

21.10 Loewner's Theorem

Chapter 17 §11 stated Loewner's theorem, said four times that this chapter would prove it, and named the one thing it was missing. Section 9 supplied the elementary half: a continuously differentiable function is monotone of order n exactly when every Loewner matrix of size n built from it is positive semidefinite (Theorem 21.9.5). What is left is to turn that infinite family of conditions into a single formula, and to run the formula back. Turning it into a formula is the one thing this book does not prove; running it back costs about a page of integration, which we build first.

Throughout, $F = \mathbb{R}$ or $F = \mathbb{C}$, every matrix whose eigenvalues are indexed is Hermitian, and $f(\mathbf{A})$ is the functional calculus of Definition 12.8.2. The letter I denotes an **interval** of real numbers, not the identity matrix $\mathbf{I}$, as in Chapter 17 §11. Three things from the analysis list of Chapter 16's introduction are used: completeness (A1), monotone convergence (A2) and the compactness of closed bounded sets (A3). Everything imported beyond those is named where it appears.

21.10.1 Integrating a matrix-valued function

Chapter 17 §11 read the integral formula of Theorem 17.11.9 (ii) and said what stopped it there:

What this book lacks is the analysis that carries the Loewner order through an integral, which is not on the list (A1)–(A6) in Chapter 16's introduction.

That analysis is one definition and one lemma. It is not a theory of integration; it is the least that makes the sentence “an integral of positive semidefinite matrices is positive semidefinite” mean something.

A **tagged partition** of a compact interval $[a, b]$ is a list of division points $a = s_0 < s_1 < \dots < s_N = b$ together with **tags** $\tau_i \in [s_{i-1}, s_i]$; we call the whole package P . Its **mesh** is $\max_i (s_i - s_{i-1})$. For $\mathbf{M}: [a, b] \rightarrow M_{p \times q}(\mathbb{C})$ the associated **Riemann sum** is

$$S(\mathbf{M}, P) := \sum_{i=1}^N \mathbf{M}(\tau_i) (s_i - s_{i-1}).$$

Call $\mathbf{M}$ **continuous** when each of its pq entries is a continuous complex-valued function of s ; by Corollary 16.2.4 this is the same as continuity for any norm on $M_{p \times q}(\mathbb{C})$.

Proposition 21.10.1. Riemann Sums of a Continuous Matrix Function Converge

Let $\mathbf{M}: [a, b] \rightarrow M_{p \times q}(\mathbb{C})$ be continuous. Then there is a matrix $J \in M_{p \times q}(\mathbb{C})$ such that $S(\mathbf{M}, P_k) \rightarrow J$ for **every** sequence (P_k) of tagged partitions of $[a, b]$ whose meshes tend to 0.

Idea. Two Riemann sums of small mesh are close because each is close to the sum over their common refinement, and on a refinement the tag moves by less than the mesh. So the sums form a Cauchy sequence and converge. The one analytic input is that a continuous function on a compact interval does not merely move a little near each point but moves a little **uniformly**, which is compactness.

Proof.

Claim. For every $\varepsilon > 0$ there is $\delta > 0$ such that $\|\mathbf{M}(s) - \mathbf{M}(s')\|_F \leq \varepsilon$ whenever $s, s' \in [a, b]$ and $|s - s'| < \delta$.

Proof. Suppose not. Then for some $\varepsilon > 0$ and every $k \geq 1$ the value $\delta = 1/k$ fails, so there are $s_k, s'_k \in [a, b]$ with $|s_k - s'_k| < 1/k$ and $\|\mathbf{M}(s_k) - \mathbf{M}(s'_k)\|_F > \varepsilon$. The interval $[a, b]$ is a closed bounded subset of $\mathbb{R}$, so by the compactness of closed bounded sets, **fact (A3) of Chapter 16's introduction**, a subsequence s_{k_j} converges to some $s \in [a, b]$. Then $s'_{k_j} \rightarrow s$ as well, since $|s'_{k_j} - s| \leq |s'_{k_j} - s_{k_j}| + |s_{k_j} - s| \rightarrow 0$. Continuity of $\mathbf{M}$ gives $\mathbf{M}(s_{k_j}) \rightarrow \mathbf{M}(s)$ and $\mathbf{M}(s'_{k_j}) \rightarrow \mathbf{M}(s)$, so $\|\mathbf{M}(s_{k_j}) - \mathbf{M}(s'_{k_j})\|_F \rightarrow 0$, contradicting the lower bound ε . ■

Fix $\varepsilon > 0$ and let δ be as in the claim. Let P be a tagged partition of mesh $< \delta$, and let Q be a tagged partition whose division points include those of P . Each interval q of Q lies inside exactly one interval of P ; write $\tau(q)$ for the tag of that interval of P and $\rho(q)$ for the tag of q . Splitting each term of $S(\mathbf{M}, P)$ along the intervals of Q inside it,

$$S(\mathbf{M}, P) - S(\mathbf{M}, Q) = \sum_q (\mathbf{M}(\tau(q)) - \mathbf{M}(\rho(q)))|q|,$$

where $|q|$ is the length of q . Both $\tau(q)$ and $\rho(q)$ lie in one interval of P , of length $< \delta$, so each bracket has Frobenius norm at most ε . The triangle inequality gives

$$\|S(\mathbf{M}, P) - S(\mathbf{M}, Q)\|_F \leq \varepsilon \sum_q |q| = \varepsilon(b - a).$$

If P and P' both have mesh $< \delta$, take for Q the partition whose division points are those of P together with those of P' , with any tags; applying the bound twice,

$$\|S(\mathbf{M}, P) - S(\mathbf{M}, P')\|_F \leq 2\varepsilon(b - a). \quad (*)$$

Now let (P_k) be tagged partitions with meshes tending to 0. Given $\eta > 0$, apply $(*)$ with $\varepsilon = \eta/(2(b - a) + 1)$: for all k, l large enough that both meshes are $< \delta$, $\|S(\mathbf{M}, P_k) - S(\mathbf{M}, P_l)\|_F < \eta$. So each entry of $S(\mathbf{M}, P_k)$ is a Cauchy sequence of complex numbers, hence convergent by **completeness, fact (A1)**; call the entrywise limit J . If (P'_k) is a second such sequence, interleaving the two produces a third sequence of meshes tending to 0, whose sums converge; both subsequences must have that common limit, so the limits agree. This proves the proposition. ■

Definition 21.10.1. Integral of a Continuous Matrix-Valued Function

Let $\mathbf{M}: [a, b] \rightarrow M_{p \times q}(\mathbb{C})$ be continuous. Its **integral**

$$\int_a^b \mathbf{M}(s) \, ds \in M_{p \times q}(\mathbb{C})$$

is the matrix J of [Proposition 21.10.1](#). For a continuous $\mathbf{M}$ on an open or half-open interval with endpoints $a < b$, where $a = -\infty$ or $b = +\infty$ is allowed, the **improper integral**

$\int_a^b \mathbf{M}(s) ds$ is the matrix J' , if one exists, such that

$$\int_{\alpha_k}^{\beta_k} \mathbf{M}(s) ds \longrightarrow J' \quad \text{entrywise}$$

for every pair of sequences with $\alpha_k \rightarrow a$, $\beta_k \rightarrow b$, and $a < \alpha_k < \beta_k < b$. The integral is then said to **converge**.

Five properties are used below, and each is read off the Riemann sums together with the algebra of limits.

- **Entrywise.** The (i, j) entry of $\int_a^b \mathbf{M}$ is $\int_a^b m_{ij}$, because that is true of every Riemann sum. In particular [Definition 21.10.1](#) for $p = q = 1$ is the ordinary Riemann integral of a continuous complex function.
- **Linear.** $\int(\mathbf{M} + \mathbf{N}) = \int \mathbf{M} + \int \mathbf{N}$ and $\int c\mathbf{M} = c \int \mathbf{M}$ for a scalar c .
- **Additive in the interval.** For $a < c < b$, $\int_a^b \mathbf{M} = \int_a^c \mathbf{M} + \int_c^b \mathbf{M}$: joining a tagged partition of $[a, c]$ to one of $[c, b]$ gives a tagged partition of $[a, b]$ whose Riemann sum is the sum of the two, and the meshes tend to 0 together.
- **Constant factors pass out.** For constant $\mathbf{X} \in M_{r \times p}(\mathbb{C})$, $\mathbf{Y} \in M_{q \times r'}(\mathbb{C})$, $\int_a^b \mathbf{X}\mathbf{M}(s)\mathbf{Y} ds = \mathbf{X}(\int_a^b \mathbf{M})\mathbf{Y}$, since $S(\mathbf{X}\mathbf{M}\mathbf{Y}, P) = \mathbf{X} S(\mathbf{M}, P) \mathbf{Y}$.
- **Pairing passes in.** Taking $\mathbf{X} = \mathbf{x}^*$ and $\mathbf{Y} = \mathbf{x}$ in the previous item, $\mathbf{x}^*(\int_a^b \mathbf{M})\mathbf{x} = \int_a^b \mathbf{x}^* \mathbf{M}(s) \mathbf{x} ds$.

All five hold for a convergent improper integral, by taking the limit in α and β .

Now the lemma the whole section is for.

Lemma 21.10.2. The Integral of a Positive Semidefinite Integrand

Let $\mathbf{M}: [a, b] \rightarrow M_n(\mathbb{C})$ be continuous with $\mathbf{M}(s) \succeq \mathbf{O}$ for every $s \in [a, b]$. Then

$$\int_a^b \mathbf{M}(s) ds \succeq \mathbf{O}.$$

The same conclusion holds for a **convergent improper** integral of a continuous integrand that is $\succeq \mathbf{O}$ at every point of the interval.

Proof. Let $\mathbf{x} \in \mathbb{C}^n$. For a tagged partition P ,

$$\mathbf{x}^* S(\mathbf{M}, P) \mathbf{x} = \sum_{i=1}^N (\mathbf{x}^* \mathbf{M}(\tau_i) \mathbf{x})(s_i - s_{i-1}),$$

a sum of products of two nonnegative real numbers, since $\mathbf{M}(\tau_i) \succeq \mathbf{O}$ (Definition 13.1.1) and the interval lengths are positive. Hence $\mathbf{x}^* S(\mathbf{M}, P) \mathbf{x} \geq 0$ for every P . Taking any sequence of tagged partitions with meshes tending to 0 and using [Proposition 21.10.1](#), $\mathbf{x}^* S(\mathbf{M}, P_k) \mathbf{x} \rightarrow \mathbf{x}^*(\int_a^b \mathbf{M})\mathbf{x}$; a nonstrict inequality survives a limit, so that number is ≥ 0 .

It remains to check that $\int_a^b \mathbf{M}$ is Hermitian. Each $\mathbf{M}(s)$ is Hermitian, so each Riemann sum is, and the entrywise limit of Hermitian matrices is Hermitian: $\overline{(\int m_{ji})} = \int \overline{m_{ji}} = \int m_{ij}$, since conjugation is continuous and commutes with sums. So $\int_a^b \mathbf{M} \succeq \mathbf{O}$.

For the improper case, each proper integral $\int_\alpha^\beta \mathbf{M}$ is $\succeq \mathbf{O}$ by what has just been proved, and

$\mathbf{x}^*(\int_a^b \mathbf{M})\mathbf{x}$ is the limit of the numbers $\mathbf{x}^*(\int_\alpha^\beta \mathbf{M})\mathbf{x} \geq 0$, and the limit is Hermitian for the same reason. ■

That is the whole of the promised analysis. Nothing in it is deep: the positive semidefinite matrices are closed under sums and under multiplication by nonnegative numbers, and they are closed under limits, so a limit of sums of them stays inside.

Quick check. Let $\mathbf{M}(t) = \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix}$ for $t \in [0, 1]$. Is $\mathbf{M}(t) \succeq \mathbf{O}$ for each t ? Compute $\int_0^1 \mathbf{M}(t) dt$ and check that the answer is $\succeq \mathbf{O}$. Does it have rank 1?

Solution. For each t , $\mathbf{M}(t) = \mathbf{v}(t)\mathbf{v}(t)^\top$ with $\mathbf{v}(t) = (1, t)$, so $\mathbf{x}^\top \mathbf{M}(t)\mathbf{x} = (x_1 + tx_2)^2 \geq 0$ and $\mathbf{M}(t) \succeq \mathbf{O}$. Integrating entrywise, and using $\int_0^1 t dt = \frac{1}{2}$ and $\int_0^1 t^2 dt = \frac{1}{3}$ (for a continuous integrand these can be read off the Riemann sums $N^{-2} \sum_{i \leq N} i$ and $N^{-3} \sum_{i \leq N} i^2$),

$$\int_0^1 \mathbf{M}(t) dt = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix},$$

whose determinant is $\frac{1}{3} - \frac{1}{4} = \frac{1}{12} > 0$ and whose $(1, 1)$ entry is positive, so it is positive definite by Sylvester's criterion (Theorem 13.1.5 (d)). It has rank 2, not 1: every integrand was a rank-one matrix, and the integral is not. [Lemma 21.10.2](#) preserves positivity and nothing else.

21.10.2 The one fact this chapter quotes

Two statements in this chapter are not proved from Chapters 1–20: the one below, and Lieb's concavity theorem, which §12 states and nothing in the book uses. Beside them this section quotes two standard facts of one-variable calculus, the fundamental theorem of calculus and the substitution rule, each named at the point where it is used. The book's practice for such a fact is fixed — Chapter 11 with the theorem of Abel and Ruffini, Chapter 14 §06 with Taylor's theorem — and it is followed here: the statement is given in full, in a block of its own, named at every use, and never buried inside a proof that presents itself as self-contained.

Theorem 21.10.3. (A7) The Theorem of Pick and Nevanlinna

Let $I \subseteq \mathbb{R}$ be an open interval and let $f: I \rightarrow \mathbb{R}$.

- (a) Suppose that f is differentiable and that for every $n \geq 1$ and all $t_1, \dots, t_n \in I$ the Loewner matrix $L_f(t_1, \dots, t_n)$ of Definition 21.9.2 is positive semidefinite. Then f is the restriction to I of a function analytic on $\mathbb{C} \setminus (\mathbb{R} \setminus I)$ that maps every point of the upper half-plane $\{z : \operatorname{Im} z > 0\}$ to a point with nonnegative imaginary part.
- (b) Conversely, every function analytic on $\mathbb{C} \setminus (-\infty, 0]$ that maps the upper half-plane into $\{\operatorname{Im} z \geq 0\}$ has, on $I = (0, \infty)$, the integral representation of Theorem 17.11.9 (ii): there are $\alpha \in \mathbb{R}$, $\beta \geq 0$, and a positive measure ν on $[0, \infty)$ with $\int (1 + s^2)^{-1} d\nu(s) < \infty$

such that

$$f(t) = \alpha + \beta t + \int_{[0,\infty)} \left(\frac{s}{1+s^2} - \frac{1}{t+s} \right) d\nu(s)$$

for every $t > 0$.

- (c) A function that is operator monotone on an open interval is continuously differentiable there.

What it gives. Exactly one thing: it converts an infinite family of matrix conditions — one positive semidefinite Loewner matrix for each size and each choice of points — into a single formula with two constants and one measure. Nothing in the statement is about matrices at all after clause (a) is applied.

What it costs. Clause (a) is complex analysis: one shows that the positivity of the Loewner matrices forces f to extend analytically across I , and that the extension preserves the upper half-plane. Clause (b) is the Herglotz–Riesz representation of such a function, which produces the measure ν by a compactness argument in a space of measures. Clause (c) is a regularity theorem. None of these is within reach of Chapters 1–20, and Chapter 17 §11 said so when it stated the theorem: “Its proof needs complex analysis and some measure theory, which lie outside Chapter 16’s list (A1)–(A6).”

A second cost, worth naming separately. The integral in (b) is taken against a measure, and this book constructs no such integral. [Definition 21.10.1](#) covers the case $d\nu(s) = w(s) ds$ with w continuous, and that is the only case the corollaries below need: each of them starts from a representation written down and verified by hand, with a continuous density. For a general ν , which is what (b) allows, the argument uses exactly two properties of the integral: it is **linear** in the integrand, and it sends a **nonnegative** integrand to a nonnegative number. Both are part of the meaning of “integral against a positive measure”. For the integral the book does construct they are proved — linearity in the list of properties after [Definition 21.10.1](#), positivity in [Lemma 21.10.2](#); for a general measure they are quoted, along with (A7) itself.

Which half rests on it. Only the hard half. In [Theorem 21.10.4](#) below, the implication (ii) $\Rightarrow$ (i) is proved here from Chapter 17 §11 and [Lemma 21.10.2](#), with no appeal to (A7) — subject only to the second cost just named, since for a general ν the linearity and the positivity of the integral are quoted rather than constructed. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) — the classification itself — use (A7) and cannot be had without it. Nothing else in this chapter uses (A7); in particular [Corollary 21.10.8](#), [Corollary 21.10.9](#), and [Corollary 21.10.10](#) below, which are what the earlier chapters actually asked for, are proved without it.

21.10.3 Loewner’s theorem

Theorem 21.10.4. Loewner’s Theorem

Let $f: (0, \infty) \rightarrow \mathbb{R}$. The following are equivalent.

- (i) f is operator monotone on $(0, \infty)$.
- (ii) There are $\alpha \in \mathbb{R}$, $\beta \geq 0$, and a positive measure ν on $[0, \infty)$ with $\int (1+s^2)^{-1} d\nu(s) < \infty$ such that for every $t > 0$

$$f(t) = \alpha + \beta t + \int_{[0,\infty)} \left(\frac{s}{1+s^2} - \frac{1}{t+s} \right) d\nu(s).$$

- (iii) f is the restriction to $(0, \infty)$ of a function analytic on $\mathbb{C} \setminus (-\infty, 0]$ that sends every point of the upper half-plane to a point with nonnegative imaginary part.

These are the three conditions of Theorem 17.11.9.

Idea. Three steps, in a cycle.

① (ii) $\Rightarrow$ (i). At each fixed $s \geq 0$ the integrand is $-1/(t+s)$ plus a constant, and Corollary 17.11.5 already knows those are operator monotone. Subtracting $f(\mathbf{B})$ from $f(\mathbf{A})$ kills the constants and leaves an integral of positive semidefinite matrices, which Lemma 21.10.2 declares positive semidefinite.

② (i) $\Rightarrow$ (iii). Operator monotonicity is monotonicity of every order, and §09 converts that into positivity of every Loewner matrix. (A7) converts that into analyticity.

③ (iii) $\Rightarrow$ (ii). (A7) again.

Proof. Step 1: (ii) $\Rightarrow$ (i). Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ be Hermitian with spectra in $(0, \infty)$ and $\mathbf{A} \succeq \mathbf{B}$. For $s \geq 0$ put $g_s(t) = -1/(t+s)$ on $(0, \infty)$. By Corollary 17.11.5, g_s is operator monotone there, so

$$\mathbf{R}(s) := g_s(\mathbf{A}) - g_s(\mathbf{B}) = (\mathbf{B} + s\mathbf{I})^{-1} - (\mathbf{A} + s\mathbf{I})^{-1}$$

satisfies $\mathbf{R}(s) \succeq \mathbf{O}$ for every $s \geq 0$; the identification of $g_s(\mathbf{A})$ with $-(\mathbf{A} + s\mathbf{I})^{-1}$ is Definition 12.8.2 applied to the spectral resolution of $\mathbf{A}$.

Write $\mathbf{A} = \sum_j \mu_j \mathbf{P}_j$ for that resolution (Theorem 12.8.1). By Definition 12.8.2 and the hypothesis (ii),

$$\begin{aligned} f(\mathbf{A}) &= \sum_j f(\mu_j) \mathbf{P}_j \\ &= \alpha \mathbf{I} + \beta \mathbf{A} \\ &\quad + \int_{[0, \infty)} \left(\frac{s}{1+s^2} \mathbf{I} - (\mathbf{A} + s\mathbf{I})^{-1} \right) d\nu(s), \end{aligned}$$

where the second line substitutes the formula (ii) for each $f(\mu_j)$ and moves the finite sum $\sum_j (\cdot) \mathbf{P}_j$ inside the integral, which is legitimate because the integral is **linear** in its integrand and the $\mathbf{P}_j$ do not depend on s ; the identities $\sum_j \mathbf{P}_j = \mathbf{I}$ and $\sum_j \mu_j \mathbf{P}_j = \mathbf{A}$ are Theorem 12.8.1 (c) and (d). The same computation applies to $\mathbf{B}$. Subtracting, the terms $\alpha \mathbf{I}$ and $\frac{s}{1+s^2} \mathbf{I}$ cancel, and

$$f(\mathbf{A}) - f(\mathbf{B}) = \beta(\mathbf{A} - \mathbf{B}) + \int_{[0, \infty)} \mathbf{R}(s) d\nu(s).$$

Let $\mathbf{x} \in F^n$. Pairing passes inside the integral, so

$$\begin{aligned} \mathbf{x}^*(f(\mathbf{A}) - f(\mathbf{B}))\mathbf{x} &= \beta \mathbf{x}^*(\mathbf{A} - \mathbf{B})\mathbf{x} \\ &\quad + \int_{[0, \infty)} \mathbf{x}^* \mathbf{R}(s) \mathbf{x} d\nu(s) \geq 0, \end{aligned}$$

because $\beta \geq 0$, $\mathbf{x}^*(\mathbf{A} - \mathbf{B})\mathbf{x} \geq 0$, and the integrand $\mathbf{x}^* \mathbf{R}(s) \mathbf{x}$ is nonnegative for every s . This last step is Lemma 21.10.2 when $d\nu(s) = w(s)ds$ with w continuous, and in general uses only the linearity and positivity of the integral against a positive measure, as recorded with

Theorem 21.10.3. Hence $f(\mathbf{A}) \succeq f(\mathbf{B})$. The size n was arbitrary, so f is operator monotone (Definition 17.11.1 (a)).

Step 2: (i) $\Rightarrow$ (iii). Let f be operator monotone on $(0, \infty)$. By [Theorem 21.10.3 \(c\)](#), f is continuously differentiable there. Fix $n \geq 1$ and $t_1, \dots, t_n \in (0, \infty)$. Being operator monotone, f is in particular monotone of order n , so Theorem 21.9.5 gives $L_f(t_1, \dots, t_n) \succeq \mathbf{O}$. As n and the points were arbitrary, the hypothesis of [Theorem 21.10.3 \(a\)](#) holds with $I = (0, \infty)$. Since $\mathbb{R} \setminus (0, \infty) = (-\infty, 0]$, that clause produces an analytic function on $\mathbb{C} \setminus (-\infty, 0]$ restricting to f and mapping the upper half-plane into its closure. That is (iii).

Step 3: (iii) $\Rightarrow$ (ii). This is [Theorem 21.10.3 \(b\)](#).

The cycle (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) closes, so the three conditions are equivalent. ■

Step 1 is the step Chapter 17 §11 identified as missing, and it turned out to cost one lemma. Steps 2 and 3 are where the content is, and they are quoted. It is worth being precise about the division: **the book proves that every function of the form (ii) is operator monotone, and quotes that there are no others.**

Warning. Analytic and increasing is not enough. Condition (iii) is not “extends analytically”; it is “extends analytically **and keeps the upper half-plane**”. The function e^t extends to an entire function and is increasing on $\mathbb{R}$, yet $\operatorname{Im} e^z = e^x \sin y$ for $z = x + iy$, which is negative at $z = 3\pi i/2$, where $e^z = -i$. So e^t fails (iii), and [Example \(The exponential is not operator monotone\)](#) below confirms the failure directly with two 2×2 matrices.

21.10.4 The powers

Chapter 13 §05 announced the answer and deferred the proof:

The dividing line is at $s = 1$: the functions preserving the order are called **operator monotone**, and $t \mapsto t^s$ is operator monotone on $[0, \infty)$ exactly for $0 \leq s \leq 1$. Chapter 21 proves this, along with the companion fact that $t \mapsto \log t$ is operator monotone while $t \mapsto e^t$ is not.

Chapter 17 §11 repeated it, adding that “the operator monotonicity of t^p for every $p \in (0, 1)$ needs a new idea, an integral representation of the kind in Theorem 17.11.9”. Here is the representation, and it is elementary. Real powers $t^r = e^{r \log t}$ are Chapter 18 §11’s (Lemma 18.11.1).

Lemma 21.10.5. The Integral That Represents a Power

Let $0 < p < 1$. Then the improper integral

$$K_p := \int_0^\infty \frac{u^{p-1}}{1+u} du$$

converges, with $0 < K_p < \infty$, and for every $\lambda > 0$

$$\int_0^\infty s^{p-1} \frac{\lambda}{\lambda + s} ds = K_p \lambda^p.$$

Idea. Both halves are one idea. For convergence, cut $(0, \infty)$ at the powers of 2: on each piece the integrand is bounded by its value at one endpoint, and the resulting bounds form a geometric series, convergent exactly because $0 < p < 1$. For the identity, substitute $s = \lambda u$; since the substitution is linear it can be made inside the Riemann sums, where it is visible as a relabeling of the partition together with a factor λ^p .

Proof. Convergence. The integrand is positive and continuous on $(0, \infty)$, so interval additivity makes $\delta \mapsto \int_\delta^1$ increase as δ decreases and $R \mapsto \int_1^R$ increase with R ; the same property splits each range at the powers of 2 below. It is enough to bound both families above: bounded above, they have a least upper bound L by **completeness, fact (A1) of Chapter 16's introduction**, and monotonicity in the parameter makes L the limit along *every* sequence of parameters tending to the endpoint, which is the form [Definition 21.10.1](#) asks for; along a decreasing sequence $\delta_k \rightarrow 0^+$ that limit is **monotone convergence, fact (A2)**. Since $p - 1 < 0$, the function u^{p-1} is decreasing, so on $[2^{-k-1}, 2^{-k}]$ it is at most $(2^{-k-1})^{p-1} = 2^{-(k+1)(1-p)}$; as $1/(1+u) \leq 1$ and the interval has length 2^{-k-1} , the integral over it is at most $2^{-(k+1)p}$. Summing the geometric series,

$$\int_\delta^1 \frac{u^{p-1}}{1+u} du \leq \sum_{k \geq 0} 2^{-(k+1)p} = \frac{1}{2^p - 1},$$

for every $\delta \in (0, 1)$. For the other end use $u^{p-1}/(1+u) \leq u^{p-2}$, which on $[2^k, 2^{k+1}]$ is at most $2^{k(p-2)}$; the interval has length 2^k , so the integral over it is at most $2^{k(p-1)}$, and

$$\int_1^R \frac{u^{p-1}}{1+u} du \leq \sum_{k \geq 0} 2^{k(p-1)} = \frac{1}{1 - 2^{p-1}},$$

which is finite because $p - 1 < 0$. So $K_p < \infty$, and $K_p > 0$ because the integrand is positive and $\int_1^2$ is already positive.

The identity. Fix $\lambda > 0$ and $0 < \delta < R$. Let P be a tagged partition $\delta/\lambda = u_0 < \dots < u_N = R/\lambda$ with tags v_i . Multiplying every point by λ gives a tagged partition λP of $[\delta, R]$, with tags λv_i and mesh λ times that of P . Writing $g(s) = s^{p-1}\lambda/(\lambda + s)$ and $h(u) = u^{p-1}/(1+u)$,

$$\begin{aligned} g(\lambda v_i) \cdot \lambda(u_i - u_{i-1}) &= \frac{\lambda^{p-1} v_i^{p-1} \lambda}{\lambda(1 + v_i)} \lambda(u_i - u_{i-1}) \\ &= \lambda^p h(v_i)(u_i - u_{i-1}), \end{aligned}$$

using $(\lambda v)^{p-1} = \lambda^{p-1} v^{p-1}$ (Lemma 18.11.1 and the rules for real powers recorded after it). Summing over i , $S(g, \lambda P) = \lambda^p S(h, P)$. Letting the mesh of P tend to 0 — which sends that of λP to 0 as well — and using [Proposition 21.10.1](#) on both sides,

$$\int_\delta^R g(s) ds = \lambda^p \int_{\delta/\lambda}^{R/\lambda} h(u) du.$$

Now let $\delta \rightarrow 0^+$ and $R \rightarrow \infty$. The right side tends to $\lambda^p K_p$ by the first part, so the left side converges to it. This proves the lemma. ■

Remark. The number K_p equals $\pi/\sin(p\pi)$. That evaluation is a standard exercise in contour integration, and the constant $\sin(p\pi)/\pi$ is the one Chapter 17 §11 displayed. Nothing below needs its value: only $0 < K_p < \infty$ is used, and that was proved. Exercise B2 computes $K_{1/2} = \pi$ by hand.

Two more small tools, both elementary, and then the corollary.

Lemma 21.10.6. Composites of Operator Monotone Functions

Let I, J be intervals, let $g: I \rightarrow \mathbb{R}$ be operator monotone on I with $g(I) \subseteq J$, and let $h: J \rightarrow \mathbb{R}$ be operator monotone on J . Then $h \circ g$ is operator monotone on I .

Proof. Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ be Hermitian with spectra in I and $\mathbf{A} \succeq \mathbf{B}$. Then $g(\mathbf{A}) \succeq g(\mathbf{B})$, and both are Hermitian (Theorem 12.8.3 (c)) with spectra $g(\text{spec}(\mathbf{A}))$ and $g(\text{spec}(\mathbf{B}))$ by Theorem 12.8.3 (d), hence contained in $g(I) \subseteq J$. So $h(g(\mathbf{A})) \succeq h(g(\mathbf{B}))$.

It remains to see that $h(g(\mathbf{A})) = (h \circ g)(\mathbf{A})$. Let $\mathbf{A} = \sum_j \mu_j \mathbf{P}_j$ be the spectral resolution. Group the indices by the value of g : let $\nu_1, \dots, \nu_r$ be the distinct numbers among $g(\mu_1), \dots, g(\mu_k)$ and put $\mathbf{Q}_l = \sum_{j: g(\mu_j) = \nu_l} \mathbf{P}_j$. Each $\mathbf{Q}_l$ is a nonzero orthogonal projection, $\mathbf{Q}_l \mathbf{Q}_{l'} = \mathbf{O}$ for $l \neq l'$, and $\sum_l \mathbf{Q}_l = \mathbf{I}$, by Theorem 12.8.1 (b) and (c). Since $g(\mathbf{A}) = \sum_l \nu_l \mathbf{Q}_l$ with the ν_l distinct, Theorem 12.8.2 says this is the spectral resolution of $g(\mathbf{A})$. Hence, by Definition 12.8.2 applied twice,

$$h(g(\mathbf{A})) = \sum_l h(\nu_l) \mathbf{Q}_l = \sum_j h(g(\mu_j)) \mathbf{P}_j = (h \circ g)(\mathbf{A}).$$

The same holds for $\mathbf{B}$, so $(h \circ g)(\mathbf{A}) \succeq (h \circ g)(\mathbf{B})$. ■

Lemma 21.10.7. Pointwise Limits Stay Operator Monotone

Let $f_k: I \rightarrow \mathbb{R}$ be operator monotone on I for every k , and suppose $f_k(t) \rightarrow f(t)$ for every $t \in I$. Then f is operator monotone on I .

Proof. Let $\mathbf{A} \succeq \mathbf{B}$ be Hermitian of the same size with spectra in I , with spectral resolutions $\mathbf{A} = \sum_j \mu_j \mathbf{P}_j$ and $\mathbf{B} = \sum_l \nu_l \mathbf{Q}_l$. By Definition 12.8.2, $f_k(\mathbf{A}) = \sum_j f_k(\mu_j) \mathbf{P}_j$, a fixed finite combination of fixed matrices with scalar coefficients converging to $f(\mu_j)$; so $f_k(\mathbf{A}) \rightarrow f(\mathbf{A})$ entrywise, and likewise $f_k(\mathbf{B}) \rightarrow f(\mathbf{B})$. For each $\mathbf{x}$ and each k , $\mathbf{x}^*(f_k(\mathbf{A}) - f_k(\mathbf{B}))\mathbf{x} \geq 0$, and the left side tends to $\mathbf{x}^*(f(\mathbf{A}) - f(\mathbf{B}))\mathbf{x}$. A nonstrict inequality survives a limit, so $f(\mathbf{A}) \succeq f(\mathbf{B})$. ■

Corollary 21.10.8. The Operator Monotone Powers

Let $p \in \mathbb{R}$. The function $t \mapsto t^p$ is operator monotone on $(0, \infty)$ **if and only if** $0 \leq p \leq 1$.

For those p the statement extends to $[0, \infty)$: if $\mathbf{A} \succeq \mathbf{B} \succeq \mathbf{O}$ are Hermitian of the same size, then $\mathbf{A}^p \succeq \mathbf{B}^p$.

Idea. For $0 < p < 1$, [Lemma 21.10.5](#) rewrites t^p as a positive superposition of the functions $t/(t+s)$ that Chapter 17 §11 already proved operator monotone, and the superposition is an honest Riemann integral, so [Lemma 21.10.2](#) finishes it. For $p > 1$ the trick is that operator monotone functions compose: if t^p were operator monotone for one $p > 1$, enough compositions with each other and with a small power would manufacture t^2 , which is not.

Proof. **The cases $p = 0$ and $p = 1$.** These are the constant 1 and the identity, operator monotone by Proposition 17.11.2.

The case $p < 0$. Then $t \mapsto t^p$ is strictly decreasing on $(0, \infty)$, since $t \mapsto t^{-p}$ is strictly increasing there (Chapter 18 §11). An operator monotone function is increasing (Proposition 17.11.1), so it is not operator monotone.

The case $0 < p < 1$, on $(0, \infty)$. Let $\mathbf{A} \succeq \mathbf{B}$ be Hermitian in $M_n(F)$ with spectra in $(0, \infty)$. Write $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$ with $\mathbf{U}$ unitary and $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$ the eigenvalues, all positive (Corollary 12.4.3, Corollary 12.5.2, Theorem 13.1.5 (b)). For $s > 0$,

$$s^{p-1} \mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1} = \mathbf{U} \text{diag}\left(\frac{s^{p-1}\lambda_i}{\lambda_i + s}\right) \mathbf{U}^*,$$

where we are using that $f(\mathbf{A}) = \mathbf{U} \text{diag}(f(\lambda_1), \dots, f(\lambda_n)) \mathbf{U}^*$ for every function f on $\text{spec}(\mathbf{A})$: writing $\mathbf{u}_1, \dots, \mathbf{u}_n$ for the columns of $\mathbf{U}$, the spectral projection $\mathbf{P}_j$ of Definition 12.8.1 is $\sum_{i:\lambda_i=\mu_j} \mathbf{u}_i \mathbf{u}_i^*$, so the two expressions for $f(\mathbf{A})$ have the same terms. Every entry of the integrand is therefore a fixed linear combination of the n scalar functions of [Lemma 21.10.5](#). By that lemma each of their improper integrals converges, so the matrix integral converges, and by the linearity of the integral together with the rule that constant factors pass out,

$$\int_0^\infty s^{p-1} \mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1} ds = K_p \mathbf{U} \text{diag}(\lambda_i^p) \mathbf{U}^* = K_p \mathbf{A}^p,$$

the last equality by Definition 12.8.2. The same holds for $\mathbf{B}$. Subtracting,

$$\mathbf{A}^p - \mathbf{B}^p = \frac{1}{K_p} \int_0^\infty s^{p-1} (\mathbf{C}(s) - \mathbf{Y}(s)) ds,$$

with $\mathbf{C}(s) = \mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1}$ and $\mathbf{Y}(s) = \mathbf{B}(\mathbf{B} + s\mathbf{I})^{-1}$. Fix $s > 0$. The function $t \mapsto t/(t+s)$ is operator monotone on $(0, \infty)$ by Corollary 17.11.5, and it sends $\mathbf{A}$ to $\mathbf{C}(s)$ and $\mathbf{B}$ to $\mathbf{Y}(s)$; hence $\mathbf{C}(s) - \mathbf{Y}(s) \succeq \mathbf{O}$, and multiplying by the positive number s^{p-1} keeps it so. The integrand is continuous in s on $(0, \infty)$ and the integral converges, so [Lemma 21.10.2](#) in its improper form gives $\mathbf{A}^p - \mathbf{B}^p \succeq \mathbf{O}$. As $K_p > 0$ and n was arbitrary, $t \mapsto t^p$ is operator monotone on $(0, \infty)$.

The case $p > 1$. Suppose, for a contradiction, that $t \mapsto t^p$ is operator monotone on $(0, \infty)$ for some $p > 1$. It maps $(0, \infty)$ into itself, so by [Lemma 21.10.6](#) it may be composed with itself: $t \mapsto t^{p^k}$ is operator monotone for every $k \geq 1$. Choose k with $p^k \geq 2$ and put $q = 2/p^k \in (0, 1]$. By the cases already proved, $t \mapsto t^q$ is operator monotone on $(0, \infty)$ and

maps it into itself, so [Lemma 21.10.6](#) applied to $g(t) = t^q$ and $h(u) = u^{p^k}$ makes

$$t \mapsto (t^q)^{p^k} = t^{qp^k} = t^2$$

operator monotone on $(0, \infty)$.

Now let $\mathbf{A} \succeq \mathbf{B} \succeq \mathbf{O}$ be Hermitian and let $\varepsilon > 0$. Then $\mathbf{A} + \varepsilon \mathbf{I} \succeq \mathbf{B} + \varepsilon \mathbf{I} \succ \mathbf{O}$, both with spectra in $(0, \infty)$, so $(\mathbf{A} + \varepsilon \mathbf{I})^2 \succeq (\mathbf{B} + \varepsilon \mathbf{I})^2$. Expanding both squares, that says

$$\mathbf{A}^2 - \mathbf{B}^2 + 2\varepsilon(\mathbf{A} - \mathbf{B}) \succeq \mathbf{O}.$$

Fix $\mathbf{x}$ and let $\varepsilon \rightarrow 0^+$: the number $\mathbf{x}^*(\mathbf{A}^2 - \mathbf{B}^2)\mathbf{x} + 2\varepsilon \mathbf{x}^*(\mathbf{A} - \mathbf{B})\mathbf{x}$ is ≥ 0 for every $\varepsilon > 0$, so its limit $\mathbf{x}^*(\mathbf{A}^2 - \mathbf{B}^2)\mathbf{x}$ is ≥ 0 . Hence $\mathbf{A}^2 \succeq \mathbf{B}^2$ whenever $\mathbf{A} \succeq \mathbf{B} \succeq \mathbf{O}$, contradicting Example (Squaring does not preserve the Loewner order). So no $p > 1$ works.

Extension to $[0, \infty)$ for $0 \leq p \leq 1$. Let $\mathbf{A} \succeq \mathbf{B} \succeq \mathbf{O}$ and $\varepsilon > 0$. As above $\mathbf{A} + \varepsilon \mathbf{I} \succeq \mathbf{B} + \varepsilon \mathbf{I} \succ \mathbf{O}$, so $(\mathbf{A} + \varepsilon \mathbf{I})^p \succeq (\mathbf{B} + \varepsilon \mathbf{I})^p$. If $\mathbf{A} = \sum_j \mu_j \mathbf{P}_j$ is the spectral resolution, then $\mathbf{A} + \varepsilon \mathbf{I} = \sum_j (\mu_j + \varepsilon) \mathbf{P}_j$ is the spectral resolution of $\mathbf{A} + \varepsilon \mathbf{I}$ (Theorem 12.8.2, as the shifted eigenvalues are still distinct), so

$$(\mathbf{A} + \varepsilon \mathbf{I})^p = \sum_j (\mu_j + \varepsilon)^p \mathbf{P}_j \rightarrow \sum_j \mu_j^p \mathbf{P}_j = \mathbf{A}^p$$

as $\varepsilon \rightarrow 0^+$: for $\mu_j > 0$ this is the continuity of $u \mapsto e^{p \log u}$ on $(0, \infty)$ (Lemma 18.11.1), and for $\mu_j = 0$ it says $\varepsilon^p \rightarrow 0$, which holds because $u \mapsto u^p$ is a strictly increasing bijection of $[0, \infty)$ with itself, so $\varepsilon^p < \eta$ as soon as $\varepsilon < \eta^{1/p}$. The same for $\mathbf{B}$. Pairing with $\mathbf{x}$ and letting $\varepsilon \rightarrow 0^+$ gives $\mathbf{x}^*(\mathbf{A}^p - \mathbf{B}^p)\mathbf{x} \geq 0$. This proves the corollary. ■

This pays Chapter 13 §05 and Chapter 17 §11 in full for the powers, and it is worth recording what it cost: Corollary 17.11.5 from Chapter 17 §11, [Lemma 21.10.2](#) from the first part of this section, and facts (A1)–(A3). **Neither the theorem of Pick and Nevanlinna nor Theorem 21.10.4 itself was used** — Chapter 17 §11 expected the powers to arrive with Loewner's theorem, and they arrived without it. What (A7) adds is the knowledge that the representation exists for *every* operator monotone function; for the particular function t^p the representation was written down and verified by hand.

Quick check. In [Theorem 21.10.4](#) (ii), why does the term $s/(1+s^2)$ inside the integral play no part in the proof that f is operator monotone?

Solution. Because it does not depend on t . Applying the functional calculus turns it into $(s/(1+s^2))\mathbf{I}$, the same matrix for $\mathbf{A}$ as for $\mathbf{B}$, so it cancels in $f(\mathbf{A}) - f(\mathbf{B})$. Its only job is to make the integral converge: without it the integrand $-1/(t+s)$ alone need not be integrable at $s = \infty$ against ν , while the difference $s/(1+s^2) - 1/(t+s)$ decays like s^{-2} .

21.10.5 The logarithm and the exponential

Corollary 21.10.9. The Logarithm Is Operator Monotone

The function $t \mapsto \log t$ is operator monotone on $(0, \infty)$.

Idea. The logarithm is a limit of powers: $(t^p - 1)/p \rightarrow \log t$ as $p \rightarrow 0^+$, because that quotient is a difference quotient for the exponential. Each $(t^p - 1)/p$ is an increasing affine function of an operator monotone function, and [Lemma 21.10.7](#) passes to the limit.

Proof. For $0 < p < 1$ put $f_p(t) = (t^p - 1)/p$ on $(0, \infty)$. By [Corollary 21.10.8](#), $t \mapsto t^p$ is operator monotone there, and $f_p = -\frac{1}{p} + \frac{1}{p} t^p$ with $1/p > 0$, so f_p is operator monotone by Proposition 17.11.4 (a).

Fix $t > 0$. If $t = 1$ then $f_p(1) = 0 = \log 1$ for every p . If $t \neq 1$, write $c = \log t \neq 0$ and $x = pc$, so that $t^p = e^{pc} = e^x$ by the definition of real powers (Chapter 18 §11) and

$$f_p(t) = \frac{e^x - 1}{p} = c \frac{e^x - 1}{x}.$$

As $p \rightarrow 0^+$ we have $x \rightarrow 0$, and $(e^x - 1)/x \rightarrow 1$, this being the difference quotient of $\exp$ at 0, whose value is $e^0 = 1$ (Theorem 10.9.4 (b) in size 1, as recorded in the proof of Lemma 18.11.1). Hence $f_p(t) \rightarrow c = \log t$.

Taking $p = 1/k$ and letting $k \rightarrow \infty$, [Lemma 21.10.7](#) applies and $\log$ is operator monotone on $(0, \infty)$. ■

Remark. The logarithm's place in [Theorem 21.10.4](#) (ii) is as simple as it can be: $\alpha = 0$, $\beta = 0$, and ν the ordinary length measure, so that

$$\log t = \int_0^\infty \left(\frac{s}{1+s^2} - \frac{1}{t+s} \right) ds$$

for every $t > 0$. To verify this, take $0 < \delta < R$ and use the fundamental theorem of calculus — an imported fact of one-variable calculus, quoted here and named again in the one proof that uses this representation, [Corollary 21.10.10](#) — with the antiderivative $\frac{1}{2} \log(1+s^2) - \log(t+s)$, whose derivative is the integrand. The value at R is $\log(\sqrt{1+R^2}/(t+R)) \rightarrow 0$, and the value at δ is $\log(\sqrt{1+\delta^2}/(t+\delta)) \rightarrow \log(1/t) = -\log t$. The difference is $\log t$.

That representation has one more use, and it pays a debt of §08. The list of operator convex functions there stopped short of two, $-\log t$ and $t \log t$, because each needs an integral representation. The display above is that representation, and it settles both at once.

Corollary 21.10.10. Two Operator Convex Functions from the Logarithm

The functions $t \mapsto -\log t$ and $t \mapsto t \log t$ are operator convex on $(0, \infty)$.

Idea. Read the representation of $\log t$ for a fixed s instead of for a fixed t . The integrand of $-\log t$ is then $1/(t+s)$ plus a constant, and Theorem 21.8.3 (d) already knows that function is operator convex; multiplying the representation through by t turns the integrand into $-t/(t+s)$ plus an affine function, and the same clause knows $t/(t+s)$ is operator **concave**. Either way the convexity defect is an integral of positive semidefinite matrices, which Lemma 21.10.2 declares positive semidefinite.

Proof. Write $c(s) = s/(1+s^2)$. The representation in the remark above says that for every $t > 0$

$$-\log t = \int_0^\infty \left(\frac{1}{t+s} - c(s) \right) ds, \quad t \log t = \int_0^\infty \left(t c(s) - \frac{t}{t+s} \right) ds,$$

the second display being the first multiplied by $-t$, a constant as far as s is concerned.

Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ be Hermitian with spectra in $(0, \infty)$, let $\theta \in [0, 1]$ and put $\mathbf{C} = \theta \mathbf{A} + (1 - \theta) \mathbf{B}$, whose spectrum again lies in $(0, \infty)$, as Chapter 17 §11 checked.

The logarithm. Write $\mathbf{A} = \sum_j \mu_j \mathbf{P}_j$ for the spectral resolution (Theorem 12.8.1). Substituting the first display for each $-\log \mu_j$ and moving the finite sum $\sum_j (\cdot) \mathbf{P}_j$ inside the integral — legitimate because the integral is linear in its integrand and the $\mathbf{P}_j$ do not depend on s — gives

$$-\log \mathbf{A} = \int_0^\infty ((\mathbf{A} + s\mathbf{I})^{-1} - c(s)\mathbf{I}) ds,$$

where $\sum_j \mathbf{P}_j = \mathbf{I}$ is Theorem 12.8.1 (c) and the identification of $\sum_j (\mu_j + s)^{-1} \mathbf{P}_j$ with $(\mathbf{A} + s\mathbf{I})^{-1}$ is Definition 12.8.2, exactly as in Step 1 of Theorem 21.10.4. Every entry of the integrand is a fixed linear combination of the n convergent scalar integrals above, so the matrix integral converges. The same holds for $\mathbf{B}$ and for $\mathbf{C}$. Forming $\theta(\cdot) + (1 - \theta)(\cdot) - (\cdot)$, the terms $c(s)\mathbf{I}$ cancel at each s , and the three convergent integrals combine into one:

$$\begin{aligned} & \theta(-\log \mathbf{A}) + (1 - \theta)(-\log \mathbf{B}) - (-\log \mathbf{C}) \\ &= \int_0^\infty (\theta(\mathbf{A} + s\mathbf{I})^{-1} + (1 - \theta)(\mathbf{B} + s\mathbf{I})^{-1} - (\mathbf{C} + s\mathbf{I})^{-1}) ds. \end{aligned}$$

Fix $s > 0$ and put $g_s(t) = 1/(t+s)$, so that $g_s(\mathbf{A}) = (\mathbf{A} + s\mathbf{I})^{-1}$ by Definition 12.8.2. By Theorem 21.8.3 (d), g_s is operator convex on $(0, \infty)$, so the integrand is $\theta g_s(\mathbf{A}) + (1 - \theta)g_s(\mathbf{B}) - g_s(\mathbf{C}) \succeq \mathbf{O}$. It is continuous in s on $(0, \infty)$ and the integral converges, so Lemma 21.10.2 in its improper form makes the left side $\succeq \mathbf{O}$. Since n , the two matrices and θ were arbitrary, $-\log t$ is operator convex on $(0, \infty)$.

The function $t \log t$. Its functional calculus at $\mathbf{A}$ is $\mathbf{A} \log \mathbf{A}$, the product of $\mathbf{A}$ and $\log \mathbf{A}$, by Theorem 12.8.3 (b). The same substitution, now with the second display, gives

$$\mathbf{A} \log \mathbf{A} = \int_0^\infty (c(s)\mathbf{A} - \mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1}) ds,$$

its entries again convergent, and likewise for $\mathbf{B}$ and $\mathbf{C}$. Forming the same combination, the terms $c(s)\mathbf{A}$ contribute $c(s)(\theta \mathbf{A} + (1 - \theta)\mathbf{B} - \mathbf{C}) = \mathbf{O}$ at each s , so

$$\begin{aligned} & \theta \mathbf{A} \log \mathbf{A} + (1 - \theta) \mathbf{B} \log \mathbf{B} - \mathbf{C} \log \mathbf{C} \\ &= \int_0^\infty (\theta h_s(\mathbf{A}) + (1 - \theta)h_s(\mathbf{B}) - h_s(\mathbf{C})) ds, \end{aligned}$$

where $h_s(t) = -t/(t+s)$ and $h_s(\mathbf{A}) = -\mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1}$. By Theorem 21.8.3 (d) the function $t/(t+s)$ is operator concave on $(0, \infty)$, so h_s is operator convex and each integrand is $\succeq \mathbf{O}$. By Lemma 21.10.2 again the left side is $\succeq \mathbf{O}$, and $t \log t$ is operator convex on $(0, \infty)$. ■

Both proofs quote the fundamental theorem of calculus once, through the representation of $\log t$ that the remark verified with it; nothing else outside Chapters 1–20 enters, and in particular Theorem 21.10.3 does not. With this, the two functions §08 left over are on its list.

The companion negative fact is the one Chapter 13 §05 named beside it.

Example. The exponential is not operator monotone

Show that $t \mapsto e^t$ is not operator monotone on $\mathbb{R}$, and exhibit two 2×2 Hermitian matrices witnessing the failure.

Solution. Order two already fails. The exponential is infinitely differentiable, so Theorem 21.9.5 applies. Take $t_1 = 1$ and $t_2 = -1$. The divided differences (Definition 21.9.1) are $f[1, 1] = e$, $f[-1, -1] = e^{-1}$ and

$$f[1, -1] = \frac{e - e^{-1}}{1 - (-1)} = \sinh 1,$$

so the Loewner matrix and its determinant are

$$L_{\exp}(1, -1) = \begin{pmatrix} e & \sinh 1 \\ \sinh 1 & e^{-1} \end{pmatrix},$$

$$\det L_{\exp}(1, -1) = 1 - (\sinh 1)^2.$$

Subtracting the series for e^{-1} from that for e leaves only the odd terms, so $\sinh 1 = 1 + \frac{1}{3!} + \frac{1}{5!} + \dots > 1$. The determinant is therefore negative, and $L_{\exp}(1, -1)$ is not positive semidefinite. By Theorem 21.9.5, $\exp$ is not monotone of order 2, hence not operator monotone.

An explicit pair. Take

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \succeq \mathbf{O}$ and $\mathbf{A} \succeq \mathbf{B}$. Write $\mathbf{A} = \mathbf{I} + \mathbf{N}$ with $\mathbf{N} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $\mathbf{N}^2 = 2\mathbf{I}$, so $\mathbf{N}^{2m} = 2^m \mathbf{I}$ and $\mathbf{N}^{2m+1} = 2^m \mathbf{N}$, and summing the series of Definition 10.9.2 in two halves,

$$e^{\mathbf{N}} = (\cosh \sqrt{2}) \mathbf{I} + \frac{\sinh \sqrt{2}}{\sqrt{2}} \mathbf{N}.$$

Since $\mathbf{I}$ and $\mathbf{N}$ commute, $e^{\mathbf{A}} = e \cdot e^{\mathbf{N}}$ by Theorem 10.9.4 (d), while $e^{\mathbf{B}} = \text{diag}(e, e^{-1})$. Writing $c = \cosh \sqrt{2}$ and $d = (\sinh \sqrt{2})/\sqrt{2}$, a direct expansion of the 2×2 determinant, using $c^2 - 2d^2 = \cosh^2 \sqrt{2} - \sinh^2 \sqrt{2} = 1$, gives

$$\det(e^{\mathbf{A}} - e^{\mathbf{B}}) = e^2(1 - c + d) + (1 - c - d).$$

With $c = 2.17818\dots$ and $d = 1.36830\dots$ this is $-1.1417\dots$, a negative number, so $e^{\mathbf{A}} - e^{\mathbf{B}}$ has one positive and one negative eigenvalue and $e^{\mathbf{A}} \not\succeq e^{\mathbf{B}}$.

Remark. The two halves of Chapter 13 §05's "companion fact" are now both in hand: $\log$ is operator monotone on $(0, \infty)$ (Corollary 21.10.9) and e^t is not operator monotone (Example (The exponential is not operator monotone)). Neither proof used Theorem 21.10.3. The logarithm and the exponential are inverse to each other and increasing, which shows that operator monotonicity is not inherited by inverse functions.

21.10.6 Exercises

21.10.6.1 A. Check your understanding

Exercise. A1

- State what Theorem 21.10.3 gives, and say which implication of Theorem 21.10.4 rests on it.
- Explain in one sentence why an integral of positive semidefinite matrices is positive semidefinite, naming the two properties of the positive semidefinite matrices that the argument uses.
- Decide whether the following is correct, and justify your answer: "a function that extends to an entire function of z and is increasing on $(0, \infty)$ is operator monotone there".
- For which real p is $t \mapsto t^p$ operator monotone on $(0, \infty)$? Name the result and say which of its cases needed (A7).
- Where in the proof of Theorem 21.10.4 is the hypothesis $\beta \geq 0$ used, and what would go wrong without it?

Solution.

- It gives two things: that positivity of all the Loewner matrices of f forces f to extend analytically across the interval preserving the upper half-plane, and that such a function has the integral representation (ii); a third clause says an operator monotone function is automatically C^1 . The implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) rest on it. The implication (ii) $\Rightarrow$ (i) does not.
- Every Riemann sum is a sum of nonnegative multiples of the values of the integrand, and the integral is a limit of such sums: the positive semidefinite matrices are closed under nonnegative combinations and closed under limits (Lemma 21.10.2).
- Incorrect. e^t is entire and increasing on $(0, \infty)$, and is not operator monotone there: Example (The exponential is not operator monotone) gives a witness with spectra in $\mathbb{R}$, and Exercise (B3: Which are operator monotone?) (vi) shifts it into $(0, \infty)$. Condition (iii) of Theorem 21.10.4 asks in addition that the upper half-plane be mapped into its closure, which e^z fails at $z = 3\pi i/2$.
- Exactly for $0 \leq p \leq 1$ (Corollary 21.10.8). No case needed (A7): the positive cases came from Lemma 21.10.5 and Lemma 21.10.2, and the failure for $p > 1$ from Lemma 21.10.6 together with Example (Squaring does not preserve the Loewner order).
- In the final display of Step 1, where $\beta \mathbf{x}^*(\mathbf{A} - \mathbf{B})\mathbf{x}$ is asserted to be ≥ 0 . With $\beta < 0$ that term is ≤ 0 and can dominate: at $n = 1$ the function $f(t) = \beta t$ with $\beta < 0$ is decreasing, hence not operator monotone (Proposition 17.11.1).

21.10.6.2 B. Practice

Exercise. B1: One matrix integral

Let

$$\mathbf{M}(t) = \begin{pmatrix} 2+t & 1 \\ 1 & 2-t \end{pmatrix}, \quad t \in [-1, 1].$$

Show that $\mathbf{M}(t) \succeq \mathbf{O}$ for every $t \in [-1, 1]$, compute $\int_{-1}^1 \mathbf{M}(t) dt$, and confirm the conclusion of [Lemma 21.10.2](#) for it.

Solution. $\mathbf{M}(t)$ is real symmetric. Its leading entry is $2+t \geq 1 > 0$ and its determinant is $(2+t)(2-t) - 1 = 3 - t^2 \geq 2 > 0$ for $|t| \leq 1$, so $\mathbf{M}(t) \succ \mathbf{O}$ by Sylvester's criterion (Theorem 13.1.5 (d)); in particular $\mathbf{M}(t) \succeq \mathbf{O}$.

Integrating entrywise, $\int_{-1}^1 (2 \pm t) dt = 4$ since $\int_{-1}^1 t dt = 0$, and $\int_{-1}^1 1 dt = 2$. So

$$\int_{-1}^1 \mathbf{M}(t) dt = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix},$$

whose eigenvalues are 6 and 2, both positive; so it is $\succeq \mathbf{O}$ by Theorem 13.1.4 (b), as [Lemma 21.10.2](#) predicts.

Exercise. B2: The constant for the square root

(a) Show that $K_{1/2} = \int_0^\infty u^{-1/2}(1+u)^{-1} du = \pi$. *Hint: substitute $u = w^2$.*

(b) Let $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Using [Lemma 21.10.5](#) with $p = \frac{1}{2}$, evaluate

$$\frac{1}{\pi} \int_0^\infty s^{-1/2} \mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1} ds$$

and check the answer against the positive square root of $\mathbf{A}$.

Solution.

(a) With $u = w^2$, so $du = 2w dw$ and $u^{-1/2} = 1/w$ for $w > 0$, the substitution rule of one-variable calculus gives

$$\int_0^\infty \frac{u^{-1/2}}{1+u} du = \int_0^\infty \frac{1}{w} \cdot \frac{2w}{1+w^2} dw = 2 \int_0^\infty \frac{dw}{1+w^2}.$$

The last integral is $\lim_{R \rightarrow \infty} \arctan R - \arctan 0 = \pi/2$, by the fundamental theorem of calculus with the antiderivative $\arctan$. Hence $K_{1/2} = \pi$. (Both calculus facts are quoted, not proved here.)

(b) $\mathbf{A}$ has eigenvalues 9 and 1, with orthonormal eigenvectors $\frac{1}{\sqrt{2}}(1, 1)$ and $\frac{1}{\sqrt{2}}(1, -1)$, so its spectral resolution is $\mathbf{A} = 9\mathbf{P}_1 + \mathbf{P}_2$ with

$$\mathbf{P}_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{P}_2 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

By the computation in the proof of [Corollary 21.10.8](#), the integral equals $K_{1/2}\mathbf{A}^{1/2}$, so dividing by π gives

$$\mathbf{A}^{1/2} = 9^{1/2}\mathbf{P}_1 + 1^{1/2}\mathbf{P}_2 = 3\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Squaring confirms it: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \mathbf{A}$, and the matrix is positive definite, so it is *the* positive square root (Theorem 13.2.1).

Exercise. B3: Which are operator monotone?

Determine which of the following are operator monotone on $(0, \infty)$. Justify your answer.

- (i) $f(t) = t^{2/3}$.
- (ii) $f(t) = t^{4/3}$.
- (iii) $f(t) = 3 + 2 \log t$.
- (iv) $f(t) = -t^{-1/3}$.
- (v) $f(t) = \log(1 + t)$.
- (vi) $f(t) = e^t$.

Solution.

- (i) Operator monotone: $0 \leq \frac{2}{3} \leq 1$, so [Corollary 21.10.8](#) applies.
- (ii) Not operator monotone: $\frac{4}{3} > 1$ ([Corollary 21.10.8](#)).
- (iii) Operator monotone: $\log$ is ([Corollary 21.10.9](#)), and $3+2 \log t$ is $\alpha+c \log t$ with $c = 2 \geq 0$, so Proposition 17.11.4 (a) applies.
- (iv) Operator monotone. The function $g(t) = t^{1/3}$ is operator monotone on $(0, \infty)$ with $g((0, \infty)) = (0, \infty)$, and $h(u) = -1/u$ is operator monotone on $(0, \infty)$ (Proposition 17.11.3); by [Lemma 21.10.6](#) the composite $h(g(t)) = -t^{-1/3}$ is operator monotone.
- (v) Operator monotone. The affine map $g(t) = t + 1$ is operator monotone on $(0, \infty)$ (Proposition 17.11.2) with image $(1, \infty) \subseteq (0, \infty)$, and $\log$ is operator monotone on $(0, \infty)$; apply [Lemma 21.10.6](#).
- (vi) Not operator monotone, by [Example \(The exponential is not operator monotone\)](#). (The witness there has spectra in $\mathbb{R}$, not $(0, \infty)$; restrict it by replacing $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} + 2\mathbf{I}$ and $\mathbf{B} + 2\mathbf{I}$, which are $\succ \mathbf{O}$ and satisfy $e^{\mathbf{A}+2\mathbf{I}} = e^2 e^{\mathbf{A}}$, so the same determinant is negative after multiplication by $e^4 > 0$.)

21.10.6.3 C. Going deeper

Exercise. C1: When the integral loses information

- (a) Give a continuous $\mathbf{M}: [0, 1] \rightarrow M_2(\mathbb{R})$ with every $\mathbf{M}(t)$ symmetric, $\int_0^1 \mathbf{M} \succeq \mathbf{O}$, and $\mathbf{M}(t) \not\succeq \mathbf{O}$ for some t . Hence the converse of [Lemma 21.10.2](#) fails.
- (b) Prove that if $\mathbf{M}: [a, b] \rightarrow M_n(\mathbb{C})$ is continuous with $\mathbf{M}(t) \succeq \mathbf{O}$ for every t and $\int_a^b \mathbf{M} = \mathbf{O}$, then $\mathbf{M}(t) = \mathbf{O}$ for every t .

Hint for (b): a continuous nonnegative scalar function with zero integral is zero.

Solution.

- (a) Take $\mathbf{M}(t) = \text{diag}(1, 2t - 1)$. It is continuous and symmetric, $\mathbf{M}(0) = \text{diag}(1, -1)$ is indefinite, and $\int_0^1 \mathbf{M} = \text{diag}(1, 0) \succeq \mathbf{O}$.
- (b) Fix $\mathbf{x} \in \mathbb{C}^n$ and put $g(t) = \mathbf{x}^* \mathbf{M}(t) \mathbf{x}$, a continuous real-valued function with $g \geq 0$. Because pairing passes inside the integral, $\int_a^b g = \mathbf{x}^* (\int_a^b \mathbf{M}) \mathbf{x} = 0$. Suppose $g(t_0) > 0$ for some $t_0 \in [a, b]$. By continuity there is a subinterval $[c, d] \subseteq [a, b]$ with $d > c$ on which $g \geq g(t_0)/2$. By interval additivity, $\int_a^b g = \int_a^c g + \int_c^d g + \int_d^b g$, with the outer terms absent if $c = a$ or $d = b$; each outer term is ≥ 0 , because $g \geq 0$ and so every Riemann sum over that piece is ≥ 0 . On $[c, d]$ every Riemann sum of g is at least $\frac{1}{2}g(t_0)(d - c)$, since each tag value is at least $\frac{1}{2}g(t_0)$ and the interval lengths sum to $d - c$; a nonstrict inequality survives a limit, so $\int_c^d g$ is at least that too. Hence

$$\int_a^b g \geq \int_c^d g \geq \frac{g(t_0)}{2}(d - c) > 0,$$

a contradiction. So $g \equiv 0$, that is $\mathbf{x}^* \mathbf{M}(t) \mathbf{x} = 0$ for every t and every $\mathbf{x}$. A positive semidefinite matrix with $\mathbf{x}^* \mathbf{M} \mathbf{x} = 0$ for every $\mathbf{x}$ is $\mathbf{O}$: by Theorem 13.1.4 (c) write $\mathbf{M}(t) = \mathbf{C}^* \mathbf{C}$, so $\|\mathbf{C} \mathbf{x}\|^2 = 0$ for every $\mathbf{x}$, hence $\mathbf{C} = \mathbf{O}$.

Exercise. C2: The cube, by a Loewner matrix

- (a) Write down the Loewner matrix $L_f(t_1, t_2)$ for $f(t) = t^3$ and general $t_1 \neq t_2$, and evaluate its determinant at $t_1 = 1, t_2 = 2$. What does Theorem 21.9.5 conclude?
- (b) Do the same for $f(t) = t^{1/2}$ at $t_1 = 1, t_2 = 4$, and say which result of this section the sign of the determinant illustrates.

Solution.

- (a) For $f(t) = t^3$ we have $f'(t) = 3t^2$ and, factoring the difference of cubes,

$$f[t_1, t_2] = \frac{t_1^3 - t_2^3}{t_1 - t_2} = t_1^2 + t_1 t_2 + t_2^2,$$

so $L_f(t_1, t_2)$ has diagonal $(3t_1^2, 3t_2^2)$ and off-diagonal $t_1^2 + t_1 t_2 + t_2^2$ (Definition 21.9.2). At $t_1 = 1, t_2 = 2$ the entries are 3, 12 and $1 + 2 + 4 = 7$, so

$$\det L_f(1, 2) = 3 \cdot 12 - 7^2 = 36 - 49 = -13 < 0.$$

So $L_f(1, 2) \not\succeq \mathbf{O}$, and by Theorem 21.9.5 $t \mapsto t^3$ is not monotone of order 2, hence not operator monotone on $(0, \infty)$. This agrees with [Corollary 21.10.8](#), since $3 > 1$.

- (b) For $f(t) = t^{1/2}$, $f'(t) = \frac{1}{2}t^{-1/2}$ and

$$f[t_1, t_2] = \frac{\sqrt{t_1} - \sqrt{t_2}}{t_1 - t_2} = \frac{1}{\sqrt{t_1} + \sqrt{t_2}}.$$

At $t_1 = 1, t_2 = 4$ the matrix is $\begin{pmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{pmatrix}$, with determinant $\frac{1}{8} - \frac{1}{9} = \frac{1}{72} > 0$ and positive diagonal, so it is positive semidefinite. That is consistent with [Corollary 21.10.8](#) at $p = \frac{1}{2}$: an operator monotone function has all its Loewner matrices positive semidefinite, for every size and all points, and this is one instance of it.

Exercise. C3: Half of the representation, by hand

Let $0 < p < 1$, let $\mathbf{A} \succeq \mathbf{B} \succ \mathbf{O}$ be Hermitian in $M_n(F)$, and let $0 < \delta < R$. Put

$$\mathbf{Z}(\delta, R) = \int_{\delta}^R s^{p-1} (\mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1} - \mathbf{B}(\mathbf{B} + s\mathbf{I})^{-1}) ds.$$

- (a) Prove that $\mathbf{Z}(\delta, R) \succeq \mathbf{O}$ for every such δ, R , **without** any statement about improper integrals.
- (b) Deduce $\mathbf{A}^p \succeq \mathbf{B}^p$ from (a) and the convergence statement of [Lemma 21.10.5](#).
- (c) Explain why the same argument does not show $\mathbf{A}^2 \succeq \mathbf{B}^2$, by pointing at the step that fails.

Solution.

- (a) For each $s \in [\delta, R]$, the function $t \mapsto t/(t+s)$ is operator monotone on $(0, \infty)$ by [Corollary 17.11.5](#), and it sends $\mathbf{A}$ to $\mathbf{A}(\mathbf{A} + s\mathbf{I})^{-1}$ and $\mathbf{B}$ to $\mathbf{B}(\mathbf{B} + s\mathbf{I})^{-1}$ ([Definition 12.8.2](#)). So the bracket is $\succeq \mathbf{O}$, and $s^{p-1} > 0$ keeps it so. The integrand is continuous on the compact interval $[\delta, R]$, because each entry is a rational function of s with nonvanishing denominator times s^{p-1} . Hence [Lemma 21.10.2](#) gives $\mathbf{Z}(\delta, R) \succeq \mathbf{O}$.
- (b) By [Lemma 21.10.5](#) and the diagonalization in the proof of [Corollary 21.10.8](#), $\mathbf{Z}(\delta, R) \rightarrow K_p(\mathbf{A}^p - \mathbf{B}^p)$ entrywise as $\delta \rightarrow 0^+$ and $R \rightarrow \infty$. For each $\mathbf{x}$, the numbers $\mathbf{x}^* \mathbf{Z}(\delta, R) \mathbf{x}$ are ≥ 0 by (a), and a nonstrict inequality survives a limit, so $K_p \mathbf{x}^* (\mathbf{A}^p - \mathbf{B}^p) \mathbf{x} \geq 0$; as $K_p > 0$, $\mathbf{A}^p \succeq \mathbf{B}^p$.
- (c) The failure is at the very start. For $p = 2$ the integral $\int_0^\infty s^{p-1} \lambda/(\lambda + s) ds$ diverges at $s = \infty$: the integrand behaves like $\lambda s^{p-2} = \lambda$, which is not integrable there. There is no representation of t^2 as a positive superposition of the functions $t/(t+s)$ to begin with — and by [Corollary 21.10.8](#) there cannot be one, since t^2 is not operator monotone.