

22.6 Projective Space

The affine geometry of the first two sections has one flaw, and it is visible in the statement of Theorem 22.2.4: the formula for the dimension of a join has two cases, because two flats may fail to meet. This section builds a geometry in which that never happens. The construction is entirely linear — a projective space is the set of lines through the origin of a vector space — and the payoff is a dimension formula with no exceptional case at all. Everything here works over an arbitrary field F , and neither analysis nor topology enters.

22.6.1 Two lines that do not meet

In the affine plane $\mathbb{A}^2(\mathbb{R})$, take

$$L_1 = \{(y_1, y_2) : y_2 - y_1 = 0\}, \quad L_2 = \{(y_1, y_2) : y_2 - y_1 = 1\}.$$

A point in both would satisfy $0 = 1$, so $L_1 \cap L_2 = \emptyset$. Both are flats of dimension 1 with the same direction space $\text{span}((1, 1))$, and their join is the whole plane. So Theorem 22.2.4 (c) is in its second case: $\dim(L_1 \vee L_2) = 1 + 1 - 1 + 1 = 2$, with the extra $+1$ paid for by the failure to meet, and there is nothing to substitute for $\dim(L_1 \cap L_2)$, because the empty set has no dimension.

The two lines do have something in common: a direction. They are going the same way, and the natural thing to say is that they meet “at infinity”. The whole of this section is the work of turning that phrase into mathematics without losing linearity. The repair, in one sentence: *stop working with points of a vector space and start working with its lines through the origin.*

22.6.2 The projective space of a vector space

Let V be a vector space over F . On $V \setminus \{0\}$ define

$$x \sim y \quad \text{if and only if} \quad y = \lambda x \text{ for some } \lambda \in F, \lambda \neq 0.$$

This is an equivalence relation (Definition 0.5.2): it is reflexive with $\lambda = 1$; symmetric, because $y = \lambda x$ with $\lambda \neq 0$ gives $x = \lambda^{-1}y$; and transitive, because $y = \lambda x$ and $z = \mu y$ give $z = (\mu\lambda)x$ with $\mu\lambda \neq 0$, a field having no zero divisors. Note that 0 had to be removed: it is equivalent only to itself, and keeping it would put a single extra point into every picture with no geometric job.

A point of projective space is a line through the origin, with the origin itself thrown away.

Definition 22.6.1. Projective Space

Let V be a finite-dimensional vector space over a field F with $\dim V = n+1$. The **projective space** of V is the quotient set (Definition 0.5.5)

$$\mathbb{P}(V) = (V \setminus \{0\})/\sim,$$

whose elements are called **points**. The point determined by $x \neq 0$ is written $[x]$. The **dimension** of $\mathbb{P}(V)$ is

$$\dim \mathbb{P}(V) = \dim V - 1 = n,$$

so that $\mathbb{P}(\{0\}) = \emptyset$ has dimension -1 . We write $\mathbb{P}^n(F) = \mathbb{P}(F^{n+1})$ for **projective n -space over F** . A projective space of dimension 1 is a **projective line**, and one of dimension 2 a

projective plane.

Two clauses need comment. The shift by one in the dimension is not a decoration: it is what makes a projective line behave like a line and a projective plane like a plane, as [Theorem 22.6.2](#) is about to confirm. And the convention $\dim \emptyset = -1$ looks like a dodge, but it is forced by the same shift — the empty set is $\mathbb{P}(\{0\})$ and $\dim\{0\} = 0$ — and it will do real work in [Theorem 22.6.6](#).

Proposition 22.6.1. Points Are Lines Through the Origin

Let V be a finite-dimensional vector space over F . The map $[x] \mapsto \text{span}(x)$ is a bijection from $\mathbb{P}(V)$ onto the set of 1-dimensional subspaces of V .

Proof. Well defined. If $[x] = [y]$ then $y = \lambda x$ with $\lambda \neq 0$, so $\text{span}(y) = \text{span}(x)$ because each of x, y is a scalar multiple of the other. Also $x \neq 0$, so $\text{span}(x)$ has dimension 1.

Injective. If $\text{span}(x) = \text{span}(y)$ with $x, y \neq 0$, then $y = \lambda x$ for some $\lambda \in F$, and $\lambda \neq 0$ since $y \neq 0$. Hence $[x] = [y]$.

Surjective. A 1-dimensional subspace has a basis (x) with $x \neq 0$, and then it is $\text{span}(x)$, the image of $[x]$. This proves the proposition. ■

So a point of $\mathbb{P}^2(F)$ is a line through the origin of F^3 , and a “line” of the projective plane will turn out to be a plane through the origin of F^3 . Every statement below can be read twice: once about $\mathbb{P}(V)$ and once about subspaces of V .

Definition 22.6.2. Homogeneous Coordinates

Let $\mathbf{x} = (x_0, x_1, \dots, x_n) \in F^{n+1}$ be nonzero. The **homogeneous coordinates** of the point $[\mathbf{x}] \in \mathbb{P}^n(F)$ are written

$$[\mathbf{x}] = [x_0 : x_1 : \cdots : x_n],$$

the colons recording that only the ratios are determined: for every $\lambda \neq 0$,

$$[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n],$$

and not all of $x_0, \dots, x_n$ are zero.

Coordinates run from 0. In F^{n+1} we index the entries and the standard basis from 0, writing $\mathbf{x} = (x_0, x_1, \dots, x_n)$ and $\mathbf{e}_0, \dots, \mathbf{e}_n$, and every projective section of this chapter does the same. Subscripts start at 1 everywhere else in the book; the exception pays for itself here, because it keeps the n of $\mathbb{P}^n(F)$ equal to the dimension while the coordinates number $n + 1$.

Examples.

- $\mathbb{P}^0(F) = \mathbb{P}(F^1)$ has exactly one point, $[1]$, since every nonzero scalar is a nonzero multiple of 1. Its dimension is 0, as a single point should have.
- $\mathbb{P}^1(F)$ consists of the points $[1 : a]$ for $a \in F$, all distinct, together with the single point $[0 : 1]$. Indeed if $x_0 \neq 0$ then $[x_0 : x_1] = [1 : x_1/x_0]$, and if $x_0 = 0$ then $x_1 \neq 0$ and $[0 : x_1] = [0 : 1]$.
- $\mathbb{P}^1(\mathbb{F}_2) = \{[1 : 0], [1 : 1], [0 : 1]\}$ has three points. The three lines through the origin of $\mathbb{F}_2^2$

are $\text{span}((1, 0))$, $\text{span}((1, 1))$, $\text{span}((0, 1))$, matching [Proposition 22.6.1](#).

- The degenerate case $\mathbb{P}(\{0\}) = \emptyset$. It is worth keeping, because intersections produce it.

Nonexample by minimal change. Over $\mathbb{R}$, replace the relation by $x \approx y$ when $y = \lambda x$ with $\lambda > 0$. This is still an equivalence relation, but the quotient is not $\mathbb{P}(V)$: every projective point $[x]$ splits into the two classes of x and $-x$. The clause that fails is the one allowing **every** nonzero λ , and what is lost is precisely the identification of a line with itself traversed backwards.

Warning. **A single homogeneous coordinate means nothing.** The expression “ $x_1 = 3$ ” is not a statement about a point of $\mathbb{P}^n(F)$, since $[1 : 3 : 0] = [2 : 6 : 0]$ and the second coordinate changed. Only two kinds of statement survive scaling: a ratio such as x_1/x_0 where $x_0 \neq 0$, and the vanishing or nonvanishing of a **homogeneous** expression, such as $x_1 = 0$ or $x_0x_2 - x_1^2 = 0$. Whenever a formula below names a coordinate, check that it survives replacing $\mathbf{x}$ by $\lambda\mathbf{x}$.

22.6.3 The affine chart, and where the missing points live

Here is the theorem that makes the construction worth doing: projective space is an affine space plus a smaller projective space.

Theorem 22.6.2. The Standard Affine Chart

Let $n \geq 1$ and let

$$U_0 = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(F) : x_0 \neq 0\},$$

$$H_\infty = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(F) : x_0 = 0\}.$$

Then:

- both conditions are independent of the representative, and $\mathbb{P}^n(F) = U_0 \sqcup H_\infty$, a disjoint union;
- the map $\iota: F^n \rightarrow U_0$, $\iota(a_1, \dots, a_n) = [1 : a_1 : \cdots : a_n]$, is a bijection, with inverse $[x_0 : \cdots : x_n] \mapsto (x_1/x_0, \dots, x_n/x_0)$;
- $H_\infty = \mathbb{P}(W)$ with $W = \{\mathbf{x} \in F^{n+1} : x_0 = 0\}$, so H_∞ has dimension $n - 1$, and $[0 : x_1 : \cdots : x_n] \mapsto [x_1 : \cdots : x_n]$ is a bijection $H_\infty \rightarrow \mathbb{P}^{n-1}(F)$.

U_0 is the **standard affine chart** and H_∞ the **hyperplane at infinity**.

Idea. Scaling multiplies x_0 by a nonzero number, so it cannot create or destroy a zero there: that single observation makes both halves well defined. On the half where $x_0 \neq 0$ we may **normalize** x_0 to 1, and once $x_0 = 1$ the remaining coordinates are no longer ambiguous, which is why the affine points come back one for one. On the other half x_0 carries no information at all, so it may be deleted, and what is left is one dimension smaller.

Proof.

- If $\mathbf{y} = \lambda\mathbf{x}$ with $\lambda \neq 0$ then $y_0 = \lambda x_0$, and a product of two elements of a field is zero only if one of them is, so $y_0 = 0$ exactly when $x_0 = 0$. Hence both defining conditions

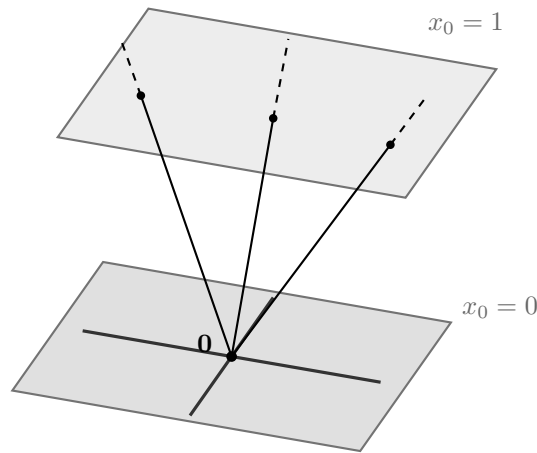
depend only on the point. Every point satisfies exactly one of them, so the union is disjoint and exhausts $\mathbb{P}^n(F)$.

- (b) *Well defined:* $(1, a_1, \dots, a_n) \neq \mathbf{0}$, and its zeroth entry is $1 \neq 0$, so $\iota(\mathbf{a}) \in U_0$.
Injective: if $[1 : \mathbf{a}] = [1 : \mathbf{a}']$ then $(1, \mathbf{a}') = \lambda(1, \mathbf{a})$ for some $\lambda \neq 0$; comparing zeroth entries gives $\lambda = 1$, hence $\mathbf{a}' = \mathbf{a}$.
Surjective: let $[\mathbf{x}] \in U_0$, so $x_0 \neq 0$ and x_0^{-1} exists. Then $x_0^{-1}\mathbf{x} = (1, x_1/x_0, \dots, x_n/x_0)$, so $[\mathbf{x}] = \iota(x_1/x_0, \dots, x_n/x_0)$. The displayed formula for the inverse is exactly this computation, and it is independent of the representative because replacing $\mathbf{x}$ by $\lambda\mathbf{x}$ multiplies numerator and denominator of each ratio by λ .
- (c) W is a subspace of F^{n+1} , being the kernel of the linear functional $\mathbf{x} \mapsto x_0$, and $\dim W = n$ by the Rank-Nullity Theorem (Theorem 2.3.1), the functional being nonzero and hence surjective onto F (Proposition 5.1.2). A nonzero $\mathbf{x}$ has $x_0 = 0$ exactly when $\mathbf{x} \in W \setminus \{\mathbf{0}\}$, so $H_\infty = \mathbb{P}(W)$ and $\dim H_\infty = n - 1$ by Definition 22.6.1. The stated map is well defined, because deleting the zeroth coordinate turns $\lambda\mathbf{x}$ into λ times the deletion and cannot produce $\mathbf{0}$ from a nonzero $\mathbf{x}$ with $x_0 = 0$; it is injective and surjective because inserting a zeroth coordinate 0 inverts it. This proves the theorem. ■

For $n = 2$ the statement reads: **the projective plane is the affine plane together with a projective line**. Writing it out, every point of $\mathbb{P}^2(F)$ is exactly one of

$$[1 : a : b] \quad (a, b \in F), \quad [0 : 1 : m] \quad (m \in F), \quad [0 : 0 : 1],$$

the first family being the affine plane through ι , and the last two being the line at infinity $H_\infty \cong \mathbb{P}^1(F)$, which has $|F| + 1$ points when F is finite. The label m is chosen deliberately: the next proposition shows $[0 : 1 : m]$ is the point where all affine lines of slope m meet.



each line through $\mathbf{0}$ is one point of $\mathbb{P}^2(F)$:
 those meeting the plane $x_0 = 1$ give the affine chart,
 those lying inside the plane $x_0 = 0$ give the line at infinity

Proposition 22.6.3. Completing an Affine Line

Let $a_1, a_2 \in F$ be not both zero, let $b \in F$, and let

$$L = \{(y_1, y_2) \in F^2 : a_1 y_1 + a_2 y_2 = b\},$$

a flat of dimension 1 in $\mathbb{A}^2$ by Theorem 22.2.10, the single row (a_1, a_2) being nonzero, with

direction space $\text{span}((-a_2, a_1))$. Let $\varphi(x_0, x_1, x_2) = a_1x_1 + a_2x_2 - bx_0$ and $\bar{L} = \mathbb{P}(\ker \varphi) \subseteq \mathbb{P}^2(F)$. Then:

- (a) $\bar{L}$ is a projective line;
- (b) $\iota(L) = \bar{L} \cap U_0$, so $\bar{L}$ contains a faithful copy of L ;
- (c) $\bar{L} \setminus U_0$ is the single point $[0 : -a_2 : a_1]$;
- (d) two such lines L, L' have the same direction space if and only if $\bar{L}$ and $\bar{L}'$ meet H_∞ in the same point.

Proof.

- (a) φ is linear and nonzero, since $\varphi(0, 1, 0) = a_1$ and $\varphi(0, 0, 1) = a_2$ are not both zero. So φ is surjective onto F (Proposition 5.1.2) and $\dim \ker \varphi = 3 - 1 = 2$ by Theorem 2.3.1. Hence $\bar{L} = \mathbb{P}(\ker \varphi)$ has dimension 1.
- (b) A point of U_0 is $\iota(y_1, y_2) = [1 : y_1 : y_2]$ by Theorem 22.6.2 (b), and $\varphi(1, y_1, y_2) = a_1y_1 + a_2y_2 - b$. This vanishes exactly when $(y_1, y_2) \in L$. Since φ is homogeneous of degree 1, the vanishing does not depend on the representative.
- (c) A point of $\bar{L} \setminus U_0$ has $x_0 = 0$ and $a_1x_1 + a_2x_2 = 0$ with $(x_1, x_2) \neq (0, 0)$. The solution set of that single nontrivial equation in F^2 is $\text{span}((-a_2, a_1))$, of dimension 1, so there is exactly one such projective point, namely $[0 : -a_2 : a_1]$.
- (d) By (c) the point in question is $[0 : -a_2 : a_1]$, which by Proposition 22.6.1 determines and is determined by $\text{span}((-a_2, a_1))$, the direction space of L . This proves the proposition. ■

Now return to the hook. The lines $L_1 : y_2 - y_1 = 0$ and $L_2 : y_2 - y_1 = 1$ have $(a_1, a_2) = (-1, 1)$ and $b = 0$ respectively $b = 1$, so

$$\bar{L}_1 : x_2 - x_1 = 0, \quad \bar{L}_2 : x_2 - x_1 - x_0 = 0.$$

Subtracting the two equations gives $x_0 = 0$, and then $x_2 = x_1$ with $(x_1, x_2) \neq (0, 0)$. So

$$\bar{L}_1 \cap \bar{L}_2 = \{[0 : 1 : 1]\},$$

exactly one point, and it lies on the line at infinity. By Proposition 22.6.3 (c) that point is $[0 : -a_2 : a_1] = [0 : -1 : -1] = [0 : 1 : 1]$ for both lines, which is their common direction $\text{span}((1, 1))$ read as a point. Two parallel lines meet at their shared direction, and nowhere else.

Quick check. In $\mathbb{P}^2(\mathbb{R})$, which point of the line at infinity do all the **vertical** affine lines $y_1 = c$ pass through, and which point do all lines of slope m pass through?

Solution. A vertical line $y_1 = c$ has $(a_1, a_2) = (1, 0)$, so by Proposition 22.6.3 (c) its point at infinity is $[0 : -a_2 : a_1] = [0 : 0 : 1]$. A line $y_2 = my_1 + k$, that is $my_1 - y_2 = -k$, has $(a_1, a_2) = (m, -1)$ and point at infinity $[0 : 1 : m]$. So the points of H_∞ are $[0 : 1 : m]$ for each slope $m \in \mathbb{R}$, plus $[0 : 0 : 1]$ for the vertical direction: one point per direction, and the vertical direction is no longer a special case.

22.6.4 Projective subspaces, and a dimension formula with one case

Definition 22.6.3. Projective Subspace

Let V be a finite-dimensional vector space over F . A **projective subspace** of $\mathbb{P}(V)$ is a subset of the form

$$\mathbb{P}(U) = \{[x] : x \in U \setminus \{0\}\}$$

for a subspace U of V . Its **dimension** is $\dim U - 1$. A projective subspace of dimension 0 is a **point**, of dimension 1 a **line**, of dimension 2 a **plane**, and of dimension $\dim \mathbb{P}(V) - 1$ a **hyperplane**.

The dimension is only well defined if U can be recovered from $\mathbb{P}(U)$. It can, and the reason is worth isolating.

Lemma 22.6.4. A Projective Subspace Remembers Its Subspace

Let U and W be subspaces of V . Then $\mathbb{P}(U) \subseteq \mathbb{P}(W)$ if and only if $U \subseteq W$; in particular $\mathbb{P}(U) = \mathbb{P}(W)$ forces $U = W$.

Proof. ($\Leftarrow$) Immediate from the definition.

($\Rightarrow$) Suppose $\mathbb{P}(U) \subseteq \mathbb{P}(W)$ and let $x \in U$. If $x = 0$ then $x \in W$. Otherwise $[x] \in \mathbb{P}(U) \subseteq \mathbb{P}(W)$, so $[x] = [w]$ for some $w \in W \setminus \{0\}$, that is $x = \lambda w$ with $\lambda \neq 0$; hence $x \in W$, W being a subspace. So $U \subseteq W$.

For the last claim, apply the equivalence in both directions and use that $\subseteq$ is antisymmetric. This proves the lemma. ■

Proposition 22.6.5. Meet and Join

Let $S = \mathbb{P}(U)$ and $T = \mathbb{P}(W)$ be projective subspaces of $\mathbb{P}(V)$. Then:

- (a) $S \cap T = \mathbb{P}(U \cap W)$, so an intersection of projective subspaces is a projective subspace;
- (b) among all projective subspaces containing $S \cup T$ there is a smallest one, namely $\mathbb{P}(U + W)$.

We write $S \vee T = \mathbb{P}(U + W)$ and call it the **join** of S and T .

Proof.

- (a) Let $x \neq 0$. By [Lemma 22.6.4](#) applied to $\text{span}(x)$, we have $[x] \in \mathbb{P}(U)$ exactly when $x \in U$. Hence $[x] \in S \cap T$ exactly when $x \in U$ and $x \in W$, that is $x \in (U \cap W) \setminus \{0\}$, which says $[x] \in \mathbb{P}(U \cap W)$.
- (b) $U \subseteq U + W$ and $W \subseteq U + W$, so $\mathbb{P}(U + W)$ contains $S \cup T$ by [Lemma 22.6.4](#). Conversely,

if $\mathbb{P}(Y) \supseteq S \cup T$ for a subspace Y , then the same lemma gives $U \subseteq Y$ and $W \subseteq Y$, hence $U + W \subseteq Y$ because $U + W$ is the smallest subspace containing both (Theorem 1.7.1), and so $\mathbb{P}(U + W) \subseteq \mathbb{P}(Y)$. This proves the proposition. ■

Theorem 22.6.6. Projective Dimension Formula

Let S and T be projective subspaces of a finite-dimensional projective space. Then

$$\dim(S \vee T) + \dim(S \cap T) = \dim S + \dim T. \quad (22.6.1)$$

There is **no exceptional case**: the identity holds for all S and T , including when $S \cap T = \emptyset$, where $\dim(S \cap T) = -1$.

Idea. There is nothing to invent. Write S and T as $\mathbb{P}(U)$ and $\mathbb{P}(W)$, apply Chapter 1's dimension formula to U and W , and subtract 1 four times. The only thing that needs watching is the bookkeeping of those four -1 's, and the reason the formula has no second case is that two of them cancel against the other two even when $U \cap W = \{0\}$ — which is precisely the situation the affine version could not handle.

Proof. By Definition 22.6.3 write $S = \mathbb{P}(U)$ and $T = \mathbb{P}(W)$; the subspaces U and W are uniquely determined by Lemma 22.6.4. By Proposition 22.6.5, $S \vee T = \mathbb{P}(U + W)$ and $S \cap T = \mathbb{P}(U \cap W)$. Hence, using Definition 22.6.3 four times and Theorem 1.7.3 once,

$$\begin{aligned} \dim(S \vee T) + \dim(S \cap T) &= (\dim(U + W) - 1) + (\dim(U \cap W) - 1) \\ &= (\dim U + \dim W) - 2 \\ &= (\dim U - 1) + (\dim W - 1) \\ &= \dim S + \dim T. \end{aligned}$$

The middle step is Theorem 1.7.3, which holds for **all** subspaces U, W with no hypothesis beyond finite dimension. In particular when $U \cap W = \{0\}$ the second term on the first line is $0 - 1 = -1$, and the computation is unaffected. This proves the theorem. ■

Compare the two formulas side by side. For flats A, B of an affine space, Theorem 22.2.4 (c) gives

$$\dim(A \vee B) = \begin{cases} \dim A + \dim B - \dim(A \cap B), & A \cap B \neq \emptyset, \\ \dim A + \dim B - \dim(\vec{A} \cap \vec{B}) + 1, & A \cap B = \emptyset, \end{cases}$$

where $\vec{A}, \vec{B}$ are the direction spaces. Two cases, and in the second the intersection has dropped out of the formula entirely, replaced by an intersection of direction spaces and an apologetic $+1$. Eq. 22.6.1 has one case and needs no directions. That is the whole reason for building projective space, and everything the rest of this chapter does with it rests on this one line.

Corollary 22.6.7. Two Lines in a Projective Plane Meet

Let P be a projective plane, that is $P = \mathbb{P}(V)$ with $\dim V = 3$. Then any two distinct lines of P meet in exactly one point.

Proof. Let $S = \mathbb{P}(U)$ and $T = \mathbb{P}(W)$ be distinct lines, so $\dim U = \dim W = 2$ and $U \neq W$ by Lemma 22.6.4. Then $U + W$ contains U properly: otherwise $W \subseteq U$, and two subspaces of the same dimension with one inside the other are equal (Theorem 1.6.8 applied inside U), contradicting $U \neq W$. So $\dim(U + W) \geq 3$, and $\dim(U + W) \leq \dim V = 3$, whence $\dim(U + W) = 3$ and $\dim(S \vee T) = 2$.

By Theorem 22.6.6,

$$\dim(S \cap T) = \dim S + \dim T - \dim(S \vee T) = 1 + 1 - 2 = 0,$$

and a projective subspace of dimension 0 is $\mathbb{P}(Y)$ with $\dim Y = 1$, which by Proposition 22.6.1 is a single point. This proves the corollary. ■

Proposition 22.6.8. Two Points Determine a Line

Let $\mathbb{P}(V)$ be a projective space of dimension at least 1. Any two distinct points of $\mathbb{P}(V)$ lie on exactly one line.

Proof. Let $[x] \neq [y]$. Then x and y are linearly independent: a dependence relation with $x, y \neq 0$ would give $y = \lambda x$ with $\lambda \neq 0$, hence $[x] = [y]$. So $U = \text{span}(x, y)$ has dimension 2 and $\mathbb{P}(U)$ is a line containing both points.

If $\mathbb{P}(Y)$ is any line containing both, then $x, y \in Y$ by Lemma 22.6.4, so $U \subseteq Y$; and $\dim Y = 2 = \dim U$, so $U = Y$ by Theorem 1.6.8. This proves the proposition. ■

The two statements just proved are each other's mirror images: "two distinct points lie on a unique line" and "two distinct lines meet in a unique point". Section 8 will show that this mirroring is not a coincidence.

Remark. The set $\mathbb{P}^2(\mathbb{R})$ is also a famous topological object, the real projective plane, and readers who have met it will be expecting words like "compact" and "nonorientable". This chapter does not use them. Nothing here is topological; every argument has been a statement about subspaces of a vector space, valid over any field, including finite ones where topology has nothing to say.

22.6.5 Counting points

Over a finite field the whole space can be counted, and the count is a useful check on the constructions above.

Proposition 22.6.9. The Number of Points of a Finite Projective Space

Let F be a finite field with q elements and let $n \geq 0$. Then

$$|\mathbb{P}^n(F)| = 1 + q + q^2 + \cdots + q^n = \frac{q^{n+1} - 1}{q - 1}.$$

More generally a projective subspace of dimension d has $(q^{d+1} - 1)/(q - 1)$ points.

Idea. Count the nonzero vectors and divide by the size of one equivalence class. Every class has the same size, $q - 1$, because scaling a fixed nonzero vector by distinct scalars gives distinct vectors. The alternative count, chart by chart, gives the geometric sum directly, and the two agreeing is a check on [Theorem 22.6.2](#).

Proof. A vector of F^{n+1} is a list of $n + 1$ entries, each chosen from F independently, so there are q^{n+1} of them and $q^{n+1} - 1$ nonzero ones.

Fix $\mathbf{x} \neq \mathbf{0}$. Its equivalence class is $\{\lambda\mathbf{x} : \lambda \in F, \lambda \neq 0\}$, and the map $\lambda \mapsto \lambda\mathbf{x}$ is injective there: if $\lambda\mathbf{x} = \mu\mathbf{x}$ then $(\lambda - \mu)\mathbf{x} = \mathbf{0}$, and $\mathbf{x} \neq \mathbf{0}$ forces $\lambda = \mu$. So every class has exactly $q - 1$ elements.

The classes partition $F^{n+1} \setminus \{\mathbf{0}\}$ (Theorem 0.5.2), so the number of classes is $(q^{n+1} - 1)/(q - 1)$. That this equals $1 + q + \cdots + q^n$ follows from

$$\begin{aligned} (q - 1)(1 + q + \cdots + q^n) &= (q + q^2 + \cdots + q^{n+1}) - (1 + q + \cdots + q^n) \\ &= q^{n+1} - 1, \end{aligned}$$

the middle terms canceling in pairs. For a projective subspace $\mathbb{P}(U)$ of dimension d , the space U has dimension $d + 1$, hence q^{d+1} elements, since a choice of basis puts it in bijection with F^{d+1} (Corollary 2.5.5); the same argument applies. This proves the proposition. ■

The geometric sum is [Theorem 22.6.2](#) in disguise: the chart contributes q^n points and the hyperplane at infinity contributes $|\mathbb{P}^{n-1}(F)|$, so $|\mathbb{P}^n| = q^n + |\mathbb{P}^{n-1}|$, and unwinding that recursion from $|\mathbb{P}^0| = 1$ gives $1 + q + \cdots + q^n$.

Example. The Fano plane

Describe $\mathbb{P}^2(\mathbb{F}_2)$ completely: its points, its lines, how many points lie on each line, and how many lines pass through each point. Verify [Corollary 22.6.7](#) and [Proposition 22.6.8](#) by direct count.

Solution. By [Proposition 22.6.9](#) there are $1 + 2 + 4 = 7$ points. Since $\mathbb{F}_2^\times = \{1\}$, each point has a **unique** nonzero representative, so the points are simply the seven nonzero vectors of $\mathbb{F}_2^3$. Number the point $[x_0 : x_1 : x_2]$ by $4x_0 + 2x_1 + x_2$:

$$\begin{aligned} 1 &= [0:0:1], & 2 &= [0:1:0], & 3 &= [0:1:1], & 4 &= [1:0:0], \\ 5 &= [1:0:1], & 6 &= [1:1:0], & 7 &= [1:1:1]. \end{aligned}$$

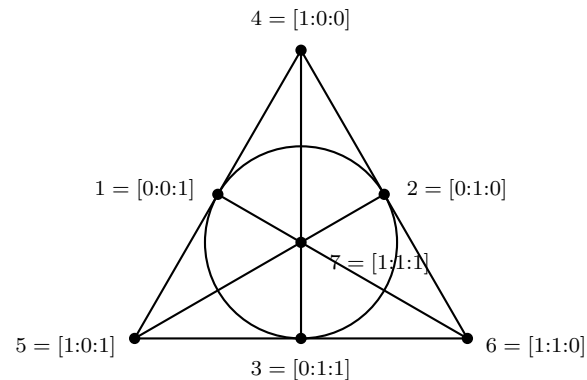
A line is $\mathbb{P}(U)$ with $\dim U = 2$, and by [Proposition 22.6.9](#) it has $(2^2 - 1)/(2 - 1) = 3$ points. Since $U = \{\mathbf{0}, \mathbf{u}, \mathbf{w}, \mathbf{u} + \mathbf{w}\}$ for any two distinct nonzero $\mathbf{u}, \mathbf{w} \in U$, the lines are exactly the triples $\{\mathbf{u}, \mathbf{w}, \mathbf{u} + \mathbf{w}\}$ of nonzero vectors. There are seven:

$$\begin{aligned} \{1, 2, 3\} : x_0 &= 0, & \{1, 4, 5\} : x_1 &= 0, \\ \{2, 4, 6\} : x_2 &= 0, & \{1, 6, 7\} : x_0 + x_1 &= 0, \\ \{2, 5, 7\} : x_0 + x_2 &= 0, & \{3, 4, 7\} : x_1 + x_2 &= 0, \\ \{3, 5, 6\} : x_0 + x_1 + x_2 &= 0. \end{aligned}$$

Each is the kernel of one of the seven nonzero linear functionals on $\mathbb{F}_2^3$, and different functionals have different kernels here because a functional is determined up to a nonzero scalar by its kernel — if $\ker \varphi = \ker \psi = U$ with $\dim U = 2$, then φ and ψ both vanish on U and so lie in the annihilator U^0 (Definition 5.2.1), which has dimension $3 - 2 = 1$ by Theorem 5.2.2 — and the only nonzero scalar here is 1. So there are exactly seven lines.

Incidences. Each of the seven labels appears in exactly three of the seven displayed triples, so every point lies on exactly 3 lines. The total is consistent: counting incident pairs (point, line through it) by lines gives $7 \cdot 3 = 21$, and counting them by points gives $7 \cdot 3 = 21$ as well.

The two theorems. There are $\binom{7}{2} = 21$ pairs of points, and each of the 7 lines contains $\binom{3}{2} = 3$ pairs, for 21 pairs in total; since no pair can be covered twice without two lines sharing two points — which [Proposition 22.6.8](#) forbids — each pair is covered exactly once. Dually there are $\binom{7}{2} = 21$ pairs of lines, and each of the 7 points lies on 3 lines and so accounts for $\binom{3}{2} = 3$ of those pairs, again 21 in total; no pair of lines is accounted for twice, since two points on both lines would again contradict [Proposition 22.6.8](#). So every pair of lines meets in exactly one point, as [Corollary 22.6.7](#) says.



the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$: 7 points, 7 lines
(three sides, three medians, one circle), 3 points on each line

In the picture the “midpoint” of two marked points really is their sum: $4 + 5 = [0:0:1] = 1$, and likewise all round. The circle is the line $x_0 = 0$, which under [Theorem 22.6.2](#) is the line at infinity, so the picture also shows $\mathbb{P}^2(\mathbb{F}_2)$ as the four affine points 4, 5, 6, 7 of $\mathbb{F}_2^2$ together with

the three directions 1, 2, 3.

Warning. **A drawing of a finite projective plane is a mnemonic, not a picture.** The seven points of the Fano plane are not points of the Euclidean plane, and the “circle” is a line, not a curve. The only content of the drawing is the incidence data: which triples of the seven labels form a line. Any picture with the same seven triples is the same object, and no metric statement about the drawing means anything.

22.6.6 Exercises

22.6.6.1 A. Check your understanding

Exercise. A1

- (a) State [Definition 22.6.1](#), and say why 0 is removed before taking the quotient.
- (b) What is $\dim \mathbb{P}(V)$ when $\dim V = 5$? What is the dimension of a hyperplane of that projective space, and of what subspace of V is it the image?
- (c) Decide whether $[2 : 4 : 6] = [1 : 2 : 3]$ in $\mathbb{P}^2(\mathbb{Q})$, and whether $[0 : 1] = [0 : 2]$ in $\mathbb{P}^1(\mathbb{F}_3)$. Justify your answers.
- (d) True or false: the condition $x_1 + x_2 = 1$ defines a subset of $\mathbb{P}^2(\mathbb{R})$. Justify your answer.
- (e) How many points does $\mathbb{P}^3(\mathbb{F}_5)$ have?

Solution.

- (a) $\mathbb{P}(V) = (V \setminus \{0\})/\sim$ with $x \sim y$ when $y = \lambda x$ for some nonzero λ , and $\dim \mathbb{P}(V) = \dim V - 1$. The vector 0 is removed because its class is $\{0\}$ alone, so it would contribute one point whose span $\text{span}(0) = \{0\}$ is the zero subspace and not a line, and [Proposition 22.6.1](#) would fail.
- (b) $\dim \mathbb{P}(V) = 4$. A hyperplane has dimension 3 and is $\mathbb{P}(U)$ for a subspace U of V with $\dim U = 4$.
- (c) Yes: $(2, 4, 6) = 2(1, 2, 3)$ and $2 \neq 0$ in $\mathbb{Q}$. Yes again: $(0, 2) = 2(0, 1)$ and $2 \neq 0$ in $\mathbb{F}_3$.
- (d) False. The expression is not homogeneous: $(1, 1, 0)$ satisfies it but $(2, 2, 0)$, which represents the same point, does not. Only the vanishing of a homogeneous expression is a condition on points.
- (e) $(5^4 - 1)/(5 - 1) = 624/4 = 156$, by [Proposition 22.6.9](#).

22.6.6.2 B. Practice

Exercise. B1: Meeting at infinity

In $\mathbb{P}^2(\mathbb{R})$, complete each pair of affine lines and find their intersection point, saying in each case whether it lies in the chart U_0 or at infinity.

- (a) $y_2 = 2y_1 + 1$ and $y_2 = 2y_1 - 3$.

- (b) $y_2 = 2y_1 + 1$ and $y_1 + y_2 = 4$.
 (c) $y_1 = 5$ and $y_1 = -2$.

Solution.

- (a) Writing them as $2y_1 - y_2 = -1$ and $2y_1 - y_2 = 3$, [Proposition 22.6.3](#) gives $\bar{L}_1 : 2x_1 - x_2 + x_0 = 0$ and $\bar{L}_2 : 2x_1 - x_2 - 3x_0 = 0$. Subtracting, $4x_0 = 0$, so $x_0 = 0$ and $x_2 = 2x_1$: the point $[0 : 1 : 2]$, at infinity. This agrees with part (c) of that proposition, since $(a_1, a_2) = (2, -1)$ gives $[0 : 1 : 2]$.
 (b) Here $\bar{L}_1 : 2x_1 - x_2 + x_0 = 0$ and $\bar{L}_2 : x_1 + x_2 - 4x_0 = 0$. Adding, $3x_1 - 3x_0 = 0$, so $x_1 = x_0$; then $x_2 = 4x_0 - x_1 = 3x_0$. Taking $x_0 = 1$ gives $[1 : 1 : 3]$, which lies in U_0 and corresponds to the affine point $(1, 3)$. Check: $3 = 2 \cdot 1 + 1$ and $1 + 3 = 4$.
 (c) $\bar{L}_1 : x_1 - 5x_0 = 0$ and $\bar{L}_2 : x_1 + 2x_0 = 0$. Subtracting, $7x_0 = 0$, so $x_0 = 0$ and then $x_1 = 0$, forcing $x_2 \neq 0$: the point $[0 : 0 : 1]$, at infinity, the vertical direction.

Exercise. B2: Joins and meets

In $\mathbb{P}^4(F)$, let S be a plane and T a plane.

- (a) What are the possible values of $\dim(S \cap T)$? Justify each.
 (b) Show that two planes in $\mathbb{P}^4(F)$ always meet.
 (c) Give an example in $\mathbb{P}^5(F)$ of two planes that do not meet.

Solution. Throughout, $S = \mathbb{P}(U)$ and $T = \mathbb{P}(W)$ with $\dim U = \dim W = 3$ inside a space of dimension 5 (for $\mathbb{P}^4$).

- (a) By [Theorem 22.6.6](#), $\dim(S \cap T) = 4 - \dim(S \vee T)$. Now $S \vee T$ contains S , so $\dim(S \vee T) \geq 2$, and it is contained in $\mathbb{P}^4$, so $\dim(S \vee T) \leq 4$. Hence $\dim(S \cap T) \in \{0, 1, 2\}$, and all three occur: 2 when $S = T$; 1 when $U \cap W$ has dimension 2, for instance $U = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $W = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4)$; and 0 for $U = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $W = \text{span}(\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$.
 (b) The computation in (a) shows $\dim(S \cap T) \geq 4 - 4 = 0 > -1$, so $S \cap T \neq \emptyset$.
 (c) In $\mathbb{P}^5(F) = \mathbb{P}(F^6)$ take $U = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $W = \text{span}(\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6)$. Then $U \cap W = \{\mathbf{0}\}$, so $S \cap T = \mathbb{P}(\{\mathbf{0}\}) = \emptyset$, and indeed [Theorem 22.6.6](#) gives $\dim(S \vee T) + (-1) = 2 + 2$, that is $\dim(S \vee T) = 5$, the whole space.

Exercise. B3: A projective plane over three elements

Let $F = \mathbb{F}_3$.

- (a) How many points does $\mathbb{P}^2(\mathbb{F}_3)$ have, and how many lines?
 (b) How many points lie on each line, and how many lines pass through each point?
 (c) List the four points of the line $x_0 = 0$.

Solution.

- (a) By [Proposition 22.6.9](#), $|\mathbb{P}^2(\mathbb{F}_3)| = 1 + 3 + 9 = 13$. A line is $\mathbb{P}(U)$ with $\dim U = 2$, equivalently the kernel of a nonzero functional on $\mathbb{F}_3^3$, and two nonzero functionals have the same kernel exactly when they are proportional (Corollary 5.6.6 applied to the 1×3 matrices, or directly: a kernel of dimension 2 determines the functional up to scale). So the lines correspond to the points of $\mathbb{P}((\mathbb{F}_3^3)^*)$, of which there are again 13.
- (b) Each line has $(3^2 - 1)/(3 - 1) = 4$ points. For the lines through a fixed point $[\mathbf{x}]$: they are the $\mathbb{P}(U)$ with $\text{span}(\mathbf{x}) \subseteq U$ and $\dim U = 2$, and every such U is $\text{span}(\mathbf{x}, \mathbf{y})$ for one of the $3^3 - 3 = 24$ vectors $\mathbf{y} \notin \text{span}(\mathbf{x})$, while two choices $\mathbf{y}, \mathbf{y}'$ give the same U exactly when $\mathbf{y}' \in U \setminus \text{span}(\mathbf{x})$, a set of $3^2 - 3 = 6$ vectors. So each point lies on $24/6 = 4$ lines. (Consistently, the incident pairs number $13 \cdot 4 = 52$ counted by lines and $13 \cdot 4 = 52$ counted by points.)
- (c) The points with $x_0 = 0$ are $[0 : x_1 : x_2]$ with $(x_1, x_2) \neq (0, 0)$, that is the points of $\mathbb{P}^1(\mathbb{F}_3)$: $[0 : 1 : 0]$, $[0 : 1 : 1]$, $[0 : 1 : 2]$ and $[0 : 0 : 1]$. Four points, as (b) requires.

22.6.6.3 C. Going deeper

Exercise. C1: Other charts

Work in $\mathbb{P}^n(F)$ and fix $j \in \{0, 1, \dots, n\}$. Let $U_j = \{[\mathbf{x}] : x_j \neq 0\}$.

- (a) Prove that U_j is well defined and in bijection with F^n .
- (b) Prove that $U_0 \cup U_1 \cup \dots \cup U_n = \mathbb{P}^n(F)$.
- (c) For $n = 1$ and $F = \mathbb{R}$, describe the point of U_1 that is missing from U_0 , and the point of U_0 missing from U_1 . Conclude that the “point at infinity” is not a property of a point, but of a chart.

Solution.

- (a) Both parts repeat [Theorem 22.6.2](#) with the index 0 replaced by j . Scaling multiplies x_j by a nonzero scalar, so the condition $x_j \neq 0$ is independent of the representative. The bijection is

$$(a_1, \dots, a_n) \mapsto [a_1 : \dots : a_j : 1 : a_{j+1} : \dots : a_n],$$

with the 1 in position j ; it is injective because comparing the j -th entries of two proportional representatives forces the scalar to be 1, and surjective because $[\mathbf{x}] = [x_j^{-1}\mathbf{x}]$ when $x_j \neq 0$.

- (b) A point is $[\mathbf{x}]$ with $\mathbf{x} \neq \mathbf{0}$, so some coordinate x_j is nonzero, and then $[\mathbf{x}] \in U_j$.
- (c) $\mathbb{P}^1(\mathbb{R}) = \{[1 : a] : a \in \mathbb{R}\} \cup \{[0 : 1]\}$. The chart U_0 omits exactly $[0 : 1]$, and the chart $U_1 = \{[b : 1] : b \in \mathbb{R}\} \cup \{[1 : 0]\}$ omits exactly $[1 : 0]$. Both omitted points are perfectly ordinary points of $\mathbb{P}^1(\mathbb{R})$; each is at infinity for one chart and in the affine part of the other. So “at infinity” describes a relation between a point and a chosen chart, not the point itself.

Exercise. C2: Counting subspaces

Let F be a finite field with q elements.

- Prove that the number of lines in $\mathbb{P}^2(F)$ equals the number of points, namely $q^2 + q + 1$.
Hint: a line is the kernel of a nonzero functional.
- Deduce that every point of $\mathbb{P}^2(F)$ lies on exactly $q + 1$ lines.
- How many points does a hyperplane of $\mathbb{P}^n(F)$ have, and how many points of $\mathbb{P}^n(F)$ lie outside it?

Solution.

- A line of $\mathbb{P}^2(F)$ is $\mathbb{P}(U)$ with $\dim U = 2$, and by Lemma 5.6.1 such a U is the kernel of a nonzero functional φ on F^3 , since $\dim U^\perp = 3 - 2 = 1$ (Theorem 5.2.2) and U is the common kernel of its annihilator. Two nonzero functionals have the same kernel exactly when they are proportional: if $\ker \varphi = \ker \psi = U$, then φ and ψ both lie in the 1-dimensional space $U^\perp$, so one is a scalar multiple of the other, and the scalar is nonzero. So lines correspond to points of $\mathbb{P}((F^3)^*)$, and $(F^3)^*$ has dimension 3 (Corollary 5.1.4); by Proposition 22.6.9 there are $q^2 + q + 1$ of them.
- Count incident pairs (point, line through it) in two ways. Each line has $q + 1$ points by Proposition 22.6.9, so the count is $(q^2 + q + 1)(q + 1)$. By symmetry among the $q^2 + q + 1$ points — or simply dividing — each point lies on $(q^2 + q + 1)(q + 1)/(q^2 + q + 1) = q + 1$ lines, provided every point lies on the same number. That last point needs an argument. The lines through a fixed $[\mathbf{x}]$ are the $\mathbb{P}(U)$ with $\text{span}(\mathbf{x}) \subseteq U$ and $\dim U = 2$, and every such U is $\text{span}(\mathbf{x}, \mathbf{y})$ for some $\mathbf{y} \notin \text{span}(\mathbf{x})$, of which there are $q^3 - q$. Two choices $\mathbf{y}, \mathbf{y}'$ give the same U exactly when $\mathbf{y}' \in U \setminus \text{span}(\mathbf{x})$, a set of $q^2 - q$ vectors. So the number of such U is $(q^3 - q)/(q^2 - q) = q + 1$, independently of $[\mathbf{x}]$.
- A hyperplane is $\mathbb{P}(U)$ with $\dim U = n$, so it has $(q^n - 1)/(q - 1) = 1 + q + \cdots + q^{n-1}$ points. Subtracting from $1 + q + \cdots + q^n$ leaves q^n points outside, in agreement with Theorem 22.6.2, whose chart U_0 is exactly the complement of a hyperplane and is in bijection with F^n .

Exercise. C3: Where the affine formula goes wrong

Let A and B be flats of $\mathbb{A}^3(F)$.

- Give two lines in $\mathbb{A}^3(F)$ that do not meet and are not parallel, and compute $\dim(A \vee B)$ from Theorem 22.2.4.
- Complete both lines to projective lines in $\mathbb{P}^3(F)$ by the recipe of Theorem 22.6.2, that is by adding the points with $x_0 = 0$ to the projective subspace each spans, and verify Eq. 22.6.1 for the completed pair.
- Explain in one sentence why the projective formula could not have an exceptional case, given Proposition 22.6.5.

Solution.

- Take $A = \{(t, 0, 0) : t \in F\}$ and $B = \{(0, s, 1) : s \in F\}$. A common point would need third coordinate 0 and 1, so $A \cap B = \emptyset$. Their direction spaces are $\text{span}(\mathbf{e}_1)$ and

$\text{span}(\mathbf{e}_2)$, which are different, so they are not parallel and $\vec{A} \cap \vec{B} = \{\mathbf{0}\}$. The second case of Theorem 22.2.4 (c) gives $\dim(A \vee B) = 1 + 1 - 0 + 1 = 3$, the whole of $\mathbb{A}^3(F)$.

- (b) Under ι of Theorem 22.6.2, A becomes $\{[1 : t : 0 : 0]\}$, whose points together with $[0 : 1 : 0 : 0]$ form $\mathbb{P}(U)$ with $U = \text{span}((1, 0, 0, 0), (0, 1, 0, 0))$; and B becomes $\{[1 : 0 : s : 1]\}$, giving $\mathbb{P}(W)$ with $W = \text{span}((1, 0, 0, 1), (0, 0, 1, 0))$. Now $U \cap W = \{\mathbf{0}\}$: a vector of U has third and fourth entries 0, while a vector $a(1, 0, 0, 1) + b(0, 0, 1, 0)$ of W has third entry b and fourth entry a , so both a and b vanish. Hence $S \cap T = \emptyset$ of dimension -1 , and $U + W$ has dimension $2 + 2 - 0 = 4$, so $\dim(S \vee T) = 3$. Then $3 + (-1) = 1 + 1$, as Eq. 22.6.1 requires. The two projective lines still do not meet, which is correct: Corollary 22.6.7 is about a projective **plane**, and these lines do not lie in one.
- (c) Because by Proposition 22.6.5 both the meet and the join of projective subspaces are again projective subspaces, coming from $U \cap W$ and $U + W$, and the vector-space formula Theorem 1.7.3 holds for those with no hypothesis whatever, the empty intersection being simply the case $U \cap W = \{\mathbf{0}\}$.