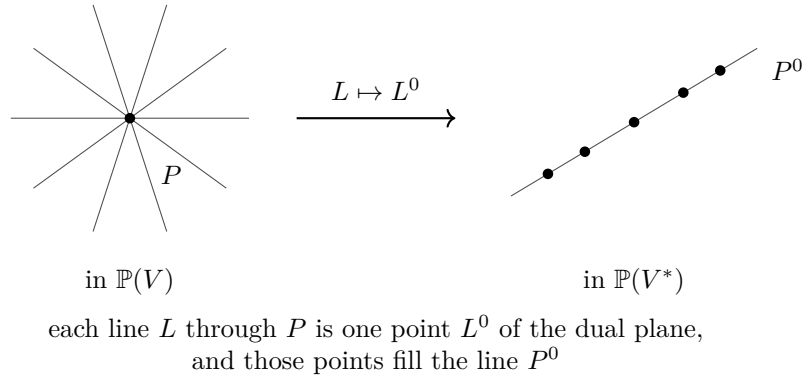
$$U := \text{ev}_V^{-1}(W^0) = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in W\},$$

a subspace of V : it is the intersection of the kernels $\ker \varphi$ over $\varphi \in W$, each a subspace by Theorem 2.2.1, and an intersection of subspaces is a subspace by Theorem 1.2.2. Since ev_V is surjective (Theorem 5.3.2), $\text{ev}_V(U) = W^0$. By Corollary 5.3.4, $(U^0)^0 = U^{00} = \text{ev}_V(U) = W^0$. Now U^0 and W are subspaces of the finite-dimensional space V^* with the same annihilator, so $U^0 = W$ by Corollary 5.3.5 applied in V^* . Hence $\mathbb{P}(U)^0 = \mathbb{P}(W) = R$. This proves the theorem. ■

Quick check. Let S be a line in $\mathbb{P}^3(F)$. What is $\dim S^0$, and what kind of object is S^0 ? What if S is a point?

Solution. Here $n = 3$ and $\dim S = 1$, so $\dim S^0 = 3 - 1 - 1 = 1$ by [Theorem 22.8.3](#) (a): S^0 is again a line, this time in the dual space $\mathbb{P}((F^4)^*)$. Lines in $\mathbb{P}^3$ are self-dual in dimension, which is one reason they are the interesting objects there. If instead S is a point, $\dim S = 0$ and $\dim S^0 = 3 - 1 - 0 = 2$: a plane in the dual space. Concretely, it is the set of all planes of $\mathbb{P}^3$ through that point.

Remark. The last sentence of the solution is the general reading of the correspondence. Since $S \subseteq P$ if and only if $P^0 \subseteq S^0$, the hyperplanes containing S are exactly the points of S^0 : *a projective subspace is named, on the dual side, by the family of hyperplanes that contain it.* For a point, that family is a hyperplane's worth of hyperplanes.



22.8.4 Dualizing twice

Applying the correspondence twice lands in $\mathbb{P}(V^{**})$, not back in $\mathbb{P}(V)$. Chapter 5 already told us what to do: the evaluation map is a canonical isomorphism $V \rightarrow V^{**}$, and it induces a projectivity.

Theorem 22.8.4. Dualizing Twice Returns the Original

Let $\dim V = n + 1 \geq 1$. The evaluation map $\text{ev}_V: V \rightarrow V^{**}$ is an isomorphism (Theorem 5.3.2), so it induces a projectivity $[\text{ev}_V]: \mathbb{P}(V) \rightarrow \mathbb{P}(V^{**})$ (Definition 22.7.1). For every projective subspace S of $\mathbb{P}(V)$,

$$S^{00} = [\text{ev}_V](S),$$

so that under the identification $\mathbb{P}(V) = \mathbb{P}(V^{**})$ given by $[\text{ev}_V]$ we have $S^{00} = S$. Moreover the inverse of the bijection of Theorem 22.8.3 (d) is $R \mapsto [\text{ev}_V]^{-1}(R^0)$.

Proof. Write $S = \mathbb{P}(U)$. By the definition of the correspondence applied twice, $S^{00} = \mathbb{P}((U^0)^0) = \mathbb{P}(\text{ev}_V(U))$, using Corollary 5.3.4. By Proposition 22.7.2, $\mathbb{P}(\text{ev}_V(U)) = [\text{ev}_V](\mathbb{P}(U)) = [\text{ev}_V](S)$.

For the last statement, let R be a projective subspace of $\mathbb{P}(V^*)$ and put $S := [\text{ev}_V]^{-1}(R^0)$, a projective subspace of $\mathbb{P}(V)$ by Proposition 22.7.2 applied to $[\text{ev}_V]^{-1}$. Then $[\text{ev}_V](S) = R^0$, so by the first part $(S^0)^0 = S^{00} = R^0$. Both S^0 and R are projective subspaces of $\mathbb{P}(V^*)$, and Theorem 22.8.3 (d) applied to the space V^* says that the correspondence there is injective, so $S^0 = R$. ■

Duality is also compatible with the maps of Section 07, which is what makes it more than a coordinate trick.

Proposition 22.8.5. Duality and Projectivities

Let $T: V \rightarrow W$ be an isomorphism of $(n + 1)$ -dimensional spaces, with dual map $T': W^* \rightarrow V^*$ (Definition 5.4.1), which is an isomorphism by Theorem 5.4.1 (d). Then for every projective subspace S of $\mathbb{P}(V)$,

$$([T](S))^0 = [(T')^{-1}](S^0).$$

Proof. Write $S = \mathbb{P}(U)$, so $[T](S) = \mathbb{P}(T(U))$ by Proposition 22.7.2. For $\psi \in W^*$,

$$\psi \in T(U)^0 \iff \psi(Tu) = 0 \text{ for all } u \in U \iff (T'\psi)(u) = 0 \text{ for all } u \in U,$$

the second step by Definition 5.4.1, and the right-hand condition says $T'\psi \in U^0$. Hence $T(U)^0 = (T')^{-1}(U^0)$. Applying $\mathbb{P}(\cdot)$ and Proposition 22.7.2 to the isomorphism $(T')^{-1}$ gives the claim. ■

22.8.5 The duality principle

We can now state precisely what “every theorem has a dual” amounts to in this book.

Theorem 22.8.6. The Duality Principle

Let $\dim V = n + 1 \geq 1$, and let $S \mapsto S^0$ be the correspondence of Theorem 22.8.3, from the projective subspaces of the n -dimensional space $\mathbb{P}(V)$ onto those of the n -dimensional space $\mathbb{P}(V^*)$. Let R, S, T be projective subspaces of $\mathbb{P}(V)$ and let d be an integer. Then:

- (a) $S \subseteq T$ if and only if $T^0 \subseteq S^0$; and $S = T$ if and only if $S^0 = T^0$;
- (b) $\dim S = d$ if and only if $\dim S^0 = n - 1 - d$;
- (c) $R = S \vee T$ if and only if $R^0 = S^0 \cap T^0$;
- (d) $R = S \cap T$ if and only if $R^0 = S^0 \vee T^0$;
- (e) as S runs over all projective subspaces of $\mathbb{P}(V)$, the subspace S^0 runs exactly once over all projective subspaces of $\mathbb{P}(V^*)$.

Proof.

- (a) The first claim is Theorem 22.8.3 (b), and the second follows by applying it in both directions, or from injectivity in Theorem 22.8.3 (d).
- (b) By Theorem 22.8.3 (a), $\dim S^0 = n - 1 - \dim S$. So $\dim S = d$ gives $\dim S^0 = n - 1 - d$; conversely $\dim S^0 = n - 1 - d$ gives $n - 1 - \dim S = n - 1 - d$, hence $\dim S = d$.
- (c) $(\Rightarrow)$ is Theorem 22.8.3 (c). $(\Leftarrow)$ If $R^0 = S^0 \cap T^0 = (S \vee T)^0$, then $R = S \vee T$ by the equality clause of (a).
- (d) The same argument with the second identity of Theorem 22.8.3 (c).
- (e) This is Theorem 22.8.3 (d). ■

How the principle is used, and what we have not proved. Every clause of Theorem 22.8.6 is an “if and only if”, and by (e) the correspondence is a bijection between the two supplies of subspaces. So a proved statement about $\mathbb{P}(V)$ whose hypotheses and conclusion are built **only** from the relations $\subseteq$ and $=$, the operations $\vee$ and $\cap$, and the dimensions of the subspaces involved, can be translated clause by clause: reverse every inclusion, exchange $\vee$ with $\cap$, and replace every dimension d by $n - 1 - d$. The translated statement then holds in $\mathbb{P}(V^*)$; and since every $(n + 1)$ -dimensional space W satisfies $W \cong (W^*)^*$ (Theorem 5.3.2), a statement true

in $\mathbb{P}(V^*)$ for **every** V is true in $\mathbb{P}(W)$ for every W , because projectivities preserve all of this structure (Proposition 22.7.2).

This is a recipe, carried out below on the statements this chapter makes, and each carrying out is a proof. It is **not** a theorem here. To make “for every statement built only from ...” itself a theorem, one has to define a formal language of incidence statements and induct on the way a statement is built, and this book has set up no such language. So we claim the principle only as a working method, and we prove each dual statement we use.

Corollary 22.8.7. Two Points, Two Lines

Let $\dim V = 3$, so that $\mathbb{P}(V)$ is a projective plane.

- (a) Two distinct points of $\mathbb{P}(V)$ lie on exactly one line.
- (b) Two distinct lines of $\mathbb{P}(V)$ meet in exactly one point.

Idea. Prove (a) by hand: two distinct points have independent representatives, so their span is a plane in V and a line in $\mathbb{P}(V)$. Then get (b) from (a) by the dictionary, in $\mathbb{P}(V^*)$, which is a projective plane of its own. Section 06 already proved both statements directly from the dimension formula (Proposition 22.6.8, Corollary 22.6.7). The point of doing (b) again this way is that duality delivers it with no geometry at all: every step is a clause of Theorem 22.8.6 with $n = 2$.

Proof.

- (a) Let $P = [v] \neq Q = [w]$. The list (v, w) is independent: otherwise w would be a scalar multiple of v , giving $P = Q$. So $\text{span}(v, w)$ has dimension 2, and by Lemma 22.8.1 (c), $P \vee Q = \mathbb{P}(\text{span}(v, w))$, a projective subspace of dimension 1, that is, a line containing both points. If L is any line containing P and Q , then $P \vee Q \subseteq L$ because the join is the smallest such subspace, and both have dimension 1, so $L = P \vee Q$ by the consequence noted after Lemma 22.8.1.
- (b) Here $n = 2$. Let $L_1 \neq L_2$ be lines of $\mathbb{P}(V)$. By Theorem 22.8.6 (b), $\dim L_i^0 = 2 - 1 - 1 = 0$, so L_1^0 and L_2^0 are points of $\mathbb{P}(V^*)$, and they are distinct by the equality clause of Theorem 22.8.6 (a). The space $\mathbb{P}(V^*)$ is a projective plane, since $\dim V^* = 3$ by Corollary 5.1.4, so part (a), applied in $\mathbb{P}(V^*)$, says that $L_1^0 \vee L_2^0$ is a line, of dimension 1.

By Theorem 22.8.6 (d), $(L_1 \cap L_2)^0 = L_1^0 \vee L_2^0$. Hence $\dim(L_1 \cap L_2)^0 = 1$, and Theorem 22.8.6 (b) read backwards gives $\dim(L_1 \cap L_2) = 2 - 1 - 1 = 0$. A projective subspace of dimension 0 is a single point. This proves that L_1 and L_2 meet in exactly one point. ■

The same bookkeeping shows that the dimension formula of Theorem 22.6.6 is **self-dual**: replacing each of $\dim(S \vee T)$, $\dim(S \cap T)$, $\dim S$, $\dim T$ by $n - 1$ minus itself and exchanging $\vee$ with $\cap$ turns

$$\dim(S \vee T) + \dim(S \cap T) = \dim S + \dim T$$

into

$$\begin{aligned} & (n - 1 - \dim(S \cap T)) + (n - 1 - \dim(S \vee T)) \\ &= (n - 1 - \dim S) + (n - 1 - \dim T), \end{aligned}$$

which is the same equation after canceling $2(n - 1)$. A self-dual theorem is one whose dual gives nothing new, and that is a useful thing to be able to detect.

Warning. The dual of “point” is not “line”; it is “hyperplane”, and which dimension that is depends on n . In the projective plane, $n = 2$, and $2 - 1 - 0 = 1$: points dualize to lines, which is why [Corollary 22.8.7](#) looks like a swap of the two words. In $\mathbb{P}^3$, $n = 3$, so points dualize to **planes** ($3 - 1 - 0 = 2$) and lines dualize to lines ($3 - 1 - 1 = 1$). The correct dual of “two distinct points lie on a unique line” in $\mathbb{P}^3$ is “two distinct planes meet in a unique line”, and that is true. The careless dual, “two distinct lines meet in a point”, is **false** in $\mathbb{P}^3$: the lines $\mathbb{P}(\text{span}(\mathbf{e}_0, \mathbf{e}_1))$ and $\mathbb{P}(\text{span}(\mathbf{e}_2, \mathbf{e}_3))$ have $\text{span}(\mathbf{e}_0, \mathbf{e}_1) \cap \text{span}(\mathbf{e}_2, \mathbf{e}_3) = \{\mathbf{0}\}$, so they do not meet at all. Always compute $n - 1 - d$; never translate by the words alone.

Pole and polar — the correspondence a conic sets up between the points and the lines of a plane — is the one piece of duality that needs a quadratic form rather than only the annihilator. It is treated in Section 10 of this chapter, where conics are, and nothing in this section depends on it.

22.8.6 Exercises

22.8.6.1 A. Check your understanding

Exercise. A1

- (a) Define the dual projective space of $\mathbb{P}(V)$, and say what its dimension is.
- (b) Let S be a plane in $\mathbb{P}^4(F)$. Compute $\dim S^0$.
- (c) True or false, with a reason: the duality correspondence of [Theorem 22.8.3](#) is a bijection from the points of $\mathbb{P}(V)$ to the points of $\mathbb{P}(V)$.
- (d) In $\mathbb{P}^2$, what is the dual of the statement “three distinct points are collinear”?
- (e) What are $\emptyset^0$ and $\mathbb{P}(V)^0$?

Solution.

- (a) It is $\mathbb{P}(V^*)$, the projective space of the dual space ([Definition 22.8.1](#)). If $\dim \mathbb{P}(V) = n$, then $\dim \mathbb{P}(V^*) = n$ as well, by [Corollary 5.1.4](#).
- (b) A plane has dimension 2 and $n = 4$, so $\dim S^0 = 4 - 1 - 2 = 1$: a line in the dual space.
- (c) False, twice over. It is a bijection between **all** projective subspaces of $\mathbb{P}(V)$ and all projective subspaces of $\mathbb{P}(V^*)$, and it sends points of $\mathbb{P}(V)$ to hyperplanes of $\mathbb{P}(V^*)$, not to points; and its target is $\mathbb{P}(V^*)$, not $\mathbb{P}(V)$ (see the warning after [Example \(A line and its point\)](#)).
- (d) With $n = 2$, points dualize to lines and a line dualizes to a point. “Three distinct points lie on one line” becomes “three distinct lines pass through one point”, that is, the three lines are concurrent.

- (e) $\emptyset^0 = \mathbb{P}(\{0\}^0) = \mathbb{P}(V^*)$, of dimension $n = n - 1 - (-1)$; and $\mathbb{P}(V)^0 = \mathbb{P}(V^0) = \mathbb{P}(\{0\}) = \emptyset$, of dimension $-1 = n - 1 - n$.

22.8.6.2 B. Practice

Exercise. B1: Lines and dual points

Work in $\mathbb{P}^2(\mathbb{Q})$.

- (a) Find the dual point of the line through $[1 : 2 : 1]$ and $[0 : 1 : 3]$.
- (b) Find the point where the lines $x_0 + x_1 - x_2 = 0$ and $2x_0 - x_1 = 0$ meet.

Solution.

- (a) We need $\mathbf{a} = (a_0, a_1, a_2)$ with $a_0 + 2a_1 + a_2 = 0$ and $a_1 + 3a_2 = 0$. From the second, $a_1 = -3a_2$; substituting, $a_0 = -2a_1 - a_2 = 6a_2 - a_2 = 5a_2$. Taking $a_2 = 1$ gives $\mathbf{a} = (5, -3, 1)$, so the line is $5x_0 - 3x_1 + x_2 = 0$ and the dual point is $[5 : -3 : 1]$. Check: $5 - 6 + 1 = 0$ and $0 - 3 + 3 = 0$. By Theorem 5.2.2 the solution space is one-dimensional, so this is the only answer up to scaling.
- (b) Solve $x_0 + x_1 - x_2 = 0$ and $2x_0 - x_1 = 0$. The second gives $x_1 = 2x_0$, and then $x_2 = x_0 + x_1 = 3x_0$. Taking $x_0 = 1$, the meet is $[1 : 2 : 3]$. Check: $1 + 2 - 3 = 0$ and $2 - 2 = 0$. The solution space is one-dimensional, consistent with Corollary 22.8.7 (b).

Exercise. B2: Dimension bookkeeping

Let $n = 5$, so $\mathbb{P}(V) = \mathbb{P}^5(F)$. For each of the following S , give $\dim S^0$, and say which of the two is larger.

- (a) S a point.
- (b) S a line.
- (c) S of dimension 2.
- (d) S a hyperplane.
- (e) $S = \emptyset$.

Then determine whether there is any S with $\dim S^0 = \dim S$. Justify your answer.

Solution. By Theorem 22.8.3 (a), $\dim S^0 = 5 - 1 - \dim S = 4 - \dim S$.

- (a) $\dim S = 0$, $\dim S^0 = 4$; the dual is larger. (b) 1 and 3; the dual is larger. (c) 2 and 2; equal. (d) 4 and 0; the original is larger. (e) -1 and 5; the dual is larger.

For the last part, $\dim S^0 = \dim S$ means $4 - \dim S = \dim S$, that is, $\dim S = 2$. So the self-dual dimension in $\mathbb{P}^5$ is 2, and part (c) is an instance. In general $\dim S^0 = \dim S$ forces $2 \dim S = n - 1$, which has a solution exactly when n is odd.

Exercise. B3: Dualize a statement

For each statement about $\mathbb{P}^3(F)$, write the dual statement, using [Theorem 22.8.6](#) with $n = 3$, and say whether both are true.

- (a) A point and a line not through it lie in exactly one plane.
- (b) Three planes with no common line meet in at most one point.

Solution. With $n = 3$, the dimension rule is $d \mapsto 2 - d$: points (0) and planes (2) exchange, lines (1) stay lines, $\emptyset$ (−1) and the whole space (3) exchange. Joins become meets.

- (a) Dual: *a plane and a line not contained in it meet in exactly one point.* Both are true. For the original: if $P \not\subseteq L$ then $P \cap L = \emptyset$, so by [Theorem 22.6.6](#) $\dim(P \vee L) = 0 + 1 - (-1) = 2$, a plane, and it is unique because any plane containing both contains $P \vee L$ and has the same dimension. Dualizing each clause by [Theorem 22.8.6](#) gives the dual statement, which is therefore also true; directly, if the plane Π does not contain L , then $\Pi \vee L$ has dimension 3, so $\dim(\Pi \cap L) = 2 + 1 - 3 = 0$.
- (b) Dualize clause by clause, as the warning after [Corollary 22.8.7](#) insists. The hypothesis “the three planes contain no common line” reads: there is no L with $\dim L = 1$ and $L \subseteq \Pi_1 \cap \Pi_2 \cap \Pi_3$. Writing $P_i = \Pi_i^0$, which are points by [Theorem 22.8.6](#) (b), and using (a) and (d) of that theorem, the hypothesis becomes: there is no L' with $\dim L' = 1$ and $P_1 \vee P_2 \vee P_3 \subseteq L'$, that is, the three points are **not** collinear. The conclusion “ $\Pi_1 \cap \Pi_2 \cap \Pi_3$ has dimension at most 0” becomes “ $P_1 \vee P_2 \vee P_3$ has dimension at least 2”. So the dual statement is: *three points not on a common line span a projective subspace of dimension at least 2*, that is, they span a plane. Both are true: three noncollinear points $[v_1], [v_2], [v_3]$ have independent representatives, since a dependence would put all three in a subspace of dimension at most 2 and hence on a common line, so their join has dimension 2 by [Lemma 22.8.1](#) (c). The original then follows by dualizing back.

22.8.6.3 C. Going deeper

Exercise. C1: Counting in a finite projective plane

Let q be a prime power and $F = \mathbb{F}_q$.

- (a) Show that a projective space of dimension d over $\mathbb{F}_q$ has $(q^{d+1} - 1)/(q - 1)$ points.
- (b) Deduce that $\mathbb{P}^2(\mathbb{F}_q)$ has as many lines as points, and that every point lies on exactly $q + 1$ lines.

Solution.

- (a) Such a space is $\mathbb{P}(U)$ with $\dim U = d + 1$. The set $U \setminus \{0\}$ has $q^{d+1} - 1$ elements, and two nonzero vectors give the same point exactly when they differ by a scalar in $\mathbb{F}_q \setminus \{0\}$, a set of size $q - 1$. So each point has exactly $q - 1$ representatives, and the number of points is $(q^{d+1} - 1)/(q - 1)$.
- (b) By [Proposition 22.8.2](#) the lines of $\mathbb{P}^2(\mathbb{F}_q)$ are in bijection with the points of $\mathbb{P}((\mathbb{F}_q^3)^*)$, which by [Corollary 5.1.4](#) is again a projective plane over $\mathbb{F}_q$. So by (a) with $d = 2$ both counts are $N := q^2 + q + 1$.

For the last claim, fix a point P . The lines of $\mathbb{P}^2(\mathbb{F}_q)$ containing P are, by the remark after [Theorem 22.8.3](#), exactly the points of P^0 , and $\dim P^0 = 2 - 1 - 0 = 1$ by [Theorem 22.8.3](#) (a). So by (a) with $d = 1$ there are $q + 1$ of them. As a check, count the incident pairs (P, L) with $P \in L$ in two ways: each line is a projective space of dimension 1 and so carries $q + 1$ points, giving $N(q + 1)$ pairs, and the other count gives $N(q + 1)$ as well.

Exercise. C2: A form makes a self-duality

Let β be a nondegenerate symmetric bilinear form on V , with $\dim V = n + 1$.

- Show that $R_\beta: V \rightarrow V^*$, $v \mapsto \beta(\cdot, v)$, is an isomorphism, and deduce that it induces a projectivity $[R_\beta]: \mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$.
- Hence define, for a projective subspace S of $\mathbb{P}(V)$, a projective subspace $S^\perp$ of $\mathbb{P}(V)$ with $\dim S^\perp = n - 1 - \dim S$, and show that the assignment $S \mapsto S^\perp$ is an inclusion-reversing bijection of the set of projective subspaces of $\mathbb{P}(V)$ with itself.
- Give an example over $\mathbb{R}$ with $n = 2$ in which $S \subseteq S^\perp$ for some point S , and explain why this cannot happen if β is positive definite.

Solution.

- By Proposition 14.1.2, β is nondegenerate exactly when R_β is an isomorphism $V \rightarrow V^*$. A projectivity is induced by any isomorphism (Definition 22.7.1), so $[R_\beta]$ is one.
- Define $S^\perp := [R_\beta]^{-1}(S^0)$. This is a projective subspace of $\mathbb{P}(V)$ by Proposition 22.7.2, of the same dimension as S^0 , namely $n - 1 - \dim S$ by [Theorem 22.8.3](#) (a). Since $[R_\beta]^{-1}$ is a bijection preserving inclusion (Proposition 22.7.2) and $S \mapsto S^0$ is an inclusion-reversing bijection onto the projective subspaces of $\mathbb{P}(V^*)$ ([Theorem 22.8.3](#) (b), (d)), the composite is an inclusion-reversing bijection of the projective subspaces of $\mathbb{P}(V)$ with themselves.
- Take $V = \mathbb{R}^3$ and $\beta(\mathbf{x}, \mathbf{y}) = x_0y_0 + x_1y_1 - x_2y_2$, which is nondegenerate since its matrix $\text{diag}(1, 1, -1)$ is invertible. Let $\mathbf{v} = (1, 0, 1)$, so $\beta(\mathbf{v}, \mathbf{v}) = 1 + 0 - 1 = 0$, and let $S = \{[\mathbf{v}]\}$. Then $R_\beta(\mathbf{v}) = \beta(\cdot, \mathbf{v})$ kills $\mathbf{v}$, so $[\mathbf{v}] \in S^\perp$, that is, $S \subseteq S^\perp$. If β were positive definite, then $\beta(\mathbf{v}, \mathbf{v}) > 0$ for every $\mathbf{v} \neq \mathbf{0}$, so no nonzero vector is annihilated by its own functional, and $S \subseteq S^\perp$ is impossible for a point S . The nonzero vectors with $\beta(\mathbf{v}, \mathbf{v}) = 0$ are the isotropic ones of Definition 14.7.1, taken for the quadratic form $q(\mathbf{v}) = \beta(\mathbf{v}, \mathbf{v})$; they are exactly the points lying on their own duals.

Exercise. C3: What the correspondence does to a chain

Let $\dim V = n + 1$ and let

$$\emptyset = S_{-1} \subsetneq S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = \mathbb{P}(V)$$

be projective subspaces with $\dim S_d = d$ for each d .

- Prove that $S_n^0 \subsetneq S_{n-1}^0 \subsetneq \cdots \subsetneq S_{-1}^0$ is again such a chain, and identify $\dim S_d^0$.
- Deduce that $S \mapsto S^0$ carries the chains of this shape in $\mathbb{P}(V)$ bijectively to those in $\mathbb{P}(V^*)$, and that it reverses their order.

Solution.

- (a) By [Theorem 22.8.6](#) (a), $S_d \subseteq S_{d+1}$ gives $S_{d+1}^0 \subseteq S_d^0$, and the inclusion is strict: if $S_{d+1}^0 = S_d^0$ then $S_{d+1} = S_d$ by the equality clause, contradicting strictness. By [Theorem 22.8.6](#) (b), $\dim S_d^0 = n - 1 - d$. As d runs from -1 to n , the value $n - 1 - d$ runs from n down to -1 , taking each value once. So writing $T_e := S_{n-1-e}^0$ gives $\dim T_e = e$ and $T_{-1} \subsetneq T_0 \subsetneq \cdots \subsetneq T_n$, a chain of the same shape in $\mathbb{P}(V^*)$.
- (b) The assignment is injective because $S \mapsto S^0$ is ([Theorem 22.8.6](#) (e)), applied term by term. It is surjective: given a chain $T_{-1} \subsetneq T_0 \subsetneq \cdots \subsetneq T_n$ with $\dim T_e = e$ in $\mathbb{P}(V^*)$, set $S_d := [\text{ev}_V]^{-1}(T_{n-1-d}^0)$, which by [Theorem 22.8.4](#) is the unique projective subspace of $\mathbb{P}(V)$ with $S_d^0 = T_{n-1-d}^0$. By [Theorem 22.8.6](#) (b), $\dim S_d = n - 1 - (n - 1 - d) = d$; and if $d < d'$ then $n - 1 - d > n - 1 - d'$, so $T_{n-1-d} \supsetneq T_{n-1-d'}$, whence $S_d \subsetneq S_{d'}$ by [Theorem 22.8.6](#) (a). So (S_d) is a chain of the required shape whose image is the given one, and the order has been reversed.