

20.2 Other Inclusion Regions

The Gershgorin discs of §01 come from one row of $\mathbf{Ax} = \lambda\mathbf{x}$, read at the largest coordinate of $\mathbf{x}$. That leaves two kinds of information unused. The first is the second largest coordinate, which the triangle inequality simply replaced by the largest; using it gives a region of ovals that is never worse than the discs. The second is the geometry of the inner product, which the discs ignore entirely: the number $\mathbf{x}^*\mathbf{Ax}$ over unit vectors $\mathbf{x}$ fills a region of the plane, the numerical range, that contains every eigenvalue. This section builds both regions. Its main theorem is that the numerical range is always convex. It ends with a rectangle for the spectrum whose sides are eigenvalues of Hermitian matrices, and with a warning about what these regions cannot do.

Throughout, matrices are complex and $n \geq 1$, and $\|\cdot\|$ on $\mathbb{C}^n$ is the Euclidean norm. As in §01, $r_i(\mathbf{A}) = \sum_{j \neq i} |a_{ij}|$ is the i -th Gershgorin radius and $D_i(\mathbf{A})$ the i -th disc (Definition 20.1.1). The plane $\mathbb{C}$ is treated as a real vector space when convexity is discussed, as Chapter 18 §01 does.

20.2.1 Using two coordinates: Brauer's ovals

In the proof of Theorem 20.1.1 every $|x_j|$ with $j \neq k$ was bounded by the largest one, $|x_k|$. That is wasteful when the second largest coordinate is much smaller. So keep two indices: p , where $|x_p|$ is largest, and $q \neq p$, where it is largest among the rest. Row p bounds $|\lambda - a_{pp}||x_p|$ by $r_p|x_q|$, since every other coordinate is at most $|x_q|$. Row q bounds $|\lambda - a_{qq}||x_q|$ by $r_q|x_p|$. Multiplying the two bounds cancels the unknown coordinates. The result is a region defined by a product of distances.

Theorem 20.2.1. Brauer's Ovals of Cassini

Let $n \geq 2$ and $\mathbf{A} \in M_n(\mathbb{C})$. For $i \neq j$ put

$$O_{ij}(\mathbf{A}) := \{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq r_i(\mathbf{A})r_j(\mathbf{A}) \}.$$

- (a) Every eigenvalue of $\mathbf{A}$ lies in $\bigcup_{i \neq j} O_{ij}(\mathbf{A})$.
- (b) $O_{ij}(\mathbf{A}) \subseteq D_i(\mathbf{A}) \cup D_j(\mathbf{A})$ for every $i \neq j$. Hence $\bigcup_{i \neq j} O_{ij}(\mathbf{A}) \subseteq \bigcup_i D_i(\mathbf{A})$.

Idea. For (a), take the two largest coordinates of an eigenvector as above and multiply the two row estimates. One case needs separate care: if the second largest coordinate is 0, there is nothing to divide by, but then $\mathbf{x}$ is a multiple of $\mathbf{e}_p$ and $\lambda = a_{pp}$, which lies in every oval through a_{pp} . For (b), a point outside both discs is farther than r_i from a_{ii} and farther than r_j from a_{jj} , so the product of its distances exceeds $r_i r_j$.

Proof.

- (a) Let $\mathbf{Ax} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. Choose p with $|x_p| = \max_j |x_j|$, which is positive, and then $q \neq p$ with $|x_q| = \max_{j \neq p} |x_j|$; this is possible because $n \geq 2$.
Case 1: $x_q = 0$. Then $x_j = 0$ for every $j \neq p$, and row p of $\mathbf{Ax} = \lambda\mathbf{x}$ reads $a_{pp}x_p = \lambda x_p$. As $x_p \neq 0$, $\lambda = a_{pp}$. The left side of the inequality defining $O_{pq}(\mathbf{A})$ is then 0, and the right side is ≥ 0 , so $\lambda \in O_{pq}(\mathbf{A})$.
Case 2: $x_q \neq 0$. Row p gives $(\lambda - a_{pp})x_p = \sum_{j \neq p} a_{pj}x_j$. Every $j \neq p$ has $|x_j| \leq |x_q|$, so

by the triangle inequality (Theorem 0.6.4)

$$|\lambda - a_{pp}| |x_p| \leq r_p(\mathbf{A}) |x_q|.$$

Row q gives $(\lambda - a_{qq})x_q = \sum_{j \neq q} a_{qj}x_j$. Every j has $|x_j| \leq |x_p|$, so

$$|\lambda - a_{qq}| |x_q| \leq r_q(\mathbf{A}) |x_p|.$$

Both sides of both inequalities are nonnegative, so they may be multiplied:

$$|\lambda - a_{pp}| |\lambda - a_{qq}| |x_p| |x_q| \leq r_p(\mathbf{A}) r_q(\mathbf{A}) |x_p| |x_q|.$$

Dividing by $|x_p| |x_q| > 0$ gives $\lambda \in O_{pq}(\mathbf{A})$.

- (b) Let $z \notin D_i(\mathbf{A}) \cup D_j(\mathbf{A})$, so $a := |z - a_{ii}| > r_i(\mathbf{A}) \geq 0$ and $c := |z - a_{jj}| > r_j(\mathbf{A}) \geq 0$. Since $c > 0$ and $r_i(\mathbf{A}) \geq 0$,

$$ac > r_i(\mathbf{A}) c \geq r_i(\mathbf{A}) r_j(\mathbf{A}),$$

so $z \notin O_{ij}(\mathbf{A})$. Taking complements, $O_{ij}(\mathbf{A}) \subseteq D_i(\mathbf{A}) \cup D_j(\mathbf{A})$, and taking the union over all pairs gives the last claim. This proves the theorem. ■

The sets $O_{ij}(\mathbf{A})$ are the **ovals of Cassini**: the points whose distances to two fixed foci have product at most a constant. Since $O_{ij}(\mathbf{A}) = O_{ji}(\mathbf{A})$, there are $n(n-1)/2$ of them to draw, against n discs. The hypothesis $n \geq 2$ is there because an oval needs two distinct indices; for $n = 1$ there are none, and the eigenvalue a_{11} must be found some other way. For $n = 2$ the single oval is as good as a region can be. The eigenvalues satisfy $(\lambda - a_{11})(\lambda - a_{22}) = a_{12}a_{21}$, so taking moduli, $|\lambda - a_{11}| |\lambda - a_{22}| = r_1 r_2$: both eigenvalues lie on the boundary curve of O_{12} .

Example. Ovals where discs fail

Let $\mathbf{A} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Show that the Gershgorin discs cannot prove $\mathbf{A}$ invertible, and that the ovals can.

Solution. The radii are $r_1 = 1$, $r_2 = 1$, $r_3 = 2$, and $D_3(\mathbf{A}) = \{|z - 1| \leq 2\}$ contains 0. So Theorem 20.1.1 (a) allows the eigenvalue 0. The column radii are 1, 2, 1, and the second column disc $|z - 4| \leq 2$ misses 0, but the third, $|z - 1| \leq 1$, contains it. So the discs cannot decide.

Now test $z = 0$ against the three ovals:

$$O_{12} : |0 - 4| |0 - 4| = 16 > 1 = r_1 r_2,$$

$$O_{13} : |0 - 4| |0 - 1| = 4 > 2 = r_1 r_3,$$

$$O_{23} : |0 - 4| |0 - 1| = 4 > 2 = r_2 r_3.$$

So 0 lies in no oval, it is not an eigenvalue by Theorem 20.2.1 (a), and $\mathbf{A}$ is invertible by Theorem 9.2.1. Indeed, expanding along the first row, $\det \mathbf{A} = 4(4 - 1) - 1(0 - 1) = 13$. The ovals also exclude the point -1 , which lies on the boundary of $D_3(\mathbf{A})$: there $|-1 - 4| |-1 - 4| =$

$25 > 1$ rules out O_{12} , and $|-1 - 4||-1 - 1| = 10 > 2$ rules out O_{13} and O_{23} .

The same test gives a criterion in the spirit of Corollary 20.1.2: if $n \geq 2$, as in Theorem 20.2.1, and $|a_{ii}||a_{jj}| > r_i(\mathbf{A})r_j(\mathbf{A})$ for **every** pair $i \neq j$, then 0 lies in no oval and $\mathbf{A}$ is invertible. One row may fail to dominate, as row 3 did above, provided every other row dominates enough to compensate.

Quick check. Why does the proof of Theorem 20.2.1 (a) choose q to maximize $|x_q|$ among the indices other than p , rather than taking any $q \neq p$?

Solution. The estimate from row p bounds every $|x_j|$, $j \neq p$, by $|x_q|$. That is true only if $|x_q|$ is the largest of them. With an arbitrary q the row- p estimate could fail, and the product of the two estimates would no longer cancel down to $r_p r_q$.

20.2.2 The numerical range

Now a different source of information. For a Hermitian $\mathbf{A}$, Chapter 17 §01 studied the Rayleigh quotient, and Proposition 17.1.1 showed that its values $\mathbf{x}^* \mathbf{A} \mathbf{x}$ on unit vectors fill exactly the interval $[\lambda_n(\mathbf{A}), \lambda_1(\mathbf{A})]$, which contains every eigenvalue. The number $\mathbf{x}^* \mathbf{A} \mathbf{x} = \langle \mathbf{A} \mathbf{x}, \mathbf{x} \rangle$ makes sense for every square matrix. Without the Hermitian hypothesis it is complex, and its values fill a region of the plane rather than an interval.

The numerical range is the set of all values $\mathbf{x}^* \mathbf{A} \mathbf{x}$ as $\mathbf{x}$ runs over the unit vectors.

Definition 20.2.1. Numerical Range

Let $\mathbf{A} \in M_n(\mathbb{C})$. The **numerical range** of $\mathbf{A}$ is the subset of $\mathbb{C}$

$$W(\mathbf{A}) := \{ \mathbf{x}^* \mathbf{A} \mathbf{x} : \mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\| = 1 \}.$$

In words: feed $\mathbf{A}$ a vector of length **exactly** 1, take the inner product of the output with the input, and collect all the complex numbers that arise. The vectors range over **complex** unit vectors, even when $\mathbf{A}$ is real, and the conjugate in $\mathbf{x}^*$ is part of the definition. $W(\mathbf{A})$ is always nonempty, since $\mathbf{e}_1$ is a unit vector, and $\mathbf{e}_i^* \mathbf{A} \mathbf{e}_i = a_{ii}$, so every diagonal entry lies in $W(\mathbf{A})$. The notation always carries its argument: $W(\mathbf{A})$, never W alone, since W also names subspaces.

Examples, simplest first.

- **$n = 1$, and scalar matrices.** For $\mathbf{A} = (a)$ the unit vectors are the scalars x with $|x| = 1$, and $\bar{x}ax = a|x|^2 = a$. So $W((a)) = \{a\}$. Likewise $W(c\mathbf{I}) = \{c\}$, since $\mathbf{x}^*(c\mathbf{I})\mathbf{x} = c\|\mathbf{x}\|^2 = c$.
- **Diagonal matrices.** For $\mathbf{A} = \text{diag}(d_1, \dots, d_n)$, $\mathbf{x}^* \mathbf{A} \mathbf{x} = \sum_i d_i |x_i|^2$. The weights $t_i = |x_i|^2$ are nonnegative with sum $\|\mathbf{x}\|^2 = 1$, and every such list of weights arises, from $x_i = \sqrt{t_i}$. So $W(\mathbf{A})$ is the set of all weighted averages of $d_1, \dots, d_n$, which is $\text{conv}\{d_1, \dots, d_n\}$ by Theorem 18.1.3. For $\text{diag}(1, i, -1)$ it is the filled triangle with these three vertices.
- **Hermitian matrices.** By Proposition 17.1.1, $W(\mathbf{A}) = [\lambda_n(\mathbf{A}), \lambda_1(\mathbf{A})]$, a segment of the real axis.
- **A Jordan block.** The next example computes $W(\mathbf{J}_2(0))$, and it is a whole disc, although the only eigenvalue is 0.

Example. The numerical range of a Jordan block

Show that $W(\mathbf{J}_2(0)) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$.

Solution. Write $\mathbf{x} = (x_1, x_2)$ with $|x_1|^2 + |x_2|^2 = 1$. Then $\mathbf{J}_2(0)\mathbf{x} = (x_2, 0)$, so $\mathbf{x}^*\mathbf{J}_2(0)\mathbf{x} = \overline{x_1}x_2$.

($\subseteq$) Since $0 \leq (|x_1| - |x_2|)^2 = 1 - 2|x_1||x_2|$, we get $|\overline{x_1}x_2| = |x_1||x_2| \leq \frac{1}{2}$.

($\supseteq$) Let $z = \rho\omega$ with $0 \leq \rho \leq \frac{1}{2}$ and $|\omega| = 1$. Since $1 - 4\rho^2 \geq 0$, we may put

$$a = \sqrt{\frac{1 + \sqrt{1 - 4\rho^2}}{2}}, \quad b = \sqrt{\frac{1 - \sqrt{1 - 4\rho^2}}{2}},$$

both real and nonnegative. Then $a^2 + b^2 = 1$ and $a^2b^2 = \frac{1}{4}(1 - (1 - 4\rho^2)) = \rho^2$, so $ab = \rho$. The unit vector $\mathbf{x} = (a, b\omega)$ gives $\mathbf{x}^*\mathbf{J}_2(0)\mathbf{x} = ab\omega = z$. Hence $W(\mathbf{J}_2(0))$ is the closed disc of radius $\frac{1}{2}$ about 0.

Nonexample by minimal change. Drop the conjugate, and collect the numbers $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ for unit $\mathbf{x}$ instead. For $\mathbf{A} = \mathbf{I}_2$ and the unit vector $\mathbf{x} = \frac{1}{\sqrt{2}}(1, i)$, $\mathbf{x}^\top \mathbf{I}_2 \mathbf{x} = \frac{1}{2}(1 + i^2) = 0$, while $\mathbf{x}^* \mathbf{I}_2 \mathbf{x} = 1$. The modified set for $\mathbf{I}_2$ contains 0, which is not an eigenvalue, while $W(\mathbf{I}_2) = \{1\}$ is exactly the spectrum. What breaks is that $\mathbf{x}^\top \mathbf{x}$ is not $\|\mathbf{x}\|^2$ over $\mathbb{C}$. The conjugate is what makes $\mathbf{x}^*(c\mathbf{I})\mathbf{x} = c$ on the unit sphere, and every property below leans on that.

Why this definition. The unit length plays the role that division by $\|\mathbf{x}\|^2$ played for the Rayleigh quotient. Without it the values $\mathbf{x}^* \mathbf{A} \mathbf{x}$ over all $\mathbf{x}$ form a set closed under multiplication by positive reals, which is all of $\mathbb{C}$ or a cone, and records nothing about the size of $\mathbf{A}$. With it, the set is bounded, and the eigenvalues are in it.

Proposition 20.2.2. Basic Properties of the Numerical Range

Let $\mathbf{A} \in M_n(\mathbb{C})$.

- (a) $\text{spec}(\mathbf{A}) \subseteq W(\mathbf{A})$.
- (b) $W(\alpha\mathbf{A} + \beta\mathbf{I}) = \alpha W(\mathbf{A}) + \beta := \{\alpha w + \beta : w \in W(\mathbf{A})\}$ for all $\alpha, \beta \in \mathbb{C}$.
- (c) $W(\mathbf{U}^* \mathbf{A} \mathbf{U}) = W(\mathbf{A})$ for every unitary $\mathbf{U} \in M_n(\mathbb{C})$.
- (d) $W(\mathbf{A})$ is compact: every sequence in $W(\mathbf{A})$ has a subsequence converging to a point of $W(\mathbf{A})$. In particular $W(\mathbf{A})$ is closed.
- (e) $W(\mathbf{A}) \subseteq \{z : |z| \leq \|\mathbf{A}\|_2\}$.

Idea. Four of the five parts are one line of algebra with a unit vector: evaluate at a unit eigenvector for (a), push the scalars or the unitary through $\mathbf{x}^*(\cdot)\mathbf{x}$ for (b) and (c), and apply Cauchy–Schwarz for (e). Only (d) needs analysis, and there the point is that $W(\mathbf{A})$ is the image of the unit sphere under a continuous map, so the compactness of the sphere passes to it.

Proof.

- (a) Let $\lambda \in \text{spec}(\mathbf{A})$, with eigenvector $\mathbf{v}$. Then $\mathbf{x} = \mathbf{v}/\|\mathbf{v}\|$ is a unit eigenvector, and $\mathbf{x}^* \mathbf{A} \mathbf{x} = \lambda \mathbf{x}^* \mathbf{x} = \lambda$.
- (b) For a unit $\mathbf{x}$, $\mathbf{x}^*(\alpha \mathbf{A} + \beta \mathbf{I})\mathbf{x} = \alpha \mathbf{x}^* \mathbf{A} \mathbf{x} + \beta \mathbf{x}^* \mathbf{x} = \alpha \mathbf{x}^* \mathbf{A} \mathbf{x} + \beta$. As $\mathbf{x}$ runs over the unit vectors, the left side runs over $W(\alpha \mathbf{A} + \beta \mathbf{I})$ and the right side over $\alpha W(\mathbf{A}) + \beta$.
- (c) For a unit $\mathbf{x}$, $\mathbf{x}^* \mathbf{U}^* \mathbf{A} \mathbf{U} \mathbf{x} = (\mathbf{U} \mathbf{x})^* \mathbf{A} (\mathbf{U} \mathbf{x})$, and $\mathbf{U} \mathbf{x}$ is a unit vector because $\mathbf{U}$ is an isometry (Theorem 11.7.3). So $W(\mathbf{U}^* \mathbf{A} \mathbf{U}) \subseteq W(\mathbf{A})$. Conversely, every unit $\mathbf{y}$ equals $\mathbf{U} \mathbf{x}$ for the unit vector $\mathbf{x} = \mathbf{U}^* \mathbf{y}$, so every value $\mathbf{y}^* \mathbf{A} \mathbf{y}$ arises, and the two sets are equal.
- (d) First, $f(\mathbf{x}) = \mathbf{x}^* \mathbf{A} \mathbf{x}$ is continuous on the unit sphere. For unit vectors $\mathbf{x}, \mathbf{y}$,

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{y})| &= |\mathbf{x}^* \mathbf{A} (\mathbf{x} - \mathbf{y}) + (\mathbf{x} - \mathbf{y})^* \mathbf{A} \mathbf{y}| \\ &\leq 2 \|\mathbf{A}\|_2 \|\mathbf{x} - \mathbf{y}\|, \end{aligned}$$

by the triangle inequality (Theorem 0.6.4), the Cauchy–Schwarz inequality (Theorem 11.1.5), and Theorem 16.3.2 (a). This is a bound of the kind fixed as the book’s continuity test in Chapter 16’s introduction. Now let (w_k) be a sequence in $W(\mathbf{A})$, with $w_k = f(\mathbf{x}_k)$ and $\|\mathbf{x}_k\| = 1$. By **the compactness of the unit sphere, fact (A3) of Chapter 16’s introduction**, a subsequence $(\mathbf{x}_{k_l})$ converges to a unit vector $\mathbf{x}$. By continuity, $w_{k_l} = f(\mathbf{x}_{k_l}) \rightarrow f(\mathbf{x}) \in W(\mathbf{A})$. If (w_k) itself converges to w , its subsequence converges to w too, and limits are unique, so $w = f(\mathbf{x}) \in W(\mathbf{A})$: the set is closed.

- (e) For a unit $\mathbf{x}$, by the Cauchy–Schwarz inequality and Theorem 16.3.2 (a), $|\mathbf{x}^* \mathbf{A} \mathbf{x}| = |\langle \mathbf{A} \mathbf{x}, \mathbf{x} \rangle| \leq \|\mathbf{A} \mathbf{x}\| \|\mathbf{x}\| \leq \|\mathbf{A}\|_2$.

■

Part (a) makes $W(\mathbf{A})$ an inclusion region for the spectrum, and (e) puts it inside the disc of radius $\|\mathbf{A}\|_2$, which upgrades $\rho(\mathbf{A}) \leq \|\mathbf{A}\|_2$ from the spectrum to the whole numerical range. Part (b) is the tool that lets us move any two points of $W(\mathbf{A})$ to 0 and 1, and the next proof depends on it. Part (c) says $W(\mathbf{A})$ depends only on the operator $T_{\mathbf{A}}$ and the inner product, not on the orthonormal basis used to write it down.

20.2.3 Convexity: the Toeplitz–Hausdorff theorem

The examples so far were all convex: a point, a segment, a triangle, a disc. That is not a coincidence. To prove it, split $\mathbf{A}$ into two Hermitian pieces, the way a complex number splits into its real and imaginary parts.

Lemma 20.2.3. Real and Imaginary Parts of a Matrix

Let $\mathbf{A} \in M_n(\mathbb{C})$, and put

$$\mathbf{H} = \frac{\mathbf{A} + \mathbf{A}^*}{2}, \quad \mathbf{K} = \frac{\mathbf{A} - \mathbf{A}^*}{2i}.$$

Then $\mathbf{H}$ and $\mathbf{K}$ are Hermitian, $\mathbf{A} = \mathbf{H} + i\mathbf{K}$, and for every $\mathbf{x} \in \mathbb{C}^n$ the numbers $\mathbf{x}^* \mathbf{H} \mathbf{x}$ and

$\mathbf{x}^* \mathbf{K} \mathbf{x}$ are real, with

$$\operatorname{Re}(\mathbf{x}^* \mathbf{A} \mathbf{x}) = \mathbf{x}^* \mathbf{H} \mathbf{x}, \quad \operatorname{Im}(\mathbf{x}^* \mathbf{A} \mathbf{x}) = \mathbf{x}^* \mathbf{K} \mathbf{x}.$$

Proof. By Definition 11.1.2, $(\mathbf{A}^*)^* = \mathbf{A}$, $(\mathbf{B} + \mathbf{C})^* = \mathbf{B}^* + \mathbf{C}^*$, and $(c\mathbf{B})^* = \bar{c}\mathbf{B}^*$. Hence $\mathbf{H}^* = \frac{1}{2}(\mathbf{A}^* + \mathbf{A}) = \mathbf{H}$ and $\mathbf{K}^* = \frac{1}{-2i}(\mathbf{A}^* - \mathbf{A}) = \frac{1}{2i}(\mathbf{A} - \mathbf{A}^*) = \mathbf{K}$. Also $\mathbf{H} + i\mathbf{K} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^*) = \mathbf{A}$. For $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* \mathbf{H} \mathbf{x} = \langle \mathbf{H} \mathbf{x}, \mathbf{x} \rangle$ is real by Proposition 11.6.7 (a), applied to the self-adjoint operator $T_{\mathbf{H}}$, and likewise $\mathbf{x}^* \mathbf{K} \mathbf{x}$. So $\mathbf{x}^* \mathbf{A} \mathbf{x} = \mathbf{x}^* \mathbf{H} \mathbf{x} + i \mathbf{x}^* \mathbf{K} \mathbf{x}$ is the splitting of a complex number into real and imaginary parts. ■

For example $\mathbf{J}_2(0)$ has $\mathbf{H} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{K} = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, both with eigenvalues $\pm \frac{1}{2}$. A Hermitian $\mathbf{A}$ has $\mathbf{K} = \mathbf{O}$, and then $W(\mathbf{A})$ lies on the real axis.

Theorem 20.2.4. Toeplitz–Hausdorff Theorem

For every $\mathbf{A} \in M_n(\mathbb{C})$, the numerical range $W(\mathbf{A})$ is a convex subset of $\mathbb{C}$.

Idea. Step roadmap. ① By Proposition 20.2.2 (b), an affine change $z \mapsto \alpha z + \beta$ carries any two points of $W(\mathbf{A})$ to 0 and 1, and carries segments to segments; so it suffices to show that $[0, 1] \subseteq W(\mathbf{B})$ whenever $0, 1 \in W(\mathbf{B})$. ② Take unit vectors $\mathbf{x}, \mathbf{y}$ with $\mathbf{x}^* \mathbf{B} \mathbf{x} = 0$ and $\mathbf{y}^* \mathbf{B} \mathbf{y} = 1$, and walk from $\mathbf{x}$ to $\mathbf{y}$ along the straight path $(1-t)\mathbf{x} + t\mathbf{y}$. Along the way the value of $\mathbf{B}$, normalized, moves continuously from 0 to 1, but it may wander off the real axis. ③ Its imaginary part along the path is $\mathbf{K}$ evaluated there, and expanding shows that only the cross term $\mathbf{x}^* \mathbf{K} \mathbf{y} + \mathbf{y}^* \mathbf{K} \mathbf{x} = 2 \operatorname{Re}(\mathbf{x}^* \mathbf{K} \mathbf{y})$ can be nonzero. Multiplying $\mathbf{y}$ by a unimodular scalar rotates $\mathbf{x}^* \mathbf{K} \mathbf{y}$ freely without changing $\mathbf{y}^* \mathbf{B} \mathbf{y}$, so we rotate it onto the imaginary axis, where its real part is 0. ④ Then the path stays on the real axis, and the intermediate value theorem fills in $[0, 1]$.

Proof. Step 1: reduction. Let $w_0, w_1 \in W(\mathbf{A})$ and $0 \leq s \leq 1$; we must show $(1-s)w_0 + sw_1 \in W(\mathbf{A})$. If $w_0 = w_1$ there is nothing to prove. Otherwise put $\alpha = 1/(w_1 - w_0)$, $\beta = -\alpha w_0$, and $\mathbf{B} = \alpha \mathbf{A} + \beta \mathbf{I}$. By Proposition 20.2.2 (b), $W(\mathbf{B}) = \alpha W(\mathbf{A}) + \beta$, which contains $\alpha w_0 + \beta = 0$ and $\alpha w_1 + \beta = 1$. Suppose we show $s \in W(\mathbf{B})$. Then $s = \alpha w + \beta$ for some $w \in W(\mathbf{A})$, and solving, $w = w_0 + s(w_1 - w_0) = (1-s)w_0 + sw_1$. So it suffices to prove: if $0, 1 \in W(\mathbf{B})$, then $[0, 1] \subseteq W(\mathbf{B})$.

Step 2: choosing the vectors. Let $\mathbf{B} = \mathbf{H} + i\mathbf{K}$ as in Lemma 20.2.3, and choose unit vectors $\mathbf{x}, \mathbf{y}_0$ with $\mathbf{x}^* \mathbf{B} \mathbf{x} = 0$ and $\mathbf{y}_0^* \mathbf{B} \mathbf{y}_0 = 1$. By Lemma 20.2.3, $\mathbf{x}^* \mathbf{H} \mathbf{x} = \mathbf{x}^* \mathbf{K} \mathbf{x} = 0$, $\mathbf{y}_0^* \mathbf{H} \mathbf{y}_0 = 1$, and $\mathbf{y}_0^* \mathbf{K} \mathbf{y}_0 = 0$. Let $c = \mathbf{x}^* \mathbf{K} \mathbf{y}_0$, and put $\gamma = 1$ if $c = 0$, and $\gamma = i\bar{c}/|c|$ otherwise. In both cases $|\gamma| = 1$ and $\gamma c = i|c|$ is purely imaginary. Set $\mathbf{y} = \gamma \mathbf{y}_0$. Then $\|\mathbf{y}\| = 1$ and $\mathbf{y}^* \mathbf{B} \mathbf{y} = |\gamma|^2 \mathbf{y}_0^* \mathbf{B} \mathbf{y}_0 = 1$, so still $\mathbf{y}^* \mathbf{H} \mathbf{y} = 1$ and $\mathbf{y}^* \mathbf{K} \mathbf{y} = 0$. Since $\mathbf{K}$ is Hermitian, $\mathbf{y}^* \mathbf{K} \mathbf{x} = (\mathbf{x}^* \mathbf{K} \mathbf{y})^* = \overline{\mathbf{x}^* \mathbf{K} \mathbf{y}}$, and so

$$\mathbf{x}^* \mathbf{K} \mathbf{y} + \mathbf{y}^* \mathbf{K} \mathbf{x} = 2 \operatorname{Re}(\mathbf{x}^* \mathbf{K} \mathbf{y}) = 2 \operatorname{Re}(\gamma c) = 0.$$

Finally, $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Otherwise, both being nonzero, $\mathbf{y} = \mu \mathbf{x}$ for a scalar μ , and then $1 = \mathbf{y}^* \mathbf{B} \mathbf{y} = |\mu|^2 \mathbf{x}^* \mathbf{B} \mathbf{x} = 0$, which is absurd.

Step 3: the path stays real. For $t \in [0, 1]$ put $\mathbf{v}(t) = (1-t)\mathbf{x} + t\mathbf{y}$. The coefficients $1-t$ and t are not both zero, so $\mathbf{v}(t) \neq \mathbf{0}$ by independence. Expanding, and using $\mathbf{x}^*\mathbf{K}\mathbf{x} = \mathbf{y}^*\mathbf{K}\mathbf{y} = 0$ and Step 2,

$$\mathbf{v}(t)^*\mathbf{K}\mathbf{v}(t) = (1-t)^2\mathbf{x}^*\mathbf{K}\mathbf{x} + t^2\mathbf{y}^*\mathbf{K}\mathbf{y} + t(1-t)(\mathbf{x}^*\mathbf{K}\mathbf{y} + \mathbf{y}^*\mathbf{K}\mathbf{x}) = 0.$$

So $\mathbf{v}(t)^*\mathbf{B}\mathbf{v}(t) = \mathbf{v}(t)^*\mathbf{H}\mathbf{v}(t)$ is real for every t . The same expansion with $\mathbf{H}$ in place of $\mathbf{K}$, and with $\|\mathbf{v}(t)\|^2 = \mathbf{v}(t)^*\mathbf{v}(t)$, gives

$$\begin{aligned}\mathbf{v}(t)^*\mathbf{H}\mathbf{v}(t) &= t^2 + 2t(1-t)\operatorname{Re}(\mathbf{x}^*\mathbf{H}\mathbf{y}), \\ \|\mathbf{v}(t)\|^2 &= (1-t)^2 + t^2 + 2t(1-t)\operatorname{Re}(\mathbf{x}^*\mathbf{y}),\end{aligned}$$

two polynomials in t with real coefficients, the second positive on $[0, 1]$.

Step 4: filling the segment. The function

$$f(t) = \frac{\mathbf{v}(t)^*\mathbf{H}\mathbf{v}(t)}{\|\mathbf{v}(t)\|^2}, \quad 0 \leq t \leq 1,$$

is a quotient of polynomials with a denominator that never vanishes, hence continuous by the algebra of limits, and $f(0) = \mathbf{x}^*\mathbf{H}\mathbf{x} = 0$, $f(1) = \mathbf{y}^*\mathbf{H}\mathbf{y} = 1$. Let $s \in [0, 1]$. By **the intermediate value theorem, fact (A5) of Chapter 16's introduction**, there is $t_s \in [0, 1]$ with $f(t_s) = s$. The vector $\mathbf{u} = \mathbf{v}(t_s)/\|\mathbf{v}(t_s)\|$ is a unit vector, and by Step 3,

$$\mathbf{u}^*\mathbf{B}\mathbf{u} = \frac{\mathbf{v}(t_s)^*\mathbf{B}\mathbf{v}(t_s)}{\|\mathbf{v}(t_s)\|^2} = \frac{\mathbf{v}(t_s)^*\mathbf{H}\mathbf{v}(t_s)}{\|\mathbf{v}(t_s)\|^2} = f(t_s) = s.$$

So $s \in W(\mathbf{B})$. By Step 1, $W(\mathbf{A})$ contains the segment between any two of its points, which is Definition 18.1.1. This proves the theorem. ■

The phase choice in Step 2 is the heart of the proof. Without it the imaginary part along the path is $2t(1-t)\operatorname{Re}(\mathbf{x}^*\mathbf{K}\mathbf{y}_0)/\|\mathbf{v}(t)\|^2$, which may be nonzero, and the path would leave the real axis between 0 and 1. The rotated vector $\gamma\mathbf{y}_0$ spans the same line as $\mathbf{y}_0$ and gives the same value of $\mathbf{B}$, so the rotation costs nothing. Only the path between $\mathbf{x}$ and $\mathbf{y}$ changes.

With convexity, $W(\mathbf{A})$ contains the convex hull of $\operatorname{spec}(\mathbf{A})$, by (a) and Definition 18.1.4, and it contains the diagonal entries, so it contains $\operatorname{conv}\{a_{11}, \dots, a_{nn}\}$ as well.

20.2.4 Normal matrices

For a diagonal matrix the numerical range is the convex hull of the diagonal entries, and those are the eigenvalues. Unitary similarity does not change W , and a normal matrix is unitarily similar to a diagonal one. So for a normal matrix $W(\mathbf{A})$ is as small as (a) and convexity allow.

Proposition 20.2.5. The Numerical Range of a Normal Matrix

Let $\mathbf{A} \in M_n(\mathbb{C})$ be normal, with eigenvalues $\lambda_1, \dots, \lambda_n$, listed with multiplicity. Then

$$W(\mathbf{A}) = \operatorname{conv}\{\lambda_1, \dots, \lambda_n\}.$$

Idea. Unitary similarity leaves W alone, by Proposition 20.2.2 (c), so we may replace $\mathbf{A}$ by a diagonal matrix of its eigenvalues. Then $\mathbf{x}^* \mathbf{A} \mathbf{x}$ is the average of the λ_i with weights $|x_i|^2$, and the two inclusions say exactly that the lists of admissible weights and of convex coefficients are the same.

Proof. By Corollary 12.4.3, $\mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{U}^*$ with $\mathbf{U}$ unitary and $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$. So $\mathbf{U}^* \mathbf{A} \mathbf{U} = \mathbf{D}$, and $W(\mathbf{A}) = W(\mathbf{D})$ by Proposition 20.2.2 (c). For a unit $\mathbf{x}$, $\mathbf{x}^* \mathbf{D} \mathbf{x} = \sum_i |x_i|^2 \lambda_i$.

($\subseteq$) The weights $|x_i|^2$ are nonnegative with sum 1, so $\mathbf{x}^* \mathbf{D} \mathbf{x}$ is a convex combination of $\lambda_1, \dots, \lambda_n$, and lies in their convex hull by Theorem 18.1.3.

($\supseteq$) By Theorem 18.1.3, a point of the hull is $\sum_{k=1}^m t_k \mu_k$ with $t_k \geq 0$, $\sum_k t_k = 1$, and each μ_k one of the λ_i . Assign each k to one index i with $\lambda_i = \mu_k$, and let s_i be the sum of the t_k assigned to i . Then $s_i \geq 0$, $\sum_i s_i = 1$, and the point is $\sum_i s_i \lambda_i$. The unit vector with $x_i = \sqrt{s_i}$ gives $\mathbf{x}^* \mathbf{D} \mathbf{x} = \sum_i s_i \lambda_i$. ■

For a Hermitian matrix this recovers Proposition 17.1.1. For a unitary matrix, whose eigenvalues lie on the unit circle, $W(\mathbf{A})$ is the convex hull of those eigenvalues: a polygon inscribed in the circle when three or more of them are distinct, and a chord or a single point when there are two or one. For a nonnormal matrix $W(\mathbf{A})$ can be much larger than $\text{conv spec}(\mathbf{A})$: by Example (The numerical range of a Jordan block), $W(\mathbf{J}_2(0))$ is a disc of radius $\frac{1}{2}$ around the single eigenvalue 0. The extra room is a measure of nonnormality, and Exercise C3 turns it into a comparison with $\|\mathbf{A}\|_2$.

Quick check. Find $W(\mathbf{A})$ for the rotation $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, regarded as a complex matrix.

Solution. $\mathbf{A}^\top \mathbf{A} = \mathbf{I}$, so $\mathbf{A}$ is real orthogonal, hence unitary and normal. Its characteristic polynomial is $x^2 + 1$, with roots $\pm i$. By Proposition 20.2.5, $W(\mathbf{A})$ is the segment from $-i$ to i on the imaginary axis. For real unit vectors $\mathbf{x}$ the value $\mathbf{x}^* \mathbf{A} \mathbf{x}$ is 0, which is why the real Rayleigh-type quotient of Chapter 17 §01 saw nothing; the complex unit vectors fill in the rest.

20.2.5 Bendixson's rectangle

By Lemma 20.2.3, the real part of $\mathbf{x}^* \mathbf{A} \mathbf{x}$ is a value of the Hermitian form of $\mathbf{H}$, and Proposition 17.1.1 confines those values to $[\lambda_n(\mathbf{H}), \lambda_1(\mathbf{H})]$. The same holds for the imaginary part and $\mathbf{K}$. Together they box in the whole numerical range, hence the spectrum.

Theorem 20.2.6. Bendixson's Theorem

Let $\mathbf{A} \in M_n(\mathbb{C})$, with $\mathbf{H} = (\mathbf{A} + \mathbf{A}^*)/2$ and $\mathbf{K} = (\mathbf{A} - \mathbf{A}^*)/(2i)$. Every $w \in W(\mathbf{A})$ satisfies

$$\begin{aligned}\lambda_n(\mathbf{H}) &\leq \operatorname{Re} w \leq \lambda_1(\mathbf{H}), \\ \lambda_n(\mathbf{K}) &\leq \operatorname{Im} w \leq \lambda_1(\mathbf{K}).\end{aligned}$$

In particular every eigenvalue λ of $\mathbf{A}$ satisfies the same inequalities with λ in place of w .

Proof. Let $w = \mathbf{x}^* \mathbf{A} \mathbf{x}$ with $\|\mathbf{x}\| = 1$. By Lemma 20.2.3, $\operatorname{Re} w = \mathbf{x}^* \mathbf{H} \mathbf{x} = \langle \mathbf{H} \mathbf{x}, \mathbf{x} \rangle$ and $\operatorname{Im} w = \mathbf{x}^* \mathbf{K} \mathbf{x}$, and $\mathbf{H}, \mathbf{K}$ are Hermitian. By Lemma 13.5.2, $\lambda_n(\mathbf{H}) \leq \langle \mathbf{H} \mathbf{x}, \mathbf{x} \rangle \leq \lambda_1(\mathbf{H})$, and likewise for $\mathbf{K}$. Every eigenvalue lies in $W(\mathbf{A})$ by Proposition 20.2.2 (a). ■

The most used consequence concerns the sign of the real parts. If $\mathbf{A} + \mathbf{A}^* \succ \mathbf{O}$, then $\lambda_n(\mathbf{H}) > 0$, so every eigenvalue of $\mathbf{A}$ has positive real part, and in particular $\mathbf{A}$ is invertible. This needs nothing about the size of the off-diagonal entries, only about their symmetric part.

Example. A rectangle beats the discs

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$. Show that every eigenvalue of $\mathbf{A}$ has positive real part, using Theorem 20.2.6, and check that the Gershgorin discs cannot show it. Compare with the eigenvalues.

Solution. Here $\mathbf{A}^* = \mathbf{A}^\top$, so

$$\mathbf{H} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{K} = \frac{1}{2i} \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3i \\ 3i & 0 \end{pmatrix}.$$

$\mathbf{H}$ has trace 4 and determinant 2, so $p_{\mathbf{H}}(x) = x^2 - 4x + 2$, with roots $2 \pm \sqrt{2}$. $\mathbf{K}$ has trace 0 and determinant $0 - (-3i)(3i) = -9$, so its eigenvalues are ± 3 . By Theorem 20.2.6, every eigenvalue lies in the rectangle

$$2 - \sqrt{2} \leq \operatorname{Re} \lambda \leq 2 + \sqrt{2}, \quad -3 \leq \operatorname{Im} \lambda \leq 3,$$

and in particular $\operatorname{Re} \lambda \geq 2 - \sqrt{2} > 0$.

The row discs are $|z - 3| \leq 2$ and $|z - 1| \leq 4$, and the second contains -3 . The column discs are $|z - 3| \leq 4$ and $|z - 1| \leq 2$, and the first contains -1 . So neither union rules out an eigenvalue with negative real part.

Directly, $p_{\mathbf{A}}(x) = x^2 - 4x + 11$, so $\lambda = 2 \pm i\sqrt{7}$. The real part 2 lies in $[2 - \sqrt{2}, 2 + \sqrt{2}]$, and the imaginary parts $\pm\sqrt{7} \approx \pm 2.65$ lie in $[-3, 3]$.

20.2.6 What these regions do not do

Every region in this section and the last is an **inclusion region**: it contains the spectrum. None of them is a perturbation bound in the sense of the rest of this chapter, which asks how far the eigenvalues move when $\mathbf{A}$ changes by a small $\mathbf{E}$. The numerical range does give a statement of

that kind. For a unit $\mathbf{x}$, $\mathbf{x}^*(\mathbf{A} + \mathbf{E})\mathbf{x} = \mathbf{x}^*\mathbf{A}\mathbf{x} + \mathbf{x}^*\mathbf{E}\mathbf{x}$, and $|\mathbf{x}^*\mathbf{E}\mathbf{x}| \leq \|\mathbf{E}\|_2$ by [Proposition 20.2.2](#) (e). Hence

$$W(\mathbf{A} + \mathbf{E}) \subseteq W(\mathbf{A}) + W(\mathbf{E}) \subseteq \{w + e : w \in W(\mathbf{A}), |e| \leq \|\mathbf{E}\|_2\},$$

and so every eigenvalue of $\mathbf{A} + \mathbf{E}$ lies within $\|\mathbf{E}\|_2$ of $W(\mathbf{A})$. This is only as good as $W(\mathbf{A})$ is small.

Warning. $W(\mathbf{A})$ can be far larger than $\text{spec}(\mathbf{A})$, so “within $\|\mathbf{E}\|_2$ of $W(\mathbf{A})$ ” is not “within $\|\mathbf{E}\|_2$ of an eigenvalue”. For $\mathbf{A} = M\mathbf{J}_2(0)$ with $M > 0$, [Proposition 20.2.2](#) (b) and [Example \(The numerical range of a Jordan block\)](#) give $W(\mathbf{A}) = \{|z| \leq M/2\}$, while $\text{spec}(\mathbf{A}) = \{0\}$. So the bound above allows the eigenvalues of $\mathbf{A} + \mathbf{E}$ to lie anywhere in $|z| \leq M/2 + \|\mathbf{E}\|_2$, and this does not shrink as $\mathbf{E} \rightarrow \mathbf{O}$. The eigenvalues do move a long way: $\mathbf{A} + \varepsilon\mathbf{e}_2\mathbf{e}_1^\top$ has characteristic polynomial $x^2 - M\varepsilon$, so for $\varepsilon > 0$ its eigenvalues $\pm\sqrt{M\varepsilon}$ have moved by $\sqrt{M\varepsilon}$, far more than ε . For a normal $\mathbf{A}$, $W(\mathbf{A}) = \text{conv spec}(\mathbf{A})$ is small, but it still contains points far from every eigenvalue, such as the midpoint of two distant ones.

The genuine perturbation bounds need a different mechanism: §03 proves that eigenvalues move continuously and at what rate in general, and §04 gives Bauer–Fike’s bound, in which the distance to the nearest **eigenvalue** is controlled.

20.2.7 Exercises

20.2.7.1 A. Check your understanding

Exercise. A1

- Define the Brauer oval $O_{ij}(\mathbf{A})$ and the numerical range $W(\mathbf{A})$, and state the Toeplitz–Hausdorff theorem.
- True or false: $W(\mathbf{A}) = \text{conv spec}(\mathbf{A})$ for every $\mathbf{A} \in M_n(\mathbb{C})$. Justify your answer.
- True or false: $W(\mathbf{A})$ is contained in the union of the Gershgorin discs of $\mathbf{A}$. Justify your answer.
- True or false: if $\mathbf{A} + \mathbf{A}^* \succ \mathbf{O}$, then $\mathbf{A}$ is invertible. Justify your answer.
- Why does [Theorem 20.2.1](#) require $n \geq 2$?

Solution.

- For $i \neq j$, $O_{ij}(\mathbf{A}) = \{z : |z - a_{ii}||z - a_{jj}| \leq r_i(\mathbf{A})r_j(\mathbf{A})\}$, and $W(\mathbf{A}) = \{\mathbf{x}^*\mathbf{A}\mathbf{x} : \|\mathbf{x}\| = 1\}$. [Theorem 20.2.4](#): for every square complex $\mathbf{A}$, $W(\mathbf{A})$ is convex.
- False. $W(\mathbf{J}_2(0))$ is the disc $|z| \leq \frac{1}{2}$ ([Example \(The numerical range of a Jordan block\)](#)), while $\text{conv spec}(\mathbf{J}_2(0)) = \{0\}$. Equality holds for normal matrices ([Proposition 20.2.5](#)).
- False. For $\mathbf{A} = \text{diag}(1, -1)$ the discs are the two points $\{1\}$ and $\{-1\}$, while $W(\mathbf{A}) = [-1, 1]$ contains 0. The discs contain the spectrum, not the numerical range.
- True. By [Theorem 20.2.6](#) every eigenvalue has $\text{Re } \lambda \geq \lambda_n(\mathbf{H}) > 0$, with $\mathbf{H} = (\mathbf{A} + \mathbf{A}^*)/2 \succ \mathbf{O}$. So 0 is not an eigenvalue, and $\mathbf{A}$ is invertible ([Theorem 9.2.1](#)).
- An oval is built from two distinct indices $i \neq j$, and the proof needs a second index $q \neq p$. For $n = 1$ there are no ovals, and the union in (a) would be empty, although

the matrix has an eigenvalue.

20.2.7.2 B. Practice

Exercise. B1: Invertibility from ovals

Let $\mathbf{A} = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

- (a) Show that neither the row discs nor the column discs of $\mathbf{A}$ exclude 0.
- (b) Use [Theorem 20.2.1](#) to show that $\mathbf{A}$ is invertible.
- (c) Check that $(1, -1, 0)$ is an eigenvector, and verify that its eigenvalue lies in $O_{12}(\mathbf{A})$.

Solution.

- (a) The row radii are 2, 2, 4, and $D_3(\mathbf{A}) = \{|z - 2| \leq 4\}$ contains 0. The column radii are 3, 3, 2, and the third column disc $|z - 2| \leq 2$ contains 0.
- (b) At $z = 0$: for O_{12} , $5 \cdot 5 = 25 > 4 = r_1 r_2$; for O_{13} and O_{23} , $5 \cdot 2 = 10 > 8 = r_1 r_3 = r_2 r_3$. So 0 lies in no oval, is not an eigenvalue by [Theorem 20.2.1](#) (a), and $\mathbf{A}$ is invertible. (Its determinant is 32.)
- (c) $\mathbf{A}(1, -1, 0) = (5 - 1, 1 - 5, 2 - 2) = (4, -4, 0)$, so the eigenvalue is 4. For O_{12} : $|4 - 5||4 - 5| = 1 \leq 4$.

Exercise. B2: A Bendixson rectangle

Let $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$. Find the rectangle of [Theorem 20.2.6](#), and hence show that every eigenvalue of $\mathbf{A}$ satisfies $0 \leq \operatorname{Re} \lambda \leq 2$ and $|\operatorname{Im} \lambda| \leq 2$. Compare with the eigenvalues.

Solution. $\mathbf{H} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, with $p_{\mathbf{H}}(x) = x^2 - 2x$, so its eigenvalues are 0 and 2. $\mathbf{K} = \frac{1}{2i}(\mathbf{A} - \mathbf{A}^\top) = \frac{1}{2i} \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$, with trace 0 and determinant -4 , so its eigenvalues are ± 2 . By [Theorem 20.2.6](#) every eigenvalue has $0 \leq \operatorname{Re} \lambda \leq 2$ and $-2 \leq \operatorname{Im} \lambda \leq 2$. Directly, $p_{\mathbf{A}}(x) = x^2 - 2x + 4$, so $\lambda = 1 \pm i\sqrt{3}$: the real part 1 lies in $[0, 2]$, and $\sqrt{3} \approx 1.73 \leq 2$.

Exercise. B3: Computing numerical ranges

Determine $W(\mathbf{A})$ for each matrix. Justify your answers.

- (a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
- (b) $\mathbf{A} = \operatorname{diag}(1, i, -1, -i)$
- (c) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Solution.

- (a) $\mathbf{A} = 2\mathbf{J}_2(0) + \mathbf{I}$, so by [Proposition 20.2.2](#) (b) and [Example \(The numerical range of a Jordan block\)](#), $W(\mathbf{A}) = 2W(\mathbf{J}_2(0)) + 1 = \{z : |z - 1| \leq 1\}$, the closed disc of radius 1 about 1.
- (b) A diagonal matrix is normal, so by [Proposition 20.2.5](#) $W(\mathbf{A}) = \text{conv}\{1, i, -1, -i\}$, the filled square with these four vertices.
- (c) $\mathbf{A}$ is real symmetric, with eigenvalues 3 and 1 (eigenvectors $(1, 1)$ and $(1, -1)$). By [Proposition 17.1.1](#), or [Proposition 20.2.5](#), $W(\mathbf{A}) = [1, 3]$.

20.2.7.3 C. Going deeper

Exercise. C1: Direct sums

- (a) Let $\mathbf{B} \in M_k(\mathbb{C})$ and $\mathbf{C} \in M_m(\mathbb{C})$. Prove that $W(\mathbf{B} \oplus \mathbf{C}) = \text{conv}(W(\mathbf{B}) \cup W(\mathbf{C}))$.
- (b) Deduce that $\mathbf{A} = \text{diag}(2, -2, 2i, -2i) \oplus \mathbf{J}_2(0)$ is not normal, but satisfies $W(\mathbf{A}) = \text{conv spec}(\mathbf{A})$.

Hint for (a): one inclusion needs [Theorem 20.2.4](#).

Solution.

- (a) ($\subseteq$) Write a unit vector of $\mathbb{C}^{k+m}$ as $\mathbf{x} = (\mathbf{y}, \mathbf{z})$, with $\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2 = 1$. Then $\mathbf{x}^*(\mathbf{B} \oplus \mathbf{C})\mathbf{x} = \mathbf{y}^*\mathbf{B}\mathbf{y} + \mathbf{z}^*\mathbf{C}\mathbf{z}$. If $\mathbf{z} = \mathbf{0}$ this is a point of $W(\mathbf{B})$, and if $\mathbf{y} = \mathbf{0}$ a point of $W(\mathbf{C})$. Otherwise, with $s = \|\mathbf{y}\|^2 \in (0, 1)$,

$$\mathbf{x}^*(\mathbf{B} \oplus \mathbf{C})\mathbf{x} = s \frac{\mathbf{y}^*\mathbf{B}\mathbf{y}}{\|\mathbf{y}\|^2} + (1 - s) \frac{\mathbf{z}^*\mathbf{C}\mathbf{z}}{\|\mathbf{z}\|^2},$$

a convex combination of a point of $W(\mathbf{B})$, at the unit vector $\mathbf{y}/\|\mathbf{y}\|$, and a point of $W(\mathbf{C})$. In every case the value lies in $\text{conv}(W(\mathbf{B}) \cup W(\mathbf{C}))$, by [Theorem 18.1.3](#).

($\supseteq$) Taking $\mathbf{x} = (\mathbf{y}, \mathbf{0})$ with $\mathbf{y}$ a unit vector shows $W(\mathbf{B}) \subseteq W(\mathbf{B} \oplus \mathbf{C})$, and likewise $W(\mathbf{C}) \subseteq W(\mathbf{B} \oplus \mathbf{C})$. By [Theorem 20.2.4](#), $W(\mathbf{B} \oplus \mathbf{C})$ is convex, so it contains the smallest convex set containing $W(\mathbf{B}) \cup W(\mathbf{C})$ ([Definition 18.1.4](#)).

- (b) $\mathbf{A}$ is not normal: its lower-right block is $\mathbf{J}_2(0)$, and $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ differ in that block, since $\mathbf{J}_2(0)^*\mathbf{J}_2(0) = \text{diag}(0, 1)$ and $\mathbf{J}_2(0)\mathbf{J}_2(0)^* = \text{diag}(1, 0)$. By (a), [Proposition 20.2.5](#), and [Example \(The numerical range of a Jordan block\)](#), $W(\mathbf{A})$ is the convex hull of the square $Q = \text{conv}\{\pm 2, \pm 2i\}$ and the disc $\{|z| \leq \frac{1}{2}\}$. The square contains the disc: $Q = \{z : |\text{Re } z| + |\text{Im } z| \leq 2\}$, and $|\text{Re } z| + |\text{Im } z| \leq 2|z| \leq 1$ on the disc. So $W(\mathbf{A}) = Q$. The spectrum is $\{\pm 2, \pm 2i, 0\}$, whose hull is also Q , since $0 \in Q$. Hence $W(\mathbf{A}) = \text{conv spec}(\mathbf{A})$, although $\mathbf{A}$ is not normal: the converse of [Proposition 20.2.5](#) fails.

Exercise. C2: When the numerical range is real, or a point

Let $\mathbf{A} \in M_n(\mathbb{C})$.

- (a) Prove that $W(\mathbf{A}) \subseteq \mathbb{R}$ if and only if $\mathbf{A}$ is Hermitian.

(b) Deduce that $W(\mathbf{A}) = \{c\}$ for a single $c \in \mathbb{C}$ if and only if $\mathbf{A} = c\mathbf{I}$.

Solution.

- (a) ($\Leftarrow$) If $\mathbf{A}$ is Hermitian, $\mathbf{x}^* \mathbf{A} \mathbf{x} = \langle \mathbf{A} \mathbf{x}, \mathbf{x} \rangle$ is real by Proposition 11.6.7 (a). ($\Rightarrow$) Write $\mathbf{A} = \mathbf{H} + i\mathbf{K}$ as in Lemma 20.2.3. If $W(\mathbf{A}) \subseteq \mathbb{R}$, then $\mathbf{x}^* \mathbf{K} \mathbf{x} = \text{Im}(\mathbf{x}^* \mathbf{A} \mathbf{x}) = 0$ for every unit $\mathbf{x}$. By Proposition 17.1.1, $\lambda_1(\mathbf{K})$ and $\lambda_n(\mathbf{K})$ are values of $\mathbf{x}^* \mathbf{K} \mathbf{x}$ on unit vectors, so both are 0, and every eigenvalue of $\mathbf{K}$ is 0. By Corollary 12.4.3, $\mathbf{K} = \mathbf{U} \mathbf{O} \mathbf{U}^* = \mathbf{O}$. Hence $\mathbf{A} = \mathbf{H}$ is Hermitian.
- (b) ($\Leftarrow$) is the example $W(c\mathbf{I}) = \{c\}$. ($\Rightarrow$) By Proposition 20.2.2 (b), $W(\mathbf{A} - c\mathbf{I}) = W(\mathbf{A}) - c = \{0\} \subseteq \mathbb{R}$, so $\mathbf{A} - c\mathbf{I}$ is Hermitian by (a). By Proposition 17.1.1 its extreme eigenvalues are values of the form on unit vectors, hence both 0, and as in (a) the Hermitian matrix $\mathbf{A} - c\mathbf{I}$ with all eigenvalues 0 is $\mathbf{O}$.

Exercise. C3: The numerical radius

For $\mathbf{A} \in M_n(\mathbb{C})$, the **numerical radius** is $w(\mathbf{A}) = \max\{|z| : z \in W(\mathbf{A})\}$.

- (a) Explain why the maximum exists, and prove that $\rho(\mathbf{A}) \leq w(\mathbf{A}) \leq \|\mathbf{A}\|_2$.
- (b) Prove that $\|\mathbf{A}\|_2 \leq 2w(\mathbf{A})$.
- (c) Show that both constants are attained: $w(\mathbf{A}) = \|\mathbf{A}\|_2$ for every normal $\mathbf{A}$, and $\|\mathbf{A}\|_2 = 2w(\mathbf{A})$ for $\mathbf{A} = \mathbf{J}_2(0)$.

Hint for (b): bound $\|\mathbf{H}\|_2$ and $\|\mathbf{K}\|_2$ separately.

Solution.

- (a) $W(\mathbf{A})$ is nonempty and compact (Proposition 20.2.2 (d)), and $z \mapsto |z|$ is continuous, so the maximum is attained by **the extreme value theorem, fact (A4) of Chapter 16's introduction**. Every eigenvalue lies in $W(\mathbf{A})$, by Proposition 20.2.2 (a), so $\rho(\mathbf{A}) \leq w(\mathbf{A})$; and $w(\mathbf{A}) \leq \|\mathbf{A}\|_2$ by Proposition 20.2.2 (e).
- (b) Let $\mathbf{A} = \mathbf{H} + i\mathbf{K}$ as in Lemma 20.2.3. By Lemma 17.3.5, $\|\mathbf{H}\|_2$ is the larger of $\lambda_1(\mathbf{H})$ and $-\lambda_n(\mathbf{H})$. By Proposition 17.1.1 there is a unit $\mathbf{x}$ with

$$\lambda_1(\mathbf{H}) = \mathbf{x}^* \mathbf{H} \mathbf{x} = \text{Re}(\mathbf{x}^* \mathbf{A} \mathbf{x}) \leq |\mathbf{x}^* \mathbf{A} \mathbf{x}| \leq w(\mathbf{A}),$$

and a unit $\mathbf{y}$ with $-\lambda_n(\mathbf{H}) = -\text{Re}(\mathbf{y}^* \mathbf{A} \mathbf{y}) \leq w(\mathbf{A})$. So $\|\mathbf{H}\|_2 \leq w(\mathbf{A})$, and the same argument with imaginary parts gives $\|\mathbf{K}\|_2 \leq w(\mathbf{A})$. As the operator norm is a norm (Theorem 16.3.2 (b)), $\|\mathbf{A}\|_2 \leq \|\mathbf{H}\|_2 + \|\mathbf{K}\|_2 \leq 2w(\mathbf{A})$.

- (c) For normal $\mathbf{A}$, $\rho(\mathbf{A}) = \|\mathbf{A}\|_2$, as Chapter 16 §04 showed after Definition 16.4.1, so the chain in (a) collapses to equalities. For $\mathbf{J}_2(0)$, Example (The numerical range of a Jordan block) gives $w(\mathbf{J}_2(0)) = \frac{1}{2}$, and $\|\mathbf{J}_2(0)\|_2 = 1$, since $\|\mathbf{J}_2(0)\mathbf{x}\| = |x_2| \leq \|\mathbf{x}\|$ with equality at $\mathbf{x} = \mathbf{e}_2$. So $\|\mathbf{J}_2(0)\|_2 = 2w(\mathbf{J}_2(0))$.