

22.2 Flats, Independence, and Coordinates

An affine space has no origin, so it has no subspaces in the sense of Chapter 1: a subspace must contain 0, and there is no 0. What it has instead are its **flats** — the lines, planes, and hyperplanes that sit inside it. This section defines them, settles when two of them meet and how big the smallest flat containing both is, puts coordinates on an affine space using $n + 1$ points rather than n vectors, and shows that in $\mathbb{A}^n(F)$ the flats are exactly the solution sets of linear systems. The join formula has two cases, and the second one is where affine geometry parts company with linear algebra.

Throughout, $\mathcal{A}$ is an affine space over a field F with direction space V (Definition 22.1.1), and from the dimension formula onward $\dim \mathcal{A} = \dim V = n$ is finite, except where a statement says otherwise.

22.2.1 Flats

A subset of $\mathcal{A}$ deserves to be called flat when it is an affine space in its own right, using the same difference map. By Definition 22.1.2 there is only one thing to require.

A flat is a nonempty set of points closed under taking affine combinations.

Definition 22.2.1. Flat

A subset $A \subseteq \mathcal{A}$ is a **flat**, or an **affine subspace** of $\mathcal{A}$, if it is **nonempty** and contains every affine combination of its own points: **whenever** $k \geq 1$, $P_1, \dots, P_k \in A$ and $t_1, \dots, t_k \in F$ satisfy $t_1 + \dots + t_k = 1$, the point $\sum_{i=1}^k t_i P_i$ lies in A .

The requirement that A be nonempty is a convention, and it is the useful one: a flat is going to have a direction space and a dimension, and the empty set has neither. Intersections of flats may of course be empty, and we will say so each time.

Proposition 22.2.1. Flats Are the Cosets

Let $A \subseteq \mathcal{A}$ be nonempty. Then A is a flat if and only if there are $P \in \mathcal{A}$ and a subspace U of V with $A = P + U := \{P + u : u \in U\}$. In that case:

- (a) $U = \{\overrightarrow{XY} : X, Y \in A\}$ is determined by A alone; it is written $\vec{A}$ and called the **direction space** of A .
- (b) $A = Q + \vec{A}$ for **every** $Q \in A$, and A , with the difference map of $\mathcal{A}$ restricted to it, is an affine space with direction space $\vec{A}$.

The **dimension** of A is $\dim A := \dim \vec{A}$. A flat of dimension 0 is a single **point**, one of dimension 1 is a **line**, one of dimension 2 is a **plane**, and, when $\dim \mathcal{A} = n$ is finite, one of dimension $n - 1$ is a **hyperplane** of $\mathcal{A}$.

Idea. One direction is a computation with origin P . For the other, fix $P \in A$ and let U be the set of displacements from P to points of A ; the three subspace axioms come from three affine combinations, with weight lists (1) , $(\lambda, 1 - \lambda)$ and $(1, 1, -1)$.

Proof. ($\Leftarrow$) Let $A = P + U$ with U a subspace, and let $P + u_1, \dots, P + u_k \in A$ with $\sum_i t_i = 1$. Computing the affine combination with origin P (Definition 22.1.2),

$$\sum_i t_i(P + u_i) = P + \sum_i t_i u_i \in P + U,$$

since U is closed under linear combinations. And $A \ni P$ is nonempty.

($\Rightarrow$) Let A be a flat, fix $P \in A$ and put $U = \{\overrightarrow{PX} : X \in A\}$. Then $0 = \overrightarrow{PP} \in U$. If $u = \overrightarrow{PX} \in U$ and $\lambda \in F$, the point $\lambda X + (1 - \lambda)P$ lies in A and equals $P + \lambda u$ when computed with origin P , so $\lambda u \in U$. If also $w = \overrightarrow{PY} \in U$, the point $X + Y - P$ lies in A (weights $1 + 1 - 1 = 1$) and equals $P + (u + w)$, so $u + w \in U$. By Theorem 1.2.1, U is a subspace, and $A = P + U$ by construction.

- (a) Let $X, Y \in A$. Then $\overrightarrow{XY} = \overrightarrow{XP} + \overrightarrow{PY} \in U$ by Chasles, since U is closed under negation and addition; so $\{\overrightarrow{XY} : X, Y \in A\} \subseteq U$. Conversely each $u \in U$ is $\overrightarrow{PX}$ for some $X \in A$, and $P \in A$. Hence the two sets are equal, and the right-hand side mentions only A .
- (b) Let $Q \in A$. For $X \in A$ we have $\overrightarrow{QX} \in \vec{A}$ by (a), so $A \subseteq Q + \vec{A}$; and for $u \in \vec{A}$, writing $u = \overrightarrow{XY}$ with $X, Y \in A$, the point $Q + u$ is the affine combination $Q + Y - X$ of points of A , hence lies in A . So $A = Q + \vec{A}$. Finally the restricted difference map takes values in $\vec{A}$ by (a), Chasles is inherited, and for fixed $P \in A$ the map $X \mapsto \overrightarrow{PX}$ is a bijection $A \rightarrow \vec{A}$, being the restriction of the bijection of (AS2) to $A = P + \vec{A}$. So (AS1) and (AS2) hold and A is an affine space with direction space $\vec{A}$.

■

Over $\mathbb{R}$, with $\mathcal{A} = V$, a flat is exactly Chapter 18's affine subspace of V (Definition 18.1.5 (b)), and the two dimensions agree: Chapter 18 also reads the subspace off the coset as the set of differences. Chapter 5's **affine hyperplane** $\mathbf{p} + H$ (the paragraph before Proposition 5.6.4) is the case $\dim A = n - 1$ in $\mathbb{A}^n(F)$. A one-point set $\{P\} = P + \{0\}$ is a flat, and $\mathcal{A} = P + V$ is a flat: the two degenerate cases.

A nonexample, by minimal change. A line is a flat; the union of two lines is not. Let $X = \{(x, y) \in \mathbb{A}^2 : xy = 0\}$, the union of the two coordinate axes. It is nonempty, and it contains every affine combination of points taken from one axis, each axis being a flat. But Definition 22.2.1 asks for all of them: the points $P = (1, 0)$ and $Q = (0, 1)$ lie in X , the weights of $\frac{1}{2}P + \frac{1}{2}Q$ add up to 1, and $\frac{1}{2}P + \frac{1}{2}Q = (\frac{1}{2}, \frac{1}{2})$ has $xy = \frac{1}{4} \neq 0$. So the closure clause fails for one pair of points, and X is not a coset of a subspace, although it is a union of two of them.

Proposition 22.2.2. Intersections of Flats

Let $(A_i)_{i \in I}$ be a nonempty family of flats of $\mathcal{A}$ and put $A = \bigcap_{i \in I} A_i$. If $A \neq \emptyset$, then A is a flat and $\vec{A} = \bigcap_{i \in I} \vec{A}_i$.

Proof. Suppose $A \neq \emptyset$ and fix $P \in A$. If $P_1, \dots, P_k \in A$ and $\sum_j t_j = 1$, then the point $\sum_j t_j P_j$ lies in each A_i , because each A_i is a flat containing all the P_j ; so it lies in A . Hence A is a flat. By Proposition 22.2.1 (b), $A_i = P + \vec{A}_i$ for each i and $A = P + \vec{A}$, so

$$\vec{A} = \{\overrightarrow{PX} : X \in A\} = \bigcap_{i \in I} \{\overrightarrow{PX} : X \in A_i\} = \bigcap_{i \in I} \vec{A}_i,$$

where the middle equality holds because $X \mapsto \overrightarrow{PX}$ is injective (AS2). ■

Since $\mathcal{A}$ itself is a flat containing every subset, the following is well defined.

Definition 22.2.2. Join of Flats

Let A, B be flats of $\mathcal{A}$. Their **join** $A \vee B$ is the intersection of all flats of $\mathcal{A}$ containing $A \cup B$.

The intersection contains A , so it is nonempty, and it is a flat by [Proposition 22.2.2](#): $A \vee B$ is the **smallest** flat containing both. A single point P is a flat, and we write $P \vee Q$, $A \vee P$ for joins in which a one-point flat $\{P\}$ appears. This is the affine analogue of $U + W$, and Chapter 18's affine hull is the same construction over $\mathbb{R}$ ([Proposition 18.1.4](#)).

Parallelism is what an affine space has and a vector space does not notice.

Definition 22.2.3. Parallel Flats

Two flats A, B of $\mathcal{A}$ are **parallel**, written $A \parallel B$, if $\vec{A} = \vec{B}$.

Proposition 22.2.3. Parallelism

Parallelism is an equivalence relation on the set of flats of $\mathcal{A}$. Two parallel flats are either equal or disjoint, and each equivalence class consists of the flats $P + U$, $P \in \mathcal{A}$, for one fixed subspace U .

Proof. Equality of direction spaces is an equivalence relation, so $\parallel$ is one. Let $A \parallel B$ with common direction U , and suppose $X \in A \cap B$. Then $A = X + U = B$ by [Proposition 22.2.1](#) (b). So A and B are equal or disjoint. Finally, the flats with direction space U are exactly the sets $P + U$ with $P \in \mathcal{A}$, again by [Proposition 22.2.1](#). ■

Parallel flats have the same dimension. Some authors also call a line parallel to a plane containing its direction; we do not, and we say “ $\vec{A} \subseteq \vec{B}$ ” when we mean that.

22.2.2 When two flats meet, and how big their join is

Here is the central computation of the section. Two subspaces of a vector space always meet, in 0 at least. Two flats need not, and the dimension formula splits accordingly.

Theorem 22.2.4. Intersection and Join of Two Flats

Let A, B be flats of $\mathcal{A}$, and choose $P \in A$ and $Q \in B$. Parts (a) and (b) need no hypothesis on $\dim \mathcal{A}$.

- (a) $A \vee B = P + (\vec{A} + \vec{B} + \text{span}(\overrightarrow{PQ}))$.
- (b) $A \cap B \neq \emptyset$ if and only if $\overrightarrow{PQ} \in \vec{A} + \vec{B}$; and then $A \cap B$ is a flat whose direction space is $\vec{A} \cap \vec{B}$.
- (c) Let $\dim \mathcal{A} = n$ be finite. If $A \cap B \neq \emptyset$, then

$$\dim(A \vee B) = \dim A + \dim B - \dim(A \cap B).$$

If $A \cap B = \emptyset$, then

$$\dim(A \vee B) = \dim A + \dim B - \dim(\vec{A} \cap \vec{B}) + 1.$$

- (d) Let $\dim \mathcal{A} = n$ be finite. If $A \cap B = \emptyset$, there is a hyperplane H of V with $A \subseteq P + H$ and $B \subseteq Q + H$, and the two flats $P + H$, $Q + H$ are parallel, distinct, and disjoint.

Idea. Everything is read off from one subspace, $W = \vec{A} + \vec{B} + \text{span}(\overrightarrow{PQ})$. The three summands are forced: the join must contain the directions of A and of B , and it must contain a way of getting from A to B . Whether the third summand is already inside the first two is exactly whether A and B meet — and that one extra dimension, present precisely when they miss each other, is the whole difference between (c)'s two cases.

Proof. Put $W = \vec{A} + \vec{B} + \text{span}(\overrightarrow{PQ})$, a subspace of V , and $C = P + W$, a flat by [Proposition 22.2.1](#).

- (a) First, $A \cup B \subseteq C$. Indeed $A = P + \vec{A} \subseteq P + W$. And each point of B is $Q + b$ with $b \in \vec{B}$, which equals $P + (\overrightarrow{PQ} + b)$ by Lemma 22.1.1 (b), and $\overrightarrow{PQ} + b \in W$. So C is one of the flats being intersected in [Definition 22.2.2](#), whence $A \vee B \subseteq C$.
Conversely, $A \vee B$ is a flat containing P , so $A \vee B = P + U$ by [Proposition 22.2.1](#) (b), where U is the direction space of $A \vee B$. Since $A \subseteq A \vee B$ and $P \in A$, part (a) of that proposition gives $\vec{A} \subseteq U$. Since $Q \in A \vee B$, also $\overrightarrow{PQ} \in U$. And for $b \in \vec{B}$ the point $Q + b$ lies in $B \subseteq A \vee B$, so $\overrightarrow{PQ} + b = \overrightarrow{P(Q+b)} \in U$, and therefore $b \in U$. Hence $W \subseteq U$, that is, $C \subseteq A \vee B$. The two inclusions give (a).
- (b) ($\Leftarrow$) Suppose $\overrightarrow{PQ} = a + b$ with $a \in \vec{A}$, $b \in \vec{B}$. Put $X = P + a$, a point of A . Then

$$\overrightarrow{QX} = \overrightarrow{QP} + \overrightarrow{PX} = -(a + b) + a = -b \in \vec{B},$$

so $X \in Q + \vec{B} = B$. Hence $X \in A \cap B$.

($\Rightarrow$) If $X \in A \cap B$, then $\overrightarrow{PX} \in \vec{A}$ and $\overrightarrow{XQ} \in \vec{B}$ by [Proposition 22.2.1](#) (a), so $\overrightarrow{PQ} = \overrightarrow{PX} + \overrightarrow{XQ} \in \vec{A} + \vec{B}$. The last clause is [Proposition 22.2.2](#) applied to the two flats A and B .

- (c) *Case 1:* $A \cap B \neq \emptyset$. By (b), $\overrightarrow{PQ} \in \vec{A} + \vec{B}$, so $W = \vec{A} + \vec{B}$. By (a) and Theorem 1.7.3,

$$\begin{aligned} \dim(A \vee B) &= \dim(\vec{A} + \vec{B}) \\ &= \dim \vec{A} + \dim \vec{B} - \dim(\vec{A} \cap \vec{B}), \end{aligned}$$

and $\dim(\vec{A} \cap \vec{B}) = \dim(A \cap B)$ by (b), which is the first formula.

Case 2: $A \cap B = \emptyset$. By (b), $\overrightarrow{PQ} \notin \vec{A} + \vec{B}$; in particular $\overrightarrow{PQ} \neq 0$. Then

$$\text{span}(\overrightarrow{PQ}) \cap (\vec{A} + \vec{B}) = \{0\},$$

since a nonzero $\lambda \overrightarrow{PQ}$ lying in $\vec{A} + \vec{B}$ would give $\overrightarrow{PQ} = \lambda^{-1}(\lambda \overrightarrow{PQ}) \in \vec{A} + \vec{B}$, λ being nonzero and hence invertible. So Theorem 1.7.3 applied to the two subspaces $\vec{A} + \vec{B}$ and $\text{span}(\overrightarrow{PQ})$ gives $\dim W = \dim(\vec{A} + \vec{B}) + 1$, and one more application, as in Case 1, gives the second formula.

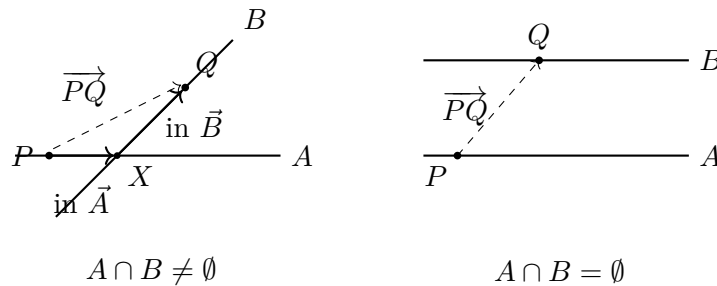
- (d) Put $U = \vec{A} + \vec{B}$ and $r = \dim U$. By (b), $\overrightarrow{PQ} \notin U$, so appending $\overrightarrow{PQ}$ to a basis $(u_1, \dots, u_r)$ of U leaves an independent list (Lemma 1.5.3). Extend it to a basis

$$(u_1, \dots, u_r, \overrightarrow{PQ}, w_1, \dots, w_{n-r-1})$$

of V by Theorem 1.6.5, and let $H = \text{span}(u_1, \dots, u_r, w_1, \dots, w_{n-r-1})$. Then $\dim H = n - 1$, so $\dim(V/H) = 1$ by Theorem 4.4.5 and H is a hyperplane (Definition 5.5.2). Now $\vec{A} \subseteq U \subseteq H$ gives $A = P + \vec{A} \subseteq P + H$, and likewise $B \subseteq Q + H$. The flats $P + H$ and $Q + H$ are parallel by Definition 22.2.3. They are distinct: if they were equal, then $Q \in P + H$ would give $\overrightarrow{PQ} \in H$, contradicting the independence of the displayed basis. Being parallel and distinct, they are disjoint by Proposition 22.2.3. This proves the theorem. ■

The two formulas in (c) are best read side by side. When the flats meet, the count is the subspace count of Theorem 1.7.3, with $A \cap B$ in the role of $\vec{A} \cap \vec{B}$. When they miss, the correction term is taken among the **directions**, where the flats always do meet, and a +1 is paid for the displacement that carries one flat to the other.

Part (b) is what the two pictures below show. On the left the displacement $\overrightarrow{PQ}$ splits as a step inside $\vec{A}$ followed by a step inside $\vec{B}$, and the flats meet; on the right it does not split, so it contributes a direction of its own to the join.



Example. Two lines in space, twice

In $\mathbb{A}^3$ let

$$A = (1, 0, 0) + \text{span}((1, 1, 0)),$$

$$B = (0, 0, 1) + \text{span}((1, -1, 0)),$$

$$B' = (0, 0, 1) + \text{span}((1, 1, 0)).$$

Decide in each case whether the lines meet, and compute $\dim(A \vee B)$ and $\dim(A \vee B')$.

Solution. Here $P = (1, 0, 0)$, $Q = (0, 0, 1)$ and $\overrightarrow{PQ} = (-1, 0, 1)$.

A and B. The directions $(1, 1, 0)$ and $(1, -1, 0)$ are independent, and $\vec{A} + \vec{B}$ is the plane of vectors with third coordinate 0. Since $\overrightarrow{PQ} = (-1, 0, 1)$ has third coordinate 1, it is not in $\vec{A} + \vec{B}$, so $A \cap B = \emptyset$ by [Theorem 22.2.4](#) (b): the lines are **skew**, that is, disjoint and not parallel. (Directly: $(1 + s, s, 0) = (t, -t, 1)$ forces $0 = 1$.) Also $\vec{A} \cap \vec{B} = \{\mathbf{0}\}$, the two directions being independent. The second formula of (c) gives

$$\dim(A \vee B) = 1 + 1 - 0 + 1 = 3,$$

so $A \vee B = \mathbb{A}^3$: two skew lines span the whole space.

A and B'. Now $\vec{A} = \vec{B}'$, so the lines are parallel, and they are distinct because $\overrightarrow{PQ} = (-1, 0, 1) \notin \vec{A}$; hence they are disjoint ([Proposition 22.2.3](#)). Here $\vec{A} \cap \vec{B}' = \vec{A}$ has dimension 1, so

$$\dim(A \vee B') = 1 + 1 - 1 + 1 = 2,$$

and $A \vee B'$ is the plane $(1, 0, 0) + \text{span}((1, 1, 0), (-1, 0, 1))$, by part (a).

Warning. There is no convention for $\dim \emptyset$ that makes one formula do both cases. It is tempting to set $\dim \emptyset = -1$ and keep writing $\dim(A \vee B) = \dim A + \dim B - \dim(A \cap B)$. For the skew lines of [Example \(Two lines in space, twice\)](#) that gives $1 + 1 - (-1) = 3$, which is right; for the parallel lines A and B' it gives 3 as well, and the correct answer is 2. The two disjoint cases differ in $\dim(\vec{A} \cap \vec{B})$, which the empty intersection cannot record. Use the second formula of [Theorem 22.2.4](#) (c), and always check first whether the flats meet.

Quick check. In $\mathbb{A}^4$, let A and B be planes (flats of dimension 2) with $\vec{A} \cap \vec{B} = \{\mathbf{0}\}$. Must they meet?

Solution. Yes. By [Theorem 1.7.3](#), $\dim(\vec{A} + \vec{B}) = 2 + 2 - 0 = 4 = \dim V$, so $\vec{A} + \vec{B} = V$ by [Theorem 1.6.8](#). Hence $\overrightarrow{PQ} \in \vec{A} + \vec{B}$ for any $P \in A$, $Q \in B$, and [Theorem 22.2.4](#) (b) gives $A \cap B \neq \emptyset$. (The dimension formula agrees: the disjoint case would predict $2 + 2 - 0 + 1 = 5 > 4$, which is impossible.) Their intersection is then a single point, since its direction space is $\vec{A} \cap \vec{B} = \{\mathbf{0}\}$.

22.2.3 Affine frames and barycentric coordinates

To put coordinates on an n -dimensional vector space we choose n vectors. To put coordinates on an n -dimensional affine space we must also say where the origin is, so we choose $n + 1$ points — and if we refuse to single out one of them, the coordinates come out symmetric in all $n + 1$. That symmetry is what makes the centroid computation below one line.

Chapter 18 §02 defined a list $(x_0, \dots, x_m)$ in a real vector space to be **affinely independent** when the only scalars with $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i x_i = \mathbf{0}$ are all zero ([Definition 18.2.1](#)). By [Definition 22.1.2](#) (b) the expression $\sum_i \lambda_i P_i$ is a well-defined vector for points of an affine space

whenever $\sum_i \lambda_i = 0$, so the same words apply here verbatim. The next lemma is Lemma 18.2.1 (a) in that setting; its proof uses nothing about $\mathbb{R}$, so we give it again over an arbitrary field.

Lemma 22.2.5. Affine Independence by Differences

Let $P_0, P_1, \dots, P_m \in \mathcal{A}$. The following are equivalent.

- (a) The only $\lambda_0, \dots, \lambda_m \in F$ with $\sum_{i=0}^m \lambda_i = 0$ and $\sum_{i=0}^m \lambda_i P_i = 0$ are $\lambda_0 = \dots = \lambda_m = 0$.
- (b) The list $(\overrightarrow{P_0 P_1}, \dots, \overrightarrow{P_0 P_m})$ is linearly independent in V .

A list satisfying these is called **affinely independent**. Condition (a) does not mention P_0 specially, so (b) holds with any P_j in place of P_0 as soon as it holds for one.

Proof. By Definition 22.1.2 (b) computed with origin P_0 , and since $\overrightarrow{P_0 P_0} = 0$,

$$\sum_{i=0}^m \lambda_i P_i = \sum_{i=0}^m \lambda_i \overrightarrow{P_0 P_i} = \sum_{i=1}^m \lambda_i \overrightarrow{P_0 P_i}.$$

Given $\lambda_1, \dots, \lambda_m$, the condition $\sum_{i=0}^m \lambda_i = 0$ holds for exactly one λ_0 , namely $\lambda_0 = -\sum_{i \geq 1} \lambda_i$; and $(\lambda_0, \dots, \lambda_m)$ is the zero list exactly when $(\lambda_1, \dots, \lambda_m)$ is. So the relations forbidden by (a) correspond, one for one, to the nontrivial linear relations among $\overrightarrow{P_0 P_1}, \dots, \overrightarrow{P_0 P_m}$, and (a) holds if and only if there are none, which is (b). ■

Over $\mathbb{R}$ with $\mathcal{A} = V$ this is Lemma 18.2.1 (a), and Chapter 18's warning applies unchanged: affine independence is **not** linear independence. The list $(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2)$ in $\mathbb{R}^2$ is linearly dependent and affinely independent, which is why the count below is $n + 1$ and not n .

Definition 22.2.4. Affine Frame

Let $\dim \mathcal{A} = n$. An **affine frame** of $\mathcal{A}$ is a list $(P_0, P_1, \dots, P_n)$ of $n + 1$ affinely independent points of $\mathcal{A}$.

By Lemma 22.2.5 the list $(\overrightarrow{P_0 P_1}, \dots, \overrightarrow{P_0 P_n})$ is then independent with $n = \dim V$ entries, hence a basis of V (Theorem 1.6.4); conversely, if it is a basis, the frame condition holds. So an affine frame is “a point together with a basis”, written symmetrically. In $\mathbb{A}^n(F)$ the **standard frame** is $(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n)$.

Theorem 22.2.6. Barycentric Coordinates

Let $\dim \mathcal{A} = n$ and let $(P_0, \dots, P_n)$ be an affine frame of $\mathcal{A}$. For every $X \in \mathcal{A}$ there are **unique** scalars $t_0, \dots, t_n \in F$ with

$$t_0 + t_1 + \dots + t_n = 1 \quad \text{and} \quad X = \sum_{i=0}^n t_i P_i.$$

They are called the **barycentric coordinates** of X with respect to the frame.

Idea. Existence is the basis expansion of $\overrightarrow{P_0X}$, with t_0 invented to make the weights add up to 1. Uniqueness is affine independence: the difference of two weight lists is an affine dependence.

Proof. Existence. The list $(\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n})$ is a basis of V , so there are unique $t_1, \dots, t_n \in F$ with $\overrightarrow{P_0X} = \sum_{i=1}^n t_i \overrightarrow{P_0P_i}$ (Theorem 1.5.1). Put $t_0 = 1 - (t_1 + \dots + t_n)$, so that $\sum_{i=0}^n t_i = 1$. Computing the affine combination with origin P_0 ,

$$\sum_{i=0}^n t_i P_i = P_0 + \sum_{i=1}^n t_i \overrightarrow{P_0P_i} = P_0 + \overrightarrow{P_0X} = X.$$

Uniqueness. Suppose $\sum_i t_i P_i = X = \sum_i t'_i P_i$ with both weight lists summing to 1, and put $\lambda_i = t_i - t'_i$. Then $\sum_i \lambda_i = 0$, so $\sum_i \lambda_i P_i$ is the vector $\sum_i \lambda_i \overrightarrow{P_0P_i}$ of Definition 22.1.2 (b), and computing both combinations with origin P_0 gives

$$\sum_i \lambda_i \overrightarrow{P_0P_i} = \overrightarrow{P_0X} - \overrightarrow{P_0X} = 0.$$

By Lemma 22.2.5 (a), every $\lambda_i = 0$. This proves the theorem. ■

Barycentric coordinates come as a **list**, not as separate numbers: they are pinned down only by the requirement that they add up to 1, so no one of them means anything on its own. Their virtue is that the frame's points are treated alike: P_j has coordinates $(0, \dots, 1, \dots, 0)$ with the 1 in slot j , for every j .

Example. Reading a triangle

In $\mathbb{A}^2$ take the frame $P_0 = (0, 0)$, $P_1 = (4, 0)$, $P_2 = (0, 3)$.

- Check that it is a frame, and find the barycentric coordinates (Theorem 22.2.6) of $X = (1, 1)$.
- Find the point with barycentric coordinates $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.
- Describe the line $P_1 \vee P_2$ in barycentric coordinates.

Solution.

- $\overrightarrow{P_0P_1} = (4, 0)$ and $\overrightarrow{P_0P_2} = (0, 3)$ are independent, so by Lemma 22.2.5 the list is affinely independent, and three points in a plane make a frame. For $X = (1, 1)$: from $(1, 1) = t_1(4, 0) + t_2(0, 3)$ we get $t_1 = \frac{1}{4}$ and $t_2 = \frac{1}{3}$, so $t_0 = 1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}$. The coordinates are $(\frac{5}{12}, \frac{1}{4}, \frac{1}{3})$. **Check:** $\frac{5}{12}(0, 0) + \frac{1}{4}(4, 0) + \frac{1}{3}(0, 3) = (1, 1)$, and $\frac{5}{12} + \frac{3}{12} + \frac{4}{12} = 1$.
- $\frac{1}{2}(0, 0) + \frac{1}{4}(4, 0) + \frac{1}{4}(0, 3) = (1, \frac{3}{4})$.
- The join of the two distinct points P_1, P_2 is $P_1 + \text{span}(\overrightarrow{P_1P_2})$ by Theorem 22.2.4 (a), a line. Its points are the affine combinations $(1-t)P_1 + tP_2$, whose barycentric coordinates are $(0, 1-t, t)$; conversely a point with $t_0 = 0$ is such a combination. So $P_1 \vee P_2 =$

$\{t_0 = 0\}$. The point X of (a) has $t_0 = \frac{5}{12} \neq 0$ and so is not on it.

The next result is the advertisement for the whole apparatus: a classical theorem of plane geometry, proved in one line of arithmetic with weights.

Proposition 22.2.7. The Medians Meet at the Centroid

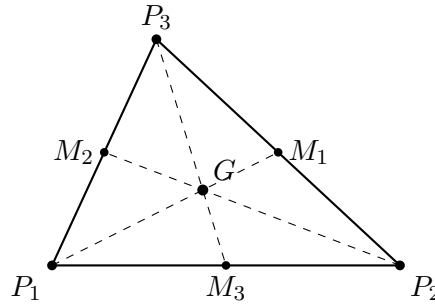
Let F be a field in which $2 \neq 0$ and $3 \neq 0$, let $\mathcal{A}$ be an affine space over F , and let (P_1, P_2, P_3) be an affinely independent list in $\mathcal{A}$. Put

$$M_1 = \frac{1}{2}P_2 + \frac{1}{2}P_3, \quad M_2 = \frac{1}{2}P_1 + \frac{1}{2}P_3, \quad M_3 = \frac{1}{2}P_1 + \frac{1}{2}P_2,$$

the **midpoints** of the sides, and let $m_i = P_i \vee M_i$ be the **medians**. Then m_1, m_2, m_3 are lines meeting in exactly one point, the **centroid**

$$G = \frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3,$$

and $\overrightarrow{P_i G} = 2\overrightarrow{GM_i}$ for each i .



Idea. Work in the plane $\pi = (P_1 \vee P_2) \vee P_3$, where (P_1, P_2, P_3) is an affine frame, and give every point its barycentric coordinates. A median is then the set of coordinate triples of one shape, and the three shapes have exactly one triple in common.

Proof. The three points are affinely independent, so $(\overrightarrow{P_1 P_2}, \overrightarrow{P_1 P_3})$ is independent by Lemma 22.2.5 and, by Theorem 22.2.4 (a) applied twice, $\pi := (P_1 \vee P_2) \vee P_3 = P_1 + \text{span}(\overrightarrow{P_1 P_2}, \overrightarrow{P_1 P_3})$ is a plane in which (P_1, P_2, P_3) is an affine frame. All the points named lie in π , being affine combinations of P_1, P_2, P_3 , so we may use the barycentric coordinates of Theorem 22.2.6 throughout; the weights $\frac{1}{2}$ and $\frac{1}{3}$ exist because 2 and 3 are nonzero, hence invertible, in F .

Each m_i is a line, and G lies on it. Take $i = 1$. In coordinates $P_1 = (1, 0, 0)$ and $M_1 = (0, \frac{1}{2}, \frac{1}{2})$, so $P_1 \neq M_1$ by the uniqueness in Theorem 22.2.6, and $m_1 = P_1 \vee M_1 = P_1 + \text{span}(\overrightarrow{P_1 M_1})$ by Theorem 22.2.4 (a), a flat of dimension 1 because $\overrightarrow{P_1 M_1} \neq 0$. Its points are the affine combinations $(1 - s)P_1 + sM_1$, $s \in F$, with coordinates

$$(1 - s, \frac{s}{2}, \frac{s}{2}).$$

At $s = \frac{2}{3}$ this is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the coordinate triple of G . So $G \in m_1$, and by the same computation with the indices permuted, $G \in m_2$ and $G \in m_3$.

No other common point. A point of $m_1 \cap m_2$ has coordinates $(1 - s, \frac{s}{2}, \frac{s}{2})$ and $(\frac{s'}{2}, 1 - s', \frac{s'}{2})$ for some $s, s' \in F$. Uniqueness in [Theorem 22.2.6](#) makes the two triples equal entry by entry. The third entries give $s = s'$, and then the first two give $1 - s = \frac{s}{2}$, that is, $\frac{3}{2}s = 1$ and $s = \frac{2}{3}$. So $m_1 \cap m_2 = \{G\}$, and a fortiori $m_1 \cap m_2 \cap m_3 = \{G\}$.

The ratio. With origin P_1 we have $\overrightarrow{P_1M_1} = \frac{1}{2}\overrightarrow{P_1P_2} + \frac{1}{2}\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1G} = \frac{1}{3}\overrightarrow{P_1P_2} + \frac{1}{3}\overrightarrow{P_1P_3}$, so $\overrightarrow{P_1G} = \frac{2}{3}\overrightarrow{P_1M_1}$ and $\overrightarrow{GM_1} = \overrightarrow{P_1M_1} - \overrightarrow{P_1G} = \frac{1}{3}\overrightarrow{P_1M_1}$. Hence $\overrightarrow{P_1G} = 2\overrightarrow{GM_1}$, and likewise for $i = 2, 3$. This proves the proposition. ■

Remark. The hypotheses on the field are not decoration. Over $\mathbb{F}_2$ there are no midpoints, because $\frac{1}{2}$ does not exist; over $\mathbb{F}_3$ there are midpoints but no centroid, because $\frac{1}{3}$ does not exist — and indeed in $\mathbb{A}^2(\mathbb{F}_3)$ the three medians of an affinely independent triple are parallel and pairwise disjoint, as Exercise C3 below computes.

22.2.4 An affine map is fixed by a frame

Barycentric coordinates make the next theorem immediate, and it is the affine counterpart of “a linear map is determined by its values on a basis”.

Theorem 22.2.8. An Affine Map Is Determined by a Frame

Let $\mathcal{A}$ have dimension n with affine frame $(P_0, \dots, P_n)$, let $\mathcal{B}$ be an affine space over the same field with direction space W , and let $Q_0, \dots, Q_n \in \mathcal{B}$ be **any** points. Then there is exactly one affine map $f: \mathcal{A} \rightarrow \mathcal{B}$ with $f(P_i) = Q_i$ for $i = 0, \dots, n$. Moreover f carries the barycentric coordinates of [Theorem 22.2.6](#) across:

$$f\left(\sum_{i=0}^n t_i P_i\right) = \sum_{i=0}^n t_i Q_i \quad \left(\sum_i t_i = 1\right).$$

Proof. Existence. The list $(\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n})$ is a basis of V , so by [Theorem 2.1.6](#) there is a linear $T: V \rightarrow W$ with $T(\overrightarrow{P_0P_i}) = \overrightarrow{Q_0Q_i}$ for $i = 1, \dots, n$. Define $f(X) = Q_0 + T(\overrightarrow{P_0X})$; it is affine with $\vec{f} = T$ by [Theorem 22.1.5 \(c\)](#). Then $f(P_0) = Q_0 + T(0) = Q_0$, and $f(P_i) = Q_0 + \overrightarrow{Q_0Q_i} = Q_i$.

Uniqueness. Let g be affine with $g(P_i) = Q_i$ for all i . By [Theorem 22.1.5 \(b\)](#),

$$\vec{g}(\overrightarrow{P_0P_i}) = \overrightarrow{g(P_0)g(P_i)} = \overrightarrow{Q_0Q_i} = T(\overrightarrow{P_0P_i}) \quad (i = 1, \dots, n),$$

so $\vec{g}$ and T agree on a basis and are equal ([Theorem 2.1.5](#)). Hence $g(X) = g(P_0) + \vec{g}(\overrightarrow{P_0X}) = Q_0 + T(\overrightarrow{P_0X}) = f(X)$ for every X .

The displayed identity is [Definition 22.1.3](#) applied to $\sum_i t_i P_i$, using $f(P_i) = Q_i$. ■

So $n + 1$ points in the right position determine an affine map of an n -dimensional space, and prescribe it freely. Fewer points never do, and not because our proof is weak.

Proposition 22.2.9. Points inside a proper flat do not determine an affine map

Let $\dim \mathcal{A} = n \geq 1$ and let $S \subseteq \mathcal{A}$ be contained in a flat A with $\dim A < n$ — for instance, any set of at most n points. Then there are two **distinct** affine maps $\mathcal{A} \rightarrow \mathcal{A}$ that agree at every point of S .

Proof. Let $P \in A$ and let $(u_1, \dots, u_d)$ be a basis of $\vec{A}$, where $d = \dim A < n$. Extend it to a basis $(u_1, \dots, u_d, w_1, \dots, w_{n-d})$ of V (Theorem 1.6.5); since $d < n$, the vector w_1 exists. Let $T: V \rightarrow V$ be the linear map fixing each u_j and each w_k with $k \geq 2$, and sending w_1 to 0 (Theorem 2.1.6), and put $g(X) = P + T(\overrightarrow{PX})$, an affine map by Theorem 22.1.5 (c).

For $X \in A$ we have $\overrightarrow{PX} \in \vec{A} = \text{span}(u_1, \dots, u_d)$, on which T is the identity, so $g(X) = P + \overrightarrow{PX} = X$. Hence g agrees with $\text{id}_{\mathcal{A}}$ on A , and so on S . But $g \neq \text{id}_{\mathcal{A}}$, since $g(P + w_1) = P + T(w_1) = P \neq P + w_1$ by Lemma 22.1.1 (c).

Finally, a nonempty set of at most n points does lie in such a flat. Put

$$D = P_0 + \text{span}(\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_m}).$$

This is the smallest flat containing $P_0, \dots, P_m$: it is a flat by Proposition 22.2.1 and contains every P_i ; and any flat containing all the P_i contains P_0 , hence equals $P_0 + U$ with each $\overrightarrow{P_0P_i} \in U$, so it contains D by Theorem 1.3.1. Thus $\dim D \leq m$, and $m + 1 \leq n$ gives $\dim D \leq n - 1 < n$. (An empty S lies in the flat $\{P\}$ for any P .)

■

Example (Two points are not enough in the plane) is an instance of this proposition, with $n = 2$ and $S = \{(0, 0), (1, 0)\}$; it exhibits its own pair of maps rather than the pair the proof constructs. The moral is the count: $n + 1$, never n . It is the same count that will govern projective transformations later in this chapter, where the answer turns out to be $n + 2$.

22.2.5 A flat is a solution set

Chapter 3 produced flats as solution sets; Chapter 5 proved that every subspace of F^n is a null space. Putting the two together identifies the flats of $\mathbb{A}^n(F)$ completely.

Theorem 22.2.10. Flats Are Exactly the Solution Sets

Let $A \subseteq \mathbb{A}^n(F)$ be nonempty and $0 \leq k \leq n$. The following are equivalent.

- (a) A is a flat of dimension k .
- (b) $A = \{\mathbf{x} \in F^n : \mathbf{M}\mathbf{x} = \mathbf{c}\}$ for some $\mathbf{M} \in M_{(n-k) \times n}(F)$ with linearly independent rows and some $\mathbf{c} \in F^{n-k}$.

In that case no system with fewer than $n - k$ equations has solution set A , and for $k < n$ the flat A is an intersection of exactly $n - k$ affine hyperplanes.

Idea. A flat is a coset $\mathbf{p} + U$ by Proposition 22.2.1, Chapter 5 turns U into a null space, and Chapter 3's “particular plus homogeneous” turns the coset into the solution set of the inhomogeneous system with the same matrix and right-hand side $\mathbf{Mp}$.

Proof.

- (a) $\Rightarrow$ (b). By Proposition 22.2.1, $A = \mathbf{p} + U$ with $U = \vec{A}$ and $\dim U = k$. By Theorem 5.6.2 (a) there is $\mathbf{M} \in M_{(n-k) \times n}(F)$ with $\text{nul}(\mathbf{M}) = U$ whose rows $\mathbf{a}_1^\top, \dots, \mathbf{a}_{n-k}^\top$ are the ones corresponding to a basis $(\varphi_{\mathbf{a}_1}, \dots, \varphi_{\mathbf{a}_{n-k}})$ of U^0 under the isomorphism $\mathbf{a} \mapsto \varphi_{\mathbf{a}}$ of Theorem 5.1.1; an injective linear map preserves independence (Theorem 2.2.5), so those rows are linearly independent. Put $\mathbf{c} = \mathbf{Mp}$. Then $\mathbf{p}$ solves $\mathbf{Mx} = \mathbf{c}$, so by Theorem 3.1.3 the solution set is $\mathbf{p} + \text{nul}(\mathbf{M}) = \mathbf{p} + U = A$.
- (b) $\Rightarrow$ (a). The set A is nonempty, so it has an element $\mathbf{p}$, and Theorem 3.1.3 gives $A = \mathbf{p} + \text{nul}(\mathbf{M})$, a coset, hence a flat by Proposition 22.2.1 with $\vec{A} = \text{nul}(\mathbf{M})$. The rows of $\mathbf{M}$ are independent, so $\text{rank } \mathbf{M} = \dim \text{row}(\mathbf{M}) = n - k$ by Theorem 3.5.5, and Theorem 3.5.7 gives $\dim A = \text{nullity}(\mathbf{M}) = n - (n - k) = k$.

Minimality. Suppose $\mathbf{B} \in M_{m \times n}(F)$ and $\mathbf{d} \in F^m$ satisfy $\{\mathbf{x} : \mathbf{Bx} = \mathbf{d}\} = A$. Fix $\mathbf{p} \in A$. For any $\mathbf{x}$, $\mathbf{Bx} = \mathbf{d}$ holds if and only if $\mathbf{B}(\mathbf{x} - \mathbf{p}) = \mathbf{0}$, so $\mathbf{x} \in A$ if and only if $\mathbf{x} - \mathbf{p} \in \text{nul}(\mathbf{B})$; that is, $A = \mathbf{p} + \text{nul}(\mathbf{B})$. Comparing with $A = \mathbf{p} + \vec{A}$ and using Proposition 22.2.1 (a), $\text{nul}(\mathbf{B}) = \vec{A} = U$. By Theorem 5.6.2 (b), $m \geq n - k$.

Hyperplanes. Let $k < n$ and write $\mathbf{a}_1^\top, \dots, \mathbf{a}_{n-k}^\top$ for the rows of a matrix $\mathbf{M}$ as in (b). Each $\varphi_{\mathbf{a}_i}$ vanishes on $U = \text{nul}(\mathbf{M})$, hence lies in U^0 (Definition 5.2.1), and the $\varphi_{\mathbf{a}_i}$ are independent because the rows are and $\mathbf{a} \mapsto \varphi_{\mathbf{a}}$ is an isomorphism (Theorem 5.1.1); as $\dim U^0 = n - k$ by Theorem 5.2.2, they are a basis of U^0 (Theorem 1.6.4). So $\mathbf{M}$ is a matrix as in Theorem 5.6.2 (a), and Proposition 5.6.4 applied to U and $\mathbf{p}$ writes $A = \mathbf{p} + U$ as the intersection of the $n - k$ affine hyperplanes $\{\mathbf{x} : \mathbf{a}_i^\top \mathbf{x} = \mathbf{a}_i^\top \mathbf{p}\}$.

■

For $k = n$ the system is empty and $A = F^n$; for $k = 0$ it has n independent equations and A is a single point. In between, the count is the familiar one: **dimension counts parameters, codimension counts independent equations**, and now it holds for flats as it did for subspaces.

Example. Equations for a plane in four-space

In Example (Equations for a plane in $\mathbb{R}^4$) the subspace $U = \text{span}((1, 1, 0, 1), (0, 1, 1, 1))$ of $\mathbb{R}^4$ was found to be $\{\mathbf{x} : x_1 - x_2 + x_3 = 0, x_4 - x_2 = 0\}$. Find equations for the flat $A = \mathbf{p} + U$ with $\mathbf{p} = (1, 0, 0, 0)$, and give two of its points.

Solution. Take $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$, whose null space is U , and $\mathbf{c} = \mathbf{Mp} = (1, 0)$. By Theorem 22.2.10,

$$A = \{\mathbf{x} \in \mathbb{R}^4 : x_1 - x_2 + x_3 = 1, x_4 - x_2 = 0\},$$

a flat of dimension $4 - 2 = 2$. Two points: $\mathbf{p} = (1, 0, 0, 0)$, with $1 - 0 + 0 = 1$ and $0 - 0 = 0$;

and $\mathbf{p} + (1, 1, 0, 1) = (2, 1, 0, 1)$, with $2 - 1 + 0 = 1$ and $1 - 1 = 0$. By the minimality clause, no single equation cuts out A , since $n - k = 2$.

That closes the affine part of the chapter's foundations. Flats are cosets, they meet or they do not, their joins obey a two-case formula, they carry symmetric coordinates, and in $\mathbb{A}^n(F)$ they are the solution sets of linear systems. The one irritant is the second case of the join formula, and the reason two coplanar lines can fail to meet: in the affine plane, parallel lines have nowhere to intersect. The projective sections later in this chapter remove that exception by adding the missing points.

22.2.6 Exercises

22.2.6.1 A. Check your understanding

Exercise. A1

- Define a flat and its direction space, and say why the direction space does not depend on the point used to compute it.
- State both cases of the dimension formula for $\dim(A \vee B)$, and say which one applies to two distinct parallel planes in $\mathbb{A}^3$.
- True or false, with a reason: the intersection of two flats is a flat.
- What are the barycentric coordinates of P_2 with respect to the frame (P_0, P_1, P_2, P_3) of a three-dimensional affine space?

Solution.

- A flat is a nonempty subset closed under affine combinations ([Definition 22.2.1](#)); equivalently a coset $P + U$ with U a subspace of V ([Proposition 22.2.1](#)). Its direction space is $\vec{A} = \{\overrightarrow{XY} : X, Y \in A\}$, a description that mentions only A , so no choice of point enters.
- If $A \cap B \neq \emptyset$, $\dim(A \vee B) = \dim A + \dim B - \dim(A \cap B)$; if $A \cap B = \emptyset$, $\dim(A \vee B) = \dim A + \dim B - \dim(\vec{A} \cap \vec{B}) + 1$. Two distinct parallel planes in $\mathbb{A}^3$ are disjoint ([Proposition 22.2.3](#)), so the second applies: $2 + 2 - 2 + 1 = 3$, and their join is $\mathbb{A}^3$.
- False as stated, because the intersection may be empty, and the empty set is not a flat ([Definition 22.2.1](#)). It is true that a **nonempty** intersection of flats is a flat ([Proposition 22.2.2](#)). Two parallel lines in $\mathbb{A}^2$ give the counterexample.
- $(0, 0, 1, 0)$: the combination $0 \cdot P_0 + 0 \cdot P_1 + 1 \cdot P_2 + 0 \cdot P_3$ has weights summing to 1 and equals P_2 , and the coordinates are unique by [Theorem 22.2.6](#).

22.2.6.2 B. Practice

Exercise. B1: Two flats in four-space

In $\mathbb{A}^4$ let $A = (1, 0, 0, 0) + \text{span}((1, 1, 0, 0), (0, 0, 1, 1))$, and let

$$B = (0, 1, 0, 0) + \text{span}((1, 0, 1, 0)), \quad B' = (2, 1, 0, 0) + \text{span}((1, 0, 1, 0)).$$

For each of B and B' , decide whether it meets A , and compute $\dim(A \vee \cdot)$. Justify your

answers with [Theorem 22.2.4](#).

Solution. Here $P = (1, 0, 0, 0)$, $\vec{A} = \text{span}((1, 1, 0, 0), (0, 0, 1, 1))$, $\vec{B} = \vec{B}' = \text{span}((1, 0, 1, 0))$. The three spanning vectors are independent (the first two have disjoint supports, and $a(1, 1, 0, 0) + b(0, 0, 1, 1) = (1, 0, 1, 0)$ would need $a = 1$ and $a = 0$), so $\dim(\vec{A} + \vec{B}) = 3$ and $\vec{A} \cap \vec{B} = \{\mathbf{0}\}$.

B . Here $Q = (0, 1, 0, 0)$ and $\overrightarrow{PQ} = (-1, 1, 0, 0)$. If $\overrightarrow{PQ} = a(1, 1, 0, 0) + b(0, 0, 1, 1) + c(1, 0, 1, 0)$, the second coordinate gives $a = 1$ and the first gives $a + c = -1$, so $c = -2$; the third then gives $b + c = 0$, so $b = 2$, and the fourth gives $b = 0$, a contradiction. So $\overrightarrow{PQ} \notin \vec{A} + \vec{B}$ and $A \cap B = \emptyset$ by part (b) of the theorem. Part (c) gives $\dim(A \vee B) = 2 + 1 - 0 + 1 = 4$, so $A \vee B = \mathbb{A}^4$.

B' . Here $Q' = (2, 1, 0, 0)$ and $\overrightarrow{PQ'} = (1, 1, 0, 0) \in \vec{A} \subseteq \vec{A} + \vec{B}'$, so $A \cap B' \neq \emptyset$. Following the proof of part (b), the meeting point is $P + (1, 1, 0, 0) = (2, 1, 0, 0) = Q'$, and indeed $Q' \in A \cap B'$. By part (b) the direction space of $A \cap B'$ is $\vec{A} \cap \vec{B}' = \{\mathbf{0}\}$, so the intersection is the single point $\{Q'\}$, of dimension 0, and $\dim(A \vee B') = 2 + 1 - 0 = 3$.

Exercise. B2: Barycentric arithmetic

In $\mathbb{A}^2$ take the frame $P_0 = (1, 1)$, $P_1 = (3, 1)$, $P_2 = (1, 4)$.

- Verify that it is a frame.
- Find the barycentric coordinates ([Theorem 22.2.6](#)) of $X = (2, 2)$.
- Find the centroid of (P_0, P_1, P_2) in ordinary coordinates, and check its barycentric coordinates.

Solution.

- $\overrightarrow{P_0P_1} = (2, 0)$ and $\overrightarrow{P_0P_2} = (0, 3)$ are independent, so the list is affinely independent by [Lemma 22.2.5](#), and it has $3 = 2 + 1$ entries.
- $\overrightarrow{P_0X} = (1, 1) = t_1(2, 0) + t_2(0, 3)$ gives $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{3}$, so $t_0 = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. The coordinates are $(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$. **Check:** $\frac{1}{6}(1, 1) + \frac{1}{2}(3, 1) + \frac{1}{3}(1, 4) = (\frac{1}{6} + \frac{3}{2} + \frac{1}{3}, \frac{1}{6} + \frac{1}{2} + \frac{4}{3}) = (2, 2)$.
- $G = \frac{1}{3}((1, 1) + (3, 1) + (1, 4)) = (\frac{5}{3}, 2)$. Its barycentric coordinates are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ by definition, and by uniqueness these are the only ones. **Check against (b)'s method:** $\overrightarrow{P_0G} = (\frac{2}{3}, 1) = \frac{1}{3}(2, 0) + \frac{1}{3}(0, 3)$, so $t_1 = t_2 = \frac{1}{3}$ and $t_0 = \frac{1}{3}$.

Exercise. B3: From parameters to equations and back

- Find a system of equations whose solution set is the line $A = (1, 2, 3) + \text{span}((1, 1, 1))$ in $\mathbb{A}^3$, using as few equations as possible.
- The set $B = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 6\}$ is a flat. Find its dimension, a point of it, and its direction space.
- Decide whether $A \subseteq B$, and compute $\dim(A \vee B)$.

Solution.

- (a) Here $k = 1$ and $n = 3$, so [Theorem 22.2.10](#) calls for $3 - 1 = 2$ equations, and no fewer will do. A functional $\mathbf{a}^\top \mathbf{x}$ vanishes on $\text{span}((1, 1, 1))$ exactly when $a_1 + a_2 + a_3 = 0$; two independent such rows are $(1, -1, 0)$ and $(0, 1, -1)$. With $\mathbf{p} = (1, 2, 3)$ the right-hand sides are $1 - 2 = -1$ and $2 - 3 = -1$, so

$$A = \{\mathbf{x} : x_1 - x_2 = -1, x_2 - x_3 = -1\}.$$

Check: $\mathbf{p}$ satisfies both, and so does $\mathbf{p} + (1, 1, 1) = (2, 3, 4)$.

- (b) One equation with a nonzero row, so $\dim B = 3 - 1 = 2$ by [Theorem 22.2.10](#), a plane. The point $(6, 0, 0)$ lies in it, and $\vec{B} = \text{nul} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \{\mathbf{x} : x_1 + x_2 + x_3 = 0\}$.
- (c) $\mathbf{p} = (1, 2, 3)$ has $1 + 2 + 3 = 6$, so $\mathbf{p} \in B$; but $(1, 1, 1) \notin \vec{B}$, since $1 + 1 + 1 = 3 \neq 0$. So $A \not\subseteq B$, while $A \cap B \neq \emptyset$ (it contains $\mathbf{p}$). Since $\vec{A} \cap \vec{B} = \{\mathbf{0}\}$, the intersection is the single point $\{\mathbf{p}\}$ and $\dim(A \vee B) = 1 + 2 - 0 = 3$: the line and the plane together span $\mathbb{A}^3$.

22.2.6.3 C. Going deeper

Exercise. C1: Affine maps and flats

Let $f: \mathcal{A} \rightarrow \mathcal{B}$ be affine and let $A \subseteq \mathcal{A}$ be a flat.

- (a) Prove that $f(A)$ is a flat of $\mathcal{B}$ whose direction space is $\vec{f}(\vec{A})$, and deduce $\dim f(A) \leq \dim A$.
- (b) Prove that if $B \subseteq \mathcal{B}$ is a flat with $f^{-1}(B) \neq \emptyset$, then $f^{-1}(B)$ is a flat with direction space $(\vec{f})^{-1}(\vec{B})$.
- (c) Hence prove that an **injective** affine map sends flats to flats of the same dimension, and lines to lines.

Solution.

- (a) $f(A)$ is nonempty. Let $P \in A$, so $A = P + \vec{A}$ ([Proposition 22.2.1](#)). For $u \in \vec{A}$, [Theorem 22.1.5 \(c\)](#) gives $f(P + u) = f(P) + \vec{f}(u)$. Hence $f(A) = f(P) + \vec{f}(\vec{A})$, and $\vec{f}(\vec{A})$ is a subspace of W , being the image of the linear map $\vec{f}$ restricted to the subspace $\vec{A}$ ([Theorem 2.2.2](#) applied to that restriction). So $f(A)$ is a flat by [Proposition 22.2.1](#), with the stated direction space, and $\dim f(A) = \dim \vec{f}(\vec{A}) \leq \dim \vec{A} = \dim A$, because the images of a basis of $\vec{A}$ span $\vec{f}(\vec{A})$ ([Theorem 2.2.4](#)) and a spanning list is at least as long as the dimension ([Theorem 1.6.3 \(b\)](#)).
- (b) Let $P \in f^{-1}(B)$, so $f(P) \in B$. For $u \in V$, $f(P + u) = f(P) + \vec{f}(u)$ lies in $B = f(P) + \vec{B}$ exactly when $\vec{f}(u) \in \vec{B}$. So $f^{-1}(B) = P + (\vec{f})^{-1}(\vec{B})$, and $(\vec{f})^{-1}(\vec{B})$ is a subspace: it contains 0 and is closed under linear combinations by linearity of $\vec{f}$ and the subspace test ([Theorem 1.2.1](#)). By [Proposition 22.2.1](#), $f^{-1}(B)$ is a flat with that direction space.
- (c) If f is injective, then $\vec{f}$ is injective: $\vec{f}(u) = 0$ gives $f(P + u) = f(P)$, hence $P + u = P$ and $u = 0$. An injective linear map carries a basis of $\vec{A}$ to a linearly independent list ([Theorem 2.2.5](#)) that spans $\vec{f}(\vec{A})$, hence to a basis of it; so $\dim f(A) = \dim \vec{f}(\vec{A}) = \dim \vec{A} = \dim A$ by (a). A line has dimension 1, so its image is a flat of dimension 1, a line.

Exercise. C2: The midpoint quadrilateral

Let F be a field with $2 \neq 0$, let $\mathcal{A}$ be an affine space over F , and let $P_1, P_2, P_3, P_4 \in \mathcal{A}$ be any four points. Put $M_{12} = \frac{1}{2}P_1 + \frac{1}{2}P_2$, and define M_{23}, M_{34}, M_{41} likewise.

- (a) Prove that $\overrightarrow{M_{12}M_{23}} = \overrightarrow{M_{41}M_{34}} = \frac{1}{2}\overrightarrow{P_1P_3}$.
- (b) Deduce that if $M_{12}, M_{23}, M_{34}, M_{41}$ are distinct, then $M_{12} \vee M_{23}$ and $M_{41} \vee M_{34}$ are parallel lines, and likewise for the other pair of opposite sides: the four midpoints form a parallelogram.
- (c) What does the statement say when P_1, P_2, P_3, P_4 are **not** coplanar, for instance in $\mathbb{A}^3$?

Solution.

- (a) Fix any origin O and write $p_i = \overrightarrow{OP_i}$. By Definition 22.1.2, $\overrightarrow{OM_{12}} = \frac{1}{2}(p_1 + p_2)$ and similarly for the others. Hence, by Proposition 22.1.3,

$$\overrightarrow{M_{12}M_{23}} = \frac{1}{2}(p_2 + p_3) - \frac{1}{2}(p_1 + p_2) = \frac{1}{2}(p_3 - p_1) = \frac{1}{2}\overrightarrow{P_1P_3},$$

and

$$\overrightarrow{M_{41}M_{34}} = \frac{1}{2}(p_3 + p_4) - \frac{1}{2}(p_4 + p_1) = \frac{1}{2}(p_3 - p_1).$$

The answer does not depend on O , as it must not.

- (b) If $M_{12} \neq M_{23}$, the join $M_{12} \vee M_{23}$ is the line $M_{12} + \text{span}(\overrightarrow{M_{12}M_{23}})$ by Theorem 22.2.4 (a). By (a) the two lines $M_{12} \vee M_{23}$ and $M_{41} \vee M_{34}$ have the same direction space $\text{span}(\frac{1}{2}\overrightarrow{P_1P_3})$, so they are parallel (Definition 22.2.3). Running (a) with the indices shifted by one gives $\overrightarrow{M_{23}M_{34}} = \overrightarrow{M_{12}M_{41}} = \frac{1}{2}\overrightarrow{P_2P_4}$, so the other two sides are parallel too.
- (c) Nothing in (a) used coplanarity, so the conclusion holds for a skew quadrilateral in $\mathbb{A}^3$ as well: the four midpoints of a skew quadrilateral are coplanar and form a parallelogram. Four noncoplanar points are affinely independent: otherwise $(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P_4})$ would be dependent (Lemma 22.2.5) and span a subspace of dimension at most 2, putting all four points in a flat of dimension at most 2. The four midpoints are then distinct, since the difference of two of them is $\sum_i \lambda_i P_i$ with $\sum_i \lambda_i = 0$ and the λ_i not all zero — for M_{12} and M_{23} the weights are $\frac{1}{2}$ at P_1 and $-\frac{1}{2}$ at P_3 — hence is a nonzero vector; so (b) applies. The two parallel lines $M_{12} \vee M_{23}$ and $M_{41} \vee M_{34}$ are also distinct: were they equal, their common direction space would contain $\overrightarrow{M_{12}M_{41}}$, which is $\frac{1}{2}\overrightarrow{P_2P_4}$ by (b), so $\frac{1}{2}\overrightarrow{P_2P_4} = \mu \cdot \frac{1}{2}\overrightarrow{P_1P_3}$ for some $\mu \in F$ (Proposition 22.2.1 (a)) — a relation $\sum_i \lambda_i P_i = 0$ with weights summing to 0 and $-\frac{1}{2} \neq 0$ at P_2 , again against affine independence. Their join is therefore a plane, of dimension $1 + 1 - 1 + 1 = 2$ by Theorem 22.2.4 (c), and it contains all four midpoints.

Exercise. C3: Medians over the field with three elements

Work in $\mathbb{A}^2(\mathbb{F}_3)$ with the affinely independent list $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (0,1)$. (Nothing is lost by this choice: by Theorem 22.2.8 any other affinely independent triple is carried to this one by an affine bijection, which preserves midpoints, being an affine combination, and carries lines to lines by Exercise (C1: Affine maps and flats).)

- (a) Compute the three midpoints M_i and the three medians $m_i = P_i \vee M_i$.

- (b) Prove that the three medians are pairwise parallel and pairwise disjoint, so that [Proposition 22.2.7](#) fails over $\mathbb{F}_3$.
- (c) Explain which step of the proof of [Proposition 22.2.7](#) breaks, and why the hypothesis $3 \neq 0$ cannot be dropped.

Solution.

- (a) In $\mathbb{F}_3$ we have $2^{-1} = 2$, since $2 \cdot 2 = 4 = 1$. So

$$M_1 = 2P_2 + 2P_3 = 2(1, 0) + 2(0, 1) = (2, 2),$$

$$M_2 = 2P_1 + 2P_3 = (0, 2),$$

$$M_3 = 2P_1 + 2P_2 = (2, 0).$$

Then $\overrightarrow{P_1M_1} = (2, 2)$, $\overrightarrow{P_2M_2} = (0, 2) - (1, 0) = (2, 2)$ and $\overrightarrow{P_3M_3} = (2, 0) - (0, 1) = (2, 2)$. So

$$m_1 = (0, 0) + \text{span}((2, 2)),$$

$$m_2 = (1, 0) + \text{span}((2, 2)),$$

$$m_3 = (0, 1) + \text{span}((2, 2)).$$

- (b) All three have the same direction space $\text{span}((2, 2)) = \text{span}((1, 1))$, so they are parallel ([Definition 22.2.3](#)). They are distinct: $\overrightarrow{P_1P_2} = (1, 0) \notin \text{span}((1, 1))$, and likewise for the other two pairs, so no two of them share a point by [Proposition 22.2.1](#) (b). Distinct parallel flats are disjoint ([Proposition 22.2.3](#)). Hence the three medians have no common point, and the conclusion of [Proposition 22.2.7](#) is false here.
- (c) The proof solved $\frac{3}{2}s = 1$ for s , which needs 3 to be invertible. Over $\mathbb{F}_3$ the point $\frac{1}{3}(P_1 + P_2 + P_3)$ does not exist, because $3 = 0$ has no inverse; the weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ cannot be written down at all. Concretely, the equation $1 - s = \frac{s}{2}$ becomes $1 - s = 2s$, that is, $1 = 3s = 0$, which has no solution: the parametrizations of m_1 and m_2 never agree. So the hypothesis $3 \neq 0$ is not an artifact of the proof; it is needed for the statement.