

17.10 Inertia, Again and Additively

Chapter 14 attached three numbers to a real symmetric matrix — how many positive, how many negative, how many zero entries appear on the diagonal after a congruence — and proved that the three do not depend on which congruence was used. The proof there was elementary and self-contained: an orthogonal basis for the form, two subspaces, a dimension count. This section proves the same theorem a second time, from the min–max description of the eigenvalues, and then shows what the second proof buys that the first could not: the three numbers become **additive across a Schur complement**.

Throughout, $F = \mathbb{R}$ or $F = \mathbb{C}$, and every matrix whose eigenvalues or inertia are named is Hermitian — real symmetric when $F = \mathbb{R}$. Its eigenvalues are real and are indexed decreasingly, $\lambda_1(\mathbf{A}) \geq \cdots \geq \lambda_n(\mathbf{A})$, as everywhere in this chapter. **Congruence** means the operation $\mathbf{A} \mapsto \mathbf{P}^* \mathbf{A} \mathbf{P}$ with $\mathbf{P}$ invertible, which sends Hermitian matrices to Hermitian matrices; over $\mathbb{R}$ the star is the plain transpose and this is exactly the relation $\mathbf{A} \simeq \mathbf{B}$ of Definition 14.2.1, and Chapter 13 already used the complex version in Proposition 13.1.2.

17.10.1 A symbol for the triple

Chapter 14 §05 named the three counts and proved them well defined, and Definition 14.5.1 records the words: the triple (n_+, n_-, n_0) is the **inertia** and the pair (n_+, n_-) the **signature**. What it did not do is give the triple a symbol, because it never needed to write an equation between two triples. This section writes several, so the symbol is worth the line it costs.

Definition 17.10.1. The Inertia Triple

Let $\mathbf{A} \in M_n(F)$ be Hermitian, with eigenvalues $\lambda_1(\mathbf{A}) \geq \cdots \geq \lambda_n(\mathbf{A})$ listed **with multiplicity**. Write $n_+(\mathbf{A})$, $n_-(\mathbf{A})$ and $n_0(\mathbf{A})$ for the numbers of indices i with $\lambda_i(\mathbf{A}) > 0$, with $\lambda_i(\mathbf{A}) < 0$ and with $\lambda_i(\mathbf{A}) = 0$, and set

$$\text{In}(\mathbf{A}) := (n_+(\mathbf{A}), n_-(\mathbf{A}), n_0(\mathbf{A})) \in \mathbb{N}^3.$$

Triples are added **entrywise**.

Clause by clause. Every eigenvalue is counted, so $n_+ + n_- + n_0 = n$ always. The counts are *with multiplicity*: $\text{In}(\mathbf{I}_n) = (n, 0, 0)$, not $(1, 0, 0)$. And $n_0(\mathbf{A}) = \dim \text{nul}(\mathbf{A})$, because a Hermitian matrix is unitarily diagonalizable (Corollary 12.4.3, or Corollary 12.5.2 when $F = \mathbb{R}$), so the eigenspace for 0 has dimension equal to the multiplicity of 0 in the list; equivalently $n_+ + n_- = \text{rank } \mathbf{A}$.

For $F = \mathbb{R}$ this is not a new object. Theorem 14.5.3 says precisely that the numbers of positive, negative and zero *eigenvalues* of a real symmetric $\mathbf{A}$ are the numbers of positive, negative and zero *diagonal entries* after any congruence to diagonal form — that is, the inertia of Definition 14.5.1. So $\text{In}(\mathbf{A})$ is the old triple, freshly abbreviated.

A few values, and the dictionary with definiteness. $\text{In}(\mathbf{O}_n) = (0, 0, n)$, and $\text{In}(\text{diag}(1, -1)) = (1, 1, 0)$. By Theorem 13.1.4, $\mathbf{A} \succeq \mathbf{O}$ exactly when every eigenvalue is ≥ 0 , that is when $n_-(\mathbf{A}) = 0$; by Theorem 13.1.5, $\mathbf{A} \succ \mathbf{O}$ exactly when $\text{In}(\mathbf{A}) = (n, 0, 0)$. Definiteness is the extreme case of inertia, and everything below degenerates to a known statement about definiteness when the triple is extreme.

17.10.2 Sylvester's law, proved again

The description of n_+ that this chapter can supply is not a count of diagonal entries at all. It is a maximum over subspaces, and it mentions no basis, no diagonalization, and no matrix.

Theorem 17.10.1. Sylvester's Law of Inertia, Second Proof

Let $\mathbf{A} \in M_n(F)$ be Hermitian. Call a subspace $W \leq F^n$ **$\mathbf{A}$ -positive** if $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle > 0$ for every $\mathbf{x} \in W$ with $\mathbf{x} \neq \mathbf{0}$. Then:

- (a) $n_+(\mathbf{A}) = \max\{\dim W : W \text{ is } \mathbf{A}\text{-positive}\}$, and the maximum is attained at the span of an orthonormal family of eigenvectors for the positive eigenvalues of $\mathbf{A}$;
- (b) for every invertible $\mathbf{P} \in M_n(F)$, the matrix $\mathbf{P}^*\mathbf{A}\mathbf{P}$ is Hermitian and

$$\text{In}(\mathbf{P}^*\mathbf{A}\mathbf{P}) = \text{In}(\mathbf{A}).$$

Over $\mathbb{R}$, part (b) is Sylvester's Law of Inertia, Theorem 14.5.1.

Idea. For (a), read Theorem 17.2.3 through a sign. Its max–min form says $\lambda_k(\mathbf{A})$ is the largest value that $R_{\mathbf{A}}$ can be *guaranteed* to exceed on a k -dimensional subspace; and “ W is $\mathbf{A}$ -positive” says exactly that $R_{\mathbf{A}}$ is guaranteed to be positive on W . So $\mathbf{A}$ -positive subspaces of dimension k exist precisely when $\lambda_k(\mathbf{A}) > 0$, which is precisely when $k \leq n_+$. For (b), the right-hand side of (a) is defined from the quadratic form alone, and a congruence is a change of variables inside that form: $\mathbf{x} \mapsto \mathbf{P}\mathbf{x}$ matches up the two families of positive subspaces, dimension for dimension.

Proof. Throughout, note that $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^*\mathbf{A}\mathbf{x}$ is real for Hermitian $\mathbf{A}$ (Proposition 11.6.7 (a), or directly: $\overline{\mathbf{x}^*\mathbf{A}\mathbf{x}} = \mathbf{x}^*\mathbf{A}^*\mathbf{x} = \mathbf{x}^*\mathbf{A}\mathbf{x}$), and that for $\mathbf{x} \neq \mathbf{0}$ the sign of $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle$ is the sign of the Rayleigh quotient $R_{\mathbf{A}}(\mathbf{x}) = \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle / \langle \mathbf{x}, \mathbf{x} \rangle$ of Definition 17.1.1, since $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 > 0$. So W is $\mathbf{A}$ -positive if and only if $R_{\mathbf{A}}(\mathbf{x}) > 0$ for every $\mathbf{x} \in W \setminus \{\mathbf{0}\}$.

(a), *the lower bound.* Put $k = n_+(\mathbf{A})$. If $k = 0$ the zero subspace is $\mathbf{A}$ -positive and there is nothing to prove, so assume $k \geq 1$. By the spectral theorem (Corollary 12.4.3, or Corollary 12.5.2 when $F = \mathbb{R}$) there is an orthonormal basis $\mathbf{q}_1, \dots, \mathbf{q}_n$ of F^n with $\mathbf{A}\mathbf{q}_i = \lambda_i(\mathbf{A})\mathbf{q}_i$, the eigenvalues in decreasing order. Put $W_0 = \text{span}(\mathbf{q}_1, \dots, \mathbf{q}_k)$, of dimension k . For $\mathbf{0} \neq \mathbf{x} = \sum_{i \leq k} c_i \mathbf{q}_i$, expanding by orthonormality,

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^k \lambda_i(\mathbf{A}) |c_i|^2 \geq \lambda_k(\mathbf{A}) \sum_{i=1}^k |c_i|^2 = \lambda_k(\mathbf{A}) \|\mathbf{x}\|^2 > 0,$$

because $\lambda_i(\mathbf{A}) \geq \lambda_k(\mathbf{A})$ for $i \leq k$ and $\lambda_k(\mathbf{A}) > 0$, the latter since $k = n_+(\mathbf{A})$ counts the positive eigenvalues of a decreasing list. So W_0 is $\mathbf{A}$ -positive of dimension $n_+(\mathbf{A})$.

(a), *the upper bound.* Let W be $\mathbf{A}$ -positive with $d = \dim W \geq 1$. The unit sphere $S_W = \{\mathbf{x} \in W : \|\mathbf{x}\| = 1\}$ is a closed bounded subset of F^n , hence compact by fact (A3) of Chapter 16's introduction, and it is nonempty because $d \geq 1$. The function $\mathbf{x} \mapsto R_{\mathbf{A}}(\mathbf{x})$ is continuous on it, so by the extreme value theorem — fact (A4) of Chapter 16's introduction — it attains a minimum $m = R_{\mathbf{A}}(\mathbf{x}_0)$ there, and $m > 0$ because $\mathbf{x}_0 \in W$ is nonzero. By homogeneity of $R_{\mathbf{A}}$, the same m is the minimum over all nonzero $\mathbf{x} \in W$. Now W is one competitor in the

max–min form of Theorem 17.2.3, so

$$\lambda_d(\mathbf{A}) = \max_{\dim W'=d} \min_{\mathbf{0} \neq \mathbf{x} \in W'} R_{\mathbf{A}}(\mathbf{x}) \geq m > 0.$$

A decreasing list with $\lambda_d > 0$ has at least d positive entries, so $d \leq n_+(\mathbf{A})$. Together with the lower bound this proves (a).

(b). Put $\mathbf{B} = \mathbf{P}^* \mathbf{A} \mathbf{P}$. Then $\mathbf{B}^* = \mathbf{P}^* \mathbf{A}^* \mathbf{P} = \mathbf{B}$, so $\mathbf{B}$ is Hermitian. For every $\mathbf{x} \in F^n$,

$$\langle \mathbf{B}\mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^* \mathbf{P}^* \mathbf{A} \mathbf{P} \mathbf{x} = (\mathbf{P}\mathbf{x})^* \mathbf{A} (\mathbf{P}\mathbf{x}) = \langle \mathbf{A}(\mathbf{P}\mathbf{x}), \mathbf{P}\mathbf{x} \rangle.$$

Since $\mathbf{P}$ is invertible, $\mathbf{x} \mapsto \mathbf{P}\mathbf{x}$ is an isomorphism of F^n ; its restriction to a subspace W is an isomorphism $W \rightarrow \mathbf{P}W$, which carries a basis to a basis (Theorem 2.5.2), so $\dim \mathbf{P}W = \dim W$. And $W \mapsto \mathbf{P}W$ is a bijection on subspaces, with inverse $W' \mapsto \mathbf{P}^{-1}W'$. Moreover W is $\mathbf{B}$ -positive if and only if $\mathbf{P}W$ is $\mathbf{A}$ -positive: the nonzero vectors of $\mathbf{P}W$ are exactly the vectors $\mathbf{P}\mathbf{x}$ with $\mathbf{0} \neq \mathbf{x} \in W$, and the display says the two quadratic forms agree at matched vectors. Hence the two sets of dimensions appearing in (a) coincide, and (a) gives $n_+(\mathbf{B}) = n_+(\mathbf{A})$.

Applying this to $-\mathbf{A}$, whose eigenvalues are the negatives of those of $\mathbf{A}$ and whose congruence by $\mathbf{P}$ is $-\mathbf{B}$, gives $n_-(\mathbf{B}) = n_+(-\mathbf{B}) = n_+(-\mathbf{A}) = n_-(\mathbf{A})$. Finally $n_0 = n - n_+ - n_-$ for both. This proves the theorem. ■

What each proof gives. Over $\mathbb{R}$ the two theorems say the same thing, since a congruence is exactly a change of basis for the underlying form (Theorem 14.2.1). They are not interchangeable.

Chapter 14’s proof uses only the existence of an orthogonal basis for the form, the invariance of rank under congruence, and the dimension formula. It mentions no inner product, no eigenvalue, and no limit — in Chapter 14 the eigenvalues enter only afterwards, in Theorem 14.5.3 — and it is the proof to give for a symmetric bilinear form on an abstract real vector space, where no inner product has been chosen and the words “Hermitian matrix” have no meaning until one is.

The proof above costs the spectral theorem and one appeal to compactness. In exchange it gives three things. First, a **description of n_+ with no basis in it at all** — a maximum over subspaces — together with an explicit optimal subspace, which is what Exercise (C1: An intrinsic description of n_+) asked the reader to establish by hand from an orthogonal basis; here it falls out of the min–max, with the eigenvalues attached. Second, it works **over $\mathbb{C}$** , for Hermitian matrices and the congruence $\mathbf{A} \mapsto \mathbf{P}^* \mathbf{A} \mathbf{P}$, which Theorem 14.5.1 does not cover. Third, it keeps the eigenvalues in view, so the perturbation theory of Section 3 applies to the inertia directly; that is the next theorem. Haynsworth’s theorem at the end of the section leans on the second gain: for complex Hermitian matrices, only this proof supplies the law of inertia it needs.

The companions of part (a) are worth stating, because one of them is a trap.

Corollary 17.10.2. What Each Count Is a Maximum Of

Let $\mathbf{A} \in M_n(F)$ be Hermitian.

- (a) $n_-(\mathbf{A}) = \max\{\dim W : \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle < 0 \text{ for every } \mathbf{0} \neq \mathbf{x} \in W\};$

- (b) $n_+(\mathbf{A}) + n_0(\mathbf{A}) = \max\{\dim W : \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \geq 0 \text{ for every } \mathbf{x} \in W\}$, and symmetrically $n_-(\mathbf{A}) + n_0(\mathbf{A})$ is the largest dimension of a subspace on which $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq 0$;
- (c) $n_0(\mathbf{A}) = \dim \text{nul}(\mathbf{A}) = n - n_+(\mathbf{A}) - n_-(\mathbf{A})$.

Idea. Each claim has a witness and a competitor, as in Courant–Fischer. The witness is the span of the eigenvectors with the right sign; any competing subspace that is too large must meet the span of the eigenvectors of the wrong sign, and on that intersection the form has the wrong sign. For (c), no maximum is involved at all — the kernel is read off the eigenvalues.

Proof.

- (a) Apply [Theorem 17.10.1](#) (a) to $-\mathbf{A}$: its eigenvalues are the negatives of those of $\mathbf{A}$, so $n_+(-\mathbf{A}) = n_-(\mathbf{A})$, and a subspace is $(-\mathbf{A})$ -positive exactly when $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle < 0$ on its nonzero vectors.
- (b) Fix, as in the previous proof, an orthonormal eigenbasis $\mathbf{q}_1, \dots, \mathbf{q}_n$ of $\mathbf{A}$ with $\mathbf{A}\mathbf{q}_i = \lambda_i(\mathbf{A})\mathbf{q}_i$ (Corollary 12.4.3). Let $t = n_+(\mathbf{A}) + n_0(\mathbf{A})$, so that $\lambda_i(\mathbf{A}) \geq 0$ exactly for $i \leq t$. On $W_1 = \text{span}(\mathbf{q}_1, \dots, \mathbf{q}_t)$ the same expansion gives $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \sum_{i \leq t} \lambda_i(\mathbf{A})|c_i|^2 \geq 0$, so the maximum is at least t . Conversely let W satisfy $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \geq 0$ throughout and suppose $\dim W > t$. Put $W_2 = \text{span}(\mathbf{q}_{t+1}, \dots, \mathbf{q}_n)$, of dimension $n - t = n_-(\mathbf{A})$, on whose nonzero vectors $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \sum_{i > t} \lambda_i(\mathbf{A})|c_i|^2 < 0$, every $\lambda_i(\mathbf{A})$ with $i > t$ being negative. Since $\dim W + \dim W_2 > t + (n - t) = n$, Lemma 17.2.1 produces $\mathbf{0} \neq \mathbf{x} \in W \cap W_2$, for which $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle$ is both ≥ 0 and < 0 . This is impossible, so $\dim W \leq t$. The statement with $\leq$ follows by applying this one to $-\mathbf{A}$.
- (c) The first equality was recorded after [Definition 17.10.1](#), and the second is $n_+ + n_- + n_0 = n$.

■

Warning. n_0 is not the largest dimension of a subspace on which the form vanishes. Parts (a) and (b) describe n_- , n_+ , and their sums with n_0 as maxima over subspaces; n_0 itself has no such description, and guessing one is the standard error here. Take $\mathbf{A} = \text{diag}(1, -1)$, so $\text{In}(\mathbf{A}) = (1, 1, 0)$ and $n_0(\mathbf{A}) = 0$. On the line $W = \text{span}((1, 1))$,

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = x_1^2 - x_2^2 = 0 \quad \text{for every } \mathbf{x} \in W,$$

so a subspace of dimension 1 carries the identically zero form while $n_0(\mathbf{A}) = 0$. What n_0 is $\dim \text{nul}(\mathbf{A})$, by (c) — a kernel, not a maximum. The largest dimension of a subspace on which the form vanishes identically is a different invariant. For a real symmetric **invertible** $\mathbf{A}$ it is the Witt index of Chapter 14 §07, which that section defines only for nondegenerate forms and bounds by $n/2$. For a singular $\mathbf{A}$ the bound fails — for $\mathbf{A} = \mathbf{O}$ the whole space qualifies — and in general the answer is $n_0 + \min(n_+, n_-)$, which [Exercise \(C1: How large can a subspace be on which the form vanishes?\)](#) computes.

Quick check. A Hermitian $\mathbf{A} \in M_5(F)$ has $\text{In}(\mathbf{A}) = (2, 2, 1)$. What is the largest dimension of a subspace on which $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle > 0$ for all nonzero $\mathbf{x}$, and the largest on which $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \geq 0$

for all $\mathbf{x}$?

Solution. 2 and 3. The first is $n_+(\mathbf{A}) = 2$ by [Theorem 17.10.1](#) (a); the second is $n_+(\mathbf{A}) + n_0(\mathbf{A}) = 2 + 1 = 3$ by [Corollary 17.10.2](#) (b). The kernel direction may be added to a positive subspace without breaking ≥ 0 , but it does break > 0 — which is the whole difference between the two numbers.

17.10.3 Inertia under a perturbation of small rank

A congruence leaves the inertia alone and does nothing else: it can move the eigenvalues anywhere compatible with their signs. Chapter 14 §05's warning displayed the pair

$$\mathbf{A} = \text{diag}(1, -1), \quad \mathbf{P}^\top \mathbf{A} \mathbf{P} = \text{diag}(4, -9) \quad (\mathbf{P} = \text{diag}(2, 3)),$$

where λ_1 rose from 1 to 4 and λ_2 fell from -1 to -9 . There is no interlacing here: $\lambda_1(\mathbf{A}) \geq \lambda_1(\mathbf{P}^\top \mathbf{A} \mathbf{P})$ already fails. A congruence *is* a perturbation, $\mathbf{P}^\top \mathbf{A} \mathbf{P} = \mathbf{A} + \mathbf{E}$ with $\mathbf{E} = \mathbf{P}^\top \mathbf{A} \mathbf{P} - \mathbf{A}$, but $\mathbf{E}$ can have full rank, and then the rank bound of Section 3 says nothing. When the perturbation has small rank, it does.

Theorem 17.10.3. Inertia Moves by at Most the Rank

Let $\mathbf{A}, \mathbf{E} \in M_n(F)$ be Hermitian with $\text{rank } \mathbf{E} \leq r$. Then

$$|n_+(\mathbf{A} + \mathbf{E}) - n_+(\mathbf{A})| \leq r, \quad |n_-(\mathbf{A} + \mathbf{E}) - n_-(\mathbf{A})| \leq r,$$

and $|n_0(\mathbf{A} + \mathbf{E}) - n_0(\mathbf{A})| \leq r$ as well.

Idea. Corollary 17.3.8 says that adding a matrix of rank r shifts each eigenvalue by at most r places in the ordered list. A count of positive eigenvalues is a count of places, so it too can move by at most r .

Proof. Put $s = n_+(\mathbf{A} + \mathbf{E})$. If $s \leq r$ then certainly $n_+(\mathbf{A}) \geq 0 \geq s - r$. Otherwise $i := s - r$ satisfies $1 \leq i$ and $i + r = s \leq n$, so Corollary 17.3.8 applies with this i and gives

$$\lambda_s(\mathbf{A} + \mathbf{E}) = \lambda_{i+r}(\mathbf{A} + \mathbf{E}) \leq \lambda_i(\mathbf{A}) = \lambda_{s-r}(\mathbf{A}).$$

Since $s = n_+(\mathbf{A} + \mathbf{E})$ we have $\lambda_s(\mathbf{A} + \mathbf{E}) > 0$, hence $\lambda_{s-r}(\mathbf{A}) > 0$, hence $n_+(\mathbf{A}) \geq s - r$. In either case

$$n_+(\mathbf{A} + \mathbf{E}) - n_+(\mathbf{A}) \leq r.$$

For the opposite inequality, write $\mathbf{A} = (\mathbf{A} + \mathbf{E}) + (-\mathbf{E})$, where $-\mathbf{E}$ is Hermitian with $\text{rank}(-\mathbf{E}) = \text{rank } \mathbf{E} \leq r$, and apply what was just proved with $\mathbf{A} + \mathbf{E}$ in the role of $\mathbf{A}$: $n_+(\mathbf{A}) - n_+(\mathbf{A} + \mathbf{E}) \leq r$. The two together give the first claim.

The second claim is the first applied to the Hermitian pair $-\mathbf{A}, -\mathbf{E}$, using $n_-(\mathbf{M}) = n_+(-\mathbf{M})$. For the third, adding the first two would only give $2r$; the sharp bound comes from kernels instead. By [Corollary 17.10.2](#) (c), $n_0(\mathbf{M}) = \dim \text{nul}(\mathbf{M})$ for every Hermitian $\mathbf{M}$. A vector killed by both $\mathbf{A}$ and $\mathbf{E}$ is killed by $\mathbf{A} + \mathbf{E}$, so $\text{nul}(\mathbf{A}) \cap \text{nul}(\mathbf{E}) \subseteq \text{nul}(\mathbf{A} + \mathbf{E})$; and $\dim \text{nul}(\mathbf{E}) =$

$n - \text{rank } \mathbf{E} \geq n - r$ by Theorem 2.3.1. The sharper form of Lemma 17.2.1 then gives

$$n_0(\mathbf{A} + \mathbf{E}) \geq \dim(\text{nul}(\mathbf{A}) \cap \text{nul}(\mathbf{E})) \geq n_0(\mathbf{A}) + (n - r) - n = n_0(\mathbf{A}) - r.$$

Applied to $\mathbf{A} = (\mathbf{A} + \mathbf{E}) + (-\mathbf{E})$ the same argument gives $n_0(\mathbf{A}) \geq n_0(\mathbf{A} + \mathbf{E}) - r$, and the two together are the claim. The bound is attained: $\mathbf{A} = \mathbf{I}_r \oplus \mathbf{O}$ and $\mathbf{E} = -\mathbf{I}_r \oplus \mathbf{O}$ give $n_0(\mathbf{A}) = n - r$ and $n_0(\mathbf{A} + \mathbf{E}) = n$. This proves the theorem. ■

Example. A rank-one perturbation, three ways

Let $\mathbf{A} = \text{diag}(3, -1, -2)$ and $\mathbf{v} = (1, 1, 0)$, and let $\mathbf{E}_t = t\mathbf{v}\mathbf{v}^\top$, which has rank 1 for $t \neq 0$. Find $\text{In}(\mathbf{A} + \mathbf{E}_t)$ for $t = 2$, $t = \frac{3}{2}$, and $t = -5$, and check Theorem 17.10.3.

Solution. $\text{In}(\mathbf{A}) = (1, 2, 0)$, read off the diagonal. Since $\mathbf{v}$ has third entry 0,

$$\mathbf{A} + \mathbf{E}_t = \begin{pmatrix} 3+t & t & 0 \\ t & t-1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 3+t & t \\ t & t-1 \end{pmatrix} \oplus (-2),$$

and the eigenvalue list of a block diagonal matrix is the two lists together (its characteristic polynomial is the product of the two, by Theorem 7.8.4). The entry -2 contributes one negative eigenvalue for every t . The 2×2 block has

$$\det = (3+t)(t-1) - t^2 = 2t - 3, \quad \text{tr} = 2t + 2,$$

and since the determinant of a real symmetric 2×2 matrix is the product of its two real eigenvalues and the trace is their sum, the eigenvalues both have the sign of the trace when the determinant is positive, have opposite signs when it is negative, and are 0 and tr when it is zero.

- $t = 2$: $\det = 1 > 0$, $\text{tr} = 6 > 0$, so the block contributes $(2, 0, 0)$ and $\text{In}(\mathbf{A} + \mathbf{E}_2) = (2, 1, 0)$.
- $t = \frac{3}{2}$: $\det = 0$, $\text{tr} = 5$, so the block contributes $(1, 0, 1)$ and $\text{In}(\mathbf{A} + \mathbf{E}_{3/2}) = (1, 1, 1)$.
- $t = -5$: $\det = -13 < 0$, so the block contributes $(1, 1, 0)$ and $\text{In}(\mathbf{A} + \mathbf{E}_{-5}) = (1, 2, 0)$, unchanged.

Against $\text{In}(\mathbf{A}) = (1, 2, 0)$: at $t = 2$ one negative direction has become positive, so n_+ and n_- each move by 1 = r and the bound of Theorem 17.10.3 is attained; at $t = \frac{3}{2}$, exactly on the way between, the direction passes through 0, so n_- and n_0 each move by 1. No rank-one perturbation of this $\mathbf{A}$ can reach $(3, 0, 0)$, which would need n_+ to move by 2.

17.10.4 Additivity across a Schur complement

Chapter 8 built the Schur complement out of block elimination and proved two things about it: $\det \mathbf{M} = \det \mathbf{A} \cdot \det(\mathbf{M}/\mathbf{A})$ (Theorem 8.3.2) and $\text{rank } \mathbf{M} = k + \text{rank}(\mathbf{M}/\mathbf{A})$ (Theorem 8.3.3). Chapter 13 then used the same elimination on Hermitian matrices — in §01, observing that it is the block $\mathbf{LDU}$ factorization written as a congruence, and in §06, to prove Theorem 13.6.1

— and each time read off a statement about positive definiteness only. Haynsworth's theorem is what that congruence is worth in general: the rank does not merely add, the *signs* add.

Theorem 17.10.4. Haynsworth's Inertia Additivity Formula

Let $1 \leq k \leq n-1$ and let

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix} \in M_n(F)$$

be Hermitian, with $\mathbf{A} \in M_k(F)$ **invertible**, $\mathbf{B} \in M_{k \times (n-k)}(F)$ and $\mathbf{D} \in M_{n-k}(F)$. Then $\mathbf{A}$ and the Schur complement $\mathbf{M}/\mathbf{A} = \mathbf{D} - \mathbf{B}^* \mathbf{A}^{-1} \mathbf{B}$ (Definition 8.3.1) are Hermitian, and

$$\text{In}(\mathbf{M}) = \text{In}(\mathbf{A}) + \text{In}(\mathbf{M}/\mathbf{A}).$$

Idea. Block elimination factors $\mathbf{M}$ as $\mathbb{L} (\mathbf{A} \oplus \mathbf{M}/\mathbf{A}) \mathbf{U}$ with $\mathbb{L}$ lower and $\mathbf{U}$ upper unit block triangular (Theorem 8.3.1). For a **Hermitian** $\mathbf{M}$ with $\mathbf{A}$ Hermitian, the two outer factors are conjugate transposes of each other, so the factorization is a congruence and not merely a factorization. Theorem 17.10.1 (b) then says the inertia of $\mathbf{M}$ is the inertia of $\mathbf{A} \oplus \mathbf{M}/\mathbf{A}$, and the inertia of a block diagonal matrix is the sum of the inertias of its blocks, because its characteristic polynomial is the product.

Proof. Since $\mathbf{M}^* = \mathbf{M}$, comparing blocks gives $\mathbf{A}^* = \mathbf{A}$ and $\mathbf{D}^* = \mathbf{D}$. As $\mathbf{A}$ is Hermitian and invertible, $(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1} = \mathbf{A}^{-1}$, so $\mathbf{A}^{-1}$ is Hermitian and

$$(\mathbf{M}/\mathbf{A})^* = \mathbf{D}^* - \mathbf{B}^* (\mathbf{A}^{-1})^* \mathbf{B} = \mathbf{D} - \mathbf{B}^* \mathbf{A}^{-1} \mathbf{B} = \mathbf{M}/\mathbf{A}.$$

Step 1: the elimination is a congruence. By Theorem 8.3.1 (a), applied with $\mathbf{C} = \mathbf{B}^*$,

$$\mathbf{M} = \mathbb{L} (\mathbf{A} \oplus (\mathbf{M}/\mathbf{A})) \mathbf{U}, \quad \mathbb{L} = \begin{pmatrix} \mathbf{I}_k & \mathbf{O} \\ \mathbf{B}^* \mathbf{A}^{-1} & \mathbf{I}_{n-k} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{I}_k & \mathbf{A}^{-1} \mathbf{B} \\ \mathbf{O} & \mathbf{I}_{n-k} \end{pmatrix}.$$

Now $\mathbf{U}^*$ has blocks $\mathbf{I}_k, \mathbf{O}$ on top and $(\mathbf{A}^{-1} \mathbf{B})^* = \mathbf{B}^* \mathbf{A}^{-1}$, $\mathbf{I}_{n-k}$ below, using that $\mathbf{A}^{-1}$ is Hermitian; that is, $\mathbf{U}^* = \mathbb{L}$. So

$$\mathbf{M} = \mathbf{U}^* (\mathbf{A} \oplus (\mathbf{M}/\mathbf{A})) \mathbf{U}.$$

The matrix $\mathbf{U}$ is invertible, with inverse $\mathbf{P} := \begin{pmatrix} \mathbf{I}_k & -\mathbf{A}^{-1} \mathbf{B} \\ \mathbf{O} & \mathbf{I}_{n-k} \end{pmatrix}$, as multiplying out confirms. Multiplying the display by $\mathbf{P}^*$ on the left and $\mathbf{P}$ on the right, and using $\mathbf{P}^* \mathbf{U}^* = (\mathbf{U} \mathbf{P})^* = \mathbf{I}_n$,

$$\mathbf{P}^* \mathbf{M} \mathbf{P} = \mathbf{A} \oplus (\mathbf{M}/\mathbf{A}). \tag{17.10.1}$$

Step 2: apply the law of inertia. $\mathbf{P}$ is invertible, so Theorem 17.10.1 (b) gives

$$\text{In}(\mathbf{M}) = \text{In}(\mathbf{P}^* \mathbf{M} \mathbf{P}) = \text{In}(\mathbf{A} \oplus (\mathbf{M}/\mathbf{A})).$$

Step 3: inertia is additive over a block diagonal sum. Let $\mathbf{X} \in M_k(F)$ and $\mathbf{Y} \in M_{n-k}(F)$ be Hermitian. The matrix $x\mathbf{I}_n - (\mathbf{X} \oplus \mathbf{Y})$ is block diagonal with blocks $x\mathbf{I}_k - \mathbf{X}$ and $x\mathbf{I}_{n-k} - \mathbf{Y}$, so by Theorem 7.8.4 its determinant is $\det(x\mathbf{I}_k - \mathbf{X}) \det(x\mathbf{I}_{n-k} - \mathbf{Y})$; that

is, $p_{\mathbf{X} \oplus \mathbf{Y}} = p_{\mathbf{X}} p_{\mathbf{Y}}$. The eigenvalue list of a Hermitian matrix with multiplicity is the list of roots of its characteristic polynomial with multiplicity, so the list for $\mathbf{X} \oplus \mathbf{Y}$ is the two lists concatenated. Counting positive, negative, and zero entries gives $\text{In}(\mathbf{X} \oplus \mathbf{Y}) = \text{In}(\mathbf{X}) + \text{In}(\mathbf{Y})$.

Applying Step 3 with $\mathbf{X} = \mathbf{A}$ and $\mathbf{Y} = \mathbf{M}/\mathbf{A}$ to the conclusion of Step 2 proves the theorem. ■

Three remarks. First, [Eq. 17.10.1](#) is the cleanest illustration in the book of what a Schur complement *is*: the block elimination of Chapter 8, performed with the same matrix on both sides, is a congruence, and $\mathbf{M}/\mathbf{A}$ is simply what the congruence leaves in the lower corner. For Hermitian $\mathbf{M}$, Chapter 8's determinant and rank formulas are both shadows of this one identity. Second, taking ranks in [Eq. 17.10.1](#) recovers Theorem 8.3.3 for Hermitian $\mathbf{M}$, since $n_+ + n_- = \text{rank}$; Haynsworth refines the rank formula by splitting each rank into its two signs. Third, the theorem needs $\mathbf{A}$ invertible and nothing else — no definiteness anywhere, which is what separates it from Chapter 13 §06's test.

Example. The Chapter 14 example, through a Schur complement

Find $\text{In}(\mathbf{M})$ for

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

the matrix of Example (A signature by symmetric elimination), using [Theorem 17.10.4](#) with the leading 2×2 block.

Solution. Take $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{B} = (2, 3)^\top$, and $\mathbf{D} = (1)$. Then $\det \mathbf{A} = 1 \neq 0$, so $\mathbf{A}$ is invertible with $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Its leading principal minors are 1 and 1, both positive, so $\mathbf{A} \succ \mathbf{O}$ by Theorem 13.1.5 and $\text{In}(\mathbf{A}) = (2, 0, 0)$.

For the Schur complement, $\mathbf{B}^\top \mathbf{A}^{-1} = (2, 3) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = (1, 1)$, so $\mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} = (1, 1) \cdot (2, 3) = 5$ and

$$\mathbf{M}/\mathbf{A} = 1 - 5 = -4, \quad \text{In}(\mathbf{M}/\mathbf{A}) = (0, 1, 0).$$

Hence $\text{In}(\mathbf{M}) = (2, 0, 0) + (0, 1, 0) = (2, 1, 0)$.

Two checks. The congruence of [Eq. 17.10.1](#) is given by $\mathbf{A}^{-1} \mathbf{B} = (1, 1)^\top$, hence

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^\top \mathbf{M} \mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

which is $\mathbf{A} \oplus (-4)$ as the theorem says; and Example (A signature by symmetric elimination) reached $\text{diag}(1, 1, -4)$ by paired row and column operations, finding the same -4 . The Schur complement of the leading block *is* the last pivot of the symmetric elimination. The eigenvalues were never needed: they are the roots of $x^3 - 4x^2 - 9x + 4$, which Example (A signature by symmetric elimination) already noted are irrational.

Example. A 4×4 inertia in two linesFind $\text{In}(\mathbf{N})$ for

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 3 \end{pmatrix}.$$

Solution. The leading 2×2 block is $\mathbf{A} = \mathbf{I}_2$, which is invertible with $\text{In}(\mathbf{A}) = (2, 0, 0)$ and $\mathbf{A}^{-1} = \mathbf{I}_2$. With

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

the Schur complement needs no inversion at all:

$$\mathbf{N}/\mathbf{A} = \mathbf{D} - \mathbf{B}^\top \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}.$$

This 2×2 matrix has determinant $2 - 1 = 1 > 0$ and trace $-3 < 0$, so both its eigenvalues are negative and $\text{In}(\mathbf{N}/\mathbf{A}) = (0, 2, 0)$. By [Theorem 17.10.4](#),

$$\text{In}(\mathbf{N}) = (2, 0, 0) + (0, 2, 0) = (2, 2, 0).$$

In particular $\mathbf{N}$ is invertible and indefinite. The characteristic polynomial of $\mathbf{N}$ is $x^4 - 6x^3 + 5x^2 + 8x + 1$, which has no rational root; the four eigenvalues were never computed, and the signature came from one 2×2 determinant and one trace.

Warning. **A must be invertible, and the off-diagonal block is not a spectator.** Take $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with $k = 1$. Neither diagonal block is invertible, so no Schur complement exists for this partition and [Theorem 17.10.4](#) does not apply — and its conclusion would be false if one tried to read the blocks directly: $\text{In}(\mathbf{M}) = (1, 1, 0)$, since the eigenvalues are ± 1 , while the two diagonal blocks contribute $(0, 0, 1) + (0, 0, 1) = (0, 0, 2)$. Even when $\mathbf{A}$ is invertible, the diagonal blocks alone decide nothing: for $\mathbf{M}_b = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ with b real, the Schur complement is $1 - b^2$, so $\text{In}(\mathbf{M}_b)$ is $(2, 0, 0)$, $(1, 0, 1)$ or $(1, 1, 0)$ according as $|b| < 1$, $= 1$ or > 1 . All the information is in the coupling.

The positive definite case of Haynsworth is a theorem Chapter 13 proved by hand, and it is now one line.

Corollary 17.10.5. Block Definiteness, a Second Proof

Let $1 \leq k \leq n-1$ and let $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix} \in M_n(F)$ be Hermitian with $\mathbf{A} \in M_k(F)$ and $\mathbf{A} \succ \mathbf{O}$. Then

$$\mathbf{M} \succ \mathbf{O} \iff \mathbf{M}/\mathbf{A} \succ \mathbf{O}, \quad \mathbf{M} \succeq \mathbf{O} \iff \mathbf{M}/\mathbf{A} \succeq \mathbf{O}.$$

Proof. $\mathbf{A} \succ \mathbf{O}$ is invertible (Theorem 13.1.5) with $\text{In}(\mathbf{A}) = (k, 0, 0)$, so [Theorem 17.10.4](#)

gives

$$\text{In}(\mathbf{M}) = (k, 0, 0) + \text{In}(\mathbf{M}/\mathbf{A}).$$

Comparing entries, $n_+(\mathbf{M}) = k + n_+(\mathbf{M}/\mathbf{A})$ and $n_-(\mathbf{M}) = n_-(\mathbf{M}/\mathbf{A})$, while $\mathbf{M}/\mathbf{A} \in M_{n-k}(F)$.

Now $\mathbf{M} \succ \mathbf{O}$ means $n_+(\mathbf{M}) = n$ (Theorem 13.1.5), that is $n_+(\mathbf{M}/\mathbf{A}) = n - k$, that is $\mathbf{M}/\mathbf{A} \succ \mathbf{O}$. And $\mathbf{M} \succeq \mathbf{O}$ means $n_-(\mathbf{M}) = 0$ (Theorem 13.1.4), that is $n_-(\mathbf{M}/\mathbf{A}) = 0$, that is $\mathbf{M}/\mathbf{A} \succeq \mathbf{O}$. This proves the corollary. ■

This is Theorem 13.6.1 (b). Chapter 13 §06 proved it with the very same congruence, and then read positivity directly off the split quadratic form $\mathbf{y}_1^* \mathbf{A} \mathbf{y}_1 + \mathbf{y}_2^* (\mathbf{M}/\mathbf{A}) \mathbf{y}_2$, with no eigenvalue in sight. The two proofs share their first step and differ in the second: Chapter 13's is shorter and needs nothing beyond Proposition 13.1.2, while the one above reads the verdict off a count, and so shows the definite case as the extreme instance of an equality that holds for every Hermitian $\mathbf{M}$ with $\mathbf{A}$ invertible, definite or not.

17.10.5 Exercises

17.10.5.1 A. Check your understanding

Exercise. A1

- Define $\text{In}(\mathbf{A})$ for a Hermitian $\mathbf{A}$, and say what $n_+ + n_-$ equals.
- State the subspace description of $n_+(\mathbf{A})$ and name the subspace that attains it.
- Determine whether the following is correct, and justify: “ $n_0(\mathbf{A})$ is the largest dimension of a subspace on which $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle$ vanishes identically.”
- State Theorem 17.10.4, with all hypotheses.
- A congruence preserves the inertia of a Hermitian matrix. Does it preserve the eigenvalues, the trace, or the rank? Justify each answer.

Solution.

- $\text{In}(\mathbf{A}) = (n_+, n_-, n_0)$, the numbers of positive, negative, and zero eigenvalues of $\mathbf{A}$ counted with multiplicity (Definition 17.10.1). Their sum is n , and $n_+ + n_- = \text{rank } \mathbf{A}$, since $n_0 = \dim \text{nul}(\mathbf{A})$ by Corollary 17.10.2 (c).
- $n_+(\mathbf{A})$ is the largest dimension of a subspace W with $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle > 0$ for every nonzero $\mathbf{x} \in W$ (Theorem 17.10.1 (a)), attained at the span of orthonormal eigenvectors for the positive eigenvalues, that is at $\text{span}(\mathbf{q}_1, \dots, \mathbf{q}_{n_+})$ in an eigenbasis ordered decreasingly.
- Incorrect. For $\mathbf{A} = \text{diag}(1, -1)$ we have $n_0(\mathbf{A}) = 0$, yet $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = x_1^2 - x_2^2$ vanishes identically on the line spanned by $(1, 1)$. The correct description is $n_0(\mathbf{A}) = \dim \text{nul}(\mathbf{A})$.
- If $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix} \in M_n(F)$ is Hermitian with $\mathbf{A} \in M_k(F)$ invertible and $1 \leq k \leq n-1$, then $\text{In}(\mathbf{M}) = \text{In}(\mathbf{A}) + \text{In}(\mathbf{M}/\mathbf{A})$, where $\mathbf{M}/\mathbf{A} = \mathbf{D} - \mathbf{B}^* \mathbf{A}^{-1} \mathbf{B}$.
- Rank: yes, since $\text{rank} = n_+ + n_-$ is determined by the inertia (and independently, congruence by an invertible matrix preserves rank by Theorem 3.5.9). Eigenvalues: no — $\text{diag}(1, -1)$ and $\text{diag}(4, -9)$ are congruent by $\mathbf{P} = \text{diag}(2, 3)$. Trace: no, by the same pair, whose traces are 0 and -5 .

17.10.5.2 B. Practice

Exercise. B1: Two inertias by Schur complement

Find the inertia of each of the following, using [Theorem 17.10.4](#) with the leading 2×2 block, and compute no eigenvalues.

$$(a) \mathbf{M} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

$$(b) \mathbf{N} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Solution.

- (a) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has leading principal minors 2 and 1, both positive, so $\mathbf{A} \succ \mathbf{O}$ (Theorem 13.1.5) and $\text{In}(\mathbf{A}) = (2, 0, 0)$; also $\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. With $\mathbf{B} = (1, 0)^\top$ and $\mathbf{D} = (-1)$,

$$\mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} = (1, 0) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1,$$

so $\mathbf{M}/\mathbf{A} = -1 - 1 = -2$ and $\text{In}(\mathbf{M}/\mathbf{A}) = (0, 1, 0)$. Hence $\text{In}(\mathbf{M}) = (2, 1, 0)$. (Consistency check: $\det \mathbf{M} = \det \mathbf{A} \cdot (\mathbf{M}/\mathbf{A}) = 1 \cdot (-2) = -2$ by Theorem 8.3.2, and a 3×3 matrix with inertia $(2, 1, 0)$ has negative determinant.)

- (b) $\mathbf{A} = \text{diag}(1, -1)$ is invertible with $\mathbf{A}^{-1} = \mathbf{A}$ and $\text{In}(\mathbf{A}) = (1, 1, 0)$. With

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

we get $\mathbf{A}^{-1} \mathbf{B} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

so

$$\mathbf{N}/\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix},$$

whose determinant is $-1 < 0$, so its two eigenvalues have opposite signs and $\text{In}(\mathbf{N}/\mathbf{A}) = (1, 1, 0)$. Hence $\text{In}(\mathbf{N}) = (1, 1, 0) + (1, 1, 0) = (2, 2, 0)$.

Exercise. B2: How much can a rank-two perturbation do?

Let $\mathbf{A} \in M_6(\mathbb{R})$ be symmetric with $\text{In}(\mathbf{A}) = (4, 1, 1)$, and let $\mathbf{E}$ be symmetric with $\text{rank } \mathbf{E} = 2$.

- (a) Which values of $n_+(\mathbf{A} + \mathbf{E})$ does [Theorem 17.10.3](#) allow?
 (b) Show that the two extreme values are both achieved, for suitable $\mathbf{A}$ and $\mathbf{E}$.

(c) Can $\mathbf{A} + \mathbf{E}$ be negative definite? Justify.

Solution.

- (a) By Theorem 17.10.3 with $r = 2$, $|n_+(\mathbf{A} + \mathbf{E}) - 4| \leq 2$, and $n_+ \leq 6$ always, so $n_+(\mathbf{A} + \mathbf{E}) \in \{2, 3, 4, 5, 6\}$.
- (b) Take $\mathbf{A} = \text{diag}(1, 1, 1, 1, -1, 0)$, which has inertia $(4, 1, 1)$. With $\mathbf{E} = \text{diag}(0, 0, 0, 0, 2, 1)$, of rank 2, we get $\mathbf{A} + \mathbf{E} = \mathbf{I}_6$, so $n_+ = 6$. With $\mathbf{E} = \text{diag}(-2, -2, 0, 0, 0, 0)$, of rank 2, we get $\mathbf{A} + \mathbf{E} = \text{diag}(-1, -1, 1, 1, -1, 0)$, so $n_+ = 2$.
- (c) No. Negative definiteness would mean $n_+(\mathbf{A} + \mathbf{E}) = 0$, and $|0 - 4| = 4 > 2$ contradicts Theorem 17.10.3.

Exercise. B3: A bordered matrix

Let $\mathbf{D} \in M_n(\mathbb{R})$ be symmetric and invertible, let $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, and put $\mathbf{M} = \begin{pmatrix} \mathbf{D} & \mathbf{b} \\ \mathbf{b}^\top & c \end{pmatrix}$.

- (a) Express $\text{In}(\mathbf{M})$ in terms of $\text{In}(\mathbf{D})$ and the scalar $s = c - \mathbf{b}^\top \mathbf{D}^{-1} \mathbf{b}$.
- (b) Take $\mathbf{D} = \text{diag}(1, -1)$ and $\mathbf{b} = (1, 1)$. Compute $\text{In}(\mathbf{M})$ for $c = 0$, $c = 1$, and $c = -1$.
- (c) For which c is $\mathbf{M}$ singular?

Solution.

- (a) By Theorem 17.10.4 with $\mathbf{A} = \mathbf{D}$, and $\mathbf{M}/\mathbf{D} = s$ a 1×1 matrix,

$$\text{In}(\mathbf{M}) = \text{In}(\mathbf{D}) + \begin{cases} (1, 0, 0) & s > 0, \\ (0, 1, 0) & s < 0, \\ (0, 0, 1) & s = 0. \end{cases}$$

- (b) Here $\mathbf{D}^{-1} = \text{diag}(1, -1) = \mathbf{D}$, so $\mathbf{b}^\top \mathbf{D}^{-1} \mathbf{b} = 1 - 1 = 0$ and $s = c$. Also $\text{In}(\mathbf{D}) = (1, 1, 0)$. Hence $\text{In}(\mathbf{M})$ is $(1, 1, 1)$ for $c = 0$, $(2, 1, 0)$ for $c = 1$, and $(1, 2, 0)$ for $c = -1$. The case $c = 0$ recovers Exercise (C3: Inertia of a block matrix) (c), which found the same triple by exhibiting the congruence directly.
- (c) $\mathbf{M}$ is singular exactly when $n_0(\mathbf{M}) \geq 1$, which by (a) happens exactly when $s = 0$, since $\mathbf{D}$ is invertible and so contributes $n_0(\mathbf{D}) = 0$. Here that is $c = 0$.

17.10.5.3 C. Going deeper

Exercise. C1: How large can a subspace be on which the form vanishes?

Let $\mathbf{A} \in M_n(F)$ be Hermitian with $\text{In}(\mathbf{A}) = (n_+, n_-, n_0)$. Call a subspace $W \leq F^n$ **totally isotropic** for $\mathbf{A}$ if $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = 0$ for every $\mathbf{x} \in W$; for real symmetric $\mathbf{A}$ this is the notion of Chapter 14 §07, in the form that section derives from polarization.

- (a) Prove that every totally isotropic W satisfies $\dim W \leq n_0 + \min(n_+, n_-)$.
- (b) Construct a totally isotropic subspace of dimension exactly $n_0 + \min(n_+, n_-)$.
- (c) Deduce that the largest dimension of a totally isotropic subspace equals n_0 if and only

if $\mathbf{A}$ is positive semidefinite or negative semidefinite.

Hint: for (a), a totally isotropic subspace satisfies both inequalities of Corollary 17.10.2 (b). For (b), pair up a positive eigenvector with a negative one.

Solution.

- (a) A totally isotropic W satisfies $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \geq 0$ throughout and also $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq 0$ throughout. By Corollary 17.10.2 (b), the first gives $\dim W \leq n_+ + n_0$ and the second $\dim W \leq n_- + n_0$. Hence $\dim W \leq n_0 + \min(n_+, n_-)$.
- (b) Fix an orthonormal eigenbasis $\mathbf{q}_1, \dots, \mathbf{q}_n$ with $\mathbf{A}\mathbf{q}_i = \lambda_i \mathbf{q}_i$ and $\lambda_1 \geq \dots \geq \lambda_n$ (Corollary 12.4.3, or Corollary 12.5.2 when $F = \mathbb{R}$), so that $\lambda_i > 0$ for $i \leq n_+$, $\lambda_i = 0$ for $n_+ < i \leq n_+ + n_0$, and $\lambda_i < 0$ for $i > n_+ + n_0$. Put $t = \min(n_+, n_-)$ and, for $1 \leq j \leq t$, let $j' = n_+ + 1 - j$. Then $j \leq n_+$ and $j' \geq n_+ + 1 - n_- > n_+ + n_0$, so $\lambda_j > 0 > \lambda_{j'}$ and the indices $1, \dots, t, 1', \dots, t'$ are $2t$ distinct indices, none of them in the zero block. Put

$$\mathbf{x}_j = \frac{\mathbf{q}_j}{\sqrt{\lambda_j}} + \frac{\mathbf{q}_{j'}}{\sqrt{-\lambda_{j'}}} \quad (1 \leq j \leq t),$$

and let W be spanned by $\mathbf{x}_1, \dots, \mathbf{x}_t$ together with the n_0 vectors $\mathbf{q}_i$ with $\lambda_i = 0$. These $t + n_0$ vectors are independent, since they are nonzero combinations of pairwise disjoint sets of vectors from a basis.

For $\mathbf{x} \in W$ write $\mathbf{x} = \sum_j c_j \mathbf{x}_j + \mathbf{z}$ with $\mathbf{A}\mathbf{z} = \mathbf{0}$. Then $\mathbf{A}\mathbf{x} = \sum_j c_j \mathbf{A}\mathbf{x}_j$ with $\mathbf{A}\mathbf{x}_j = \sqrt{\lambda_j} \mathbf{q}_j - \sqrt{-\lambda_{j'}} \mathbf{q}_{j'}$, and

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^* \mathbf{A}\mathbf{x} = \sum_{j,k} \overline{c_k} c_j \mathbf{x}_k^* \mathbf{A}\mathbf{x}_j + \sum_j c_j \mathbf{z}^* \mathbf{A}\mathbf{x}_j.$$

Each $\mathbf{z}^* \mathbf{A}\mathbf{x}_j = 0$, because $\mathbf{z}$ lies in the span of eigenvectors orthogonal to $\mathbf{q}_j$ and $\mathbf{q}_{j'}$. For $k \neq j$, $\mathbf{x}_k^* \mathbf{A}\mathbf{x}_j = 0$, because $\mathbf{x}_k$ and $\mathbf{A}\mathbf{x}_j$ are combinations of disjoint sets of orthonormal vectors. And

$$\mathbf{x}_j^* \mathbf{A}\mathbf{x}_j = \frac{\sqrt{\lambda_j}}{\sqrt{\lambda_j}} - \frac{\sqrt{-\lambda_{j'}}}{\sqrt{-\lambda_{j'}}} = 1 - 1 = 0.$$

Hence $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = 0$ for every $\mathbf{x} \in W$, and W is totally isotropic of dimension $n_0 + \min(n_+, n_-)$.

- (c) By (a) and (b) the largest totally isotropic dimension is $n_0 + \min(n_+, n_-)$, which equals n_0 exactly when $\min(n_+, n_-) = 0$, that is when $n_- = 0$ or $n_+ = 0$, that is when $\mathbf{A} \succeq \mathbf{0}$ or $-\mathbf{A} \succeq \mathbf{0}$ (Theorem 13.1.4).

Exercise. C2: The inertia of a Hermitian dilation

Let $\mathbf{A} \in M_{m \times n}(F)$ have rank r , and let $\mathcal{H}(\mathbf{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0} \end{pmatrix} \in M_{m+n}(F)$ be the Hermitian dilation of Proposition 17.9.3.

- (a) Prove that $\text{In}(\mathcal{H}(\mathbf{A})) = (r, r, m + n - 2r)$.
- (b) Deduce a formula for $\text{rank } \mathbf{A}$ in terms of the inertia of $\mathcal{H}(\mathbf{A})$, and check that it gives $\text{rank } \mathbf{0} = 0$.
- (c) Verify (a) for $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

Solution.

- (a) By Proposition 17.9.3 the eigenvalue list of $\mathcal{H}(\mathbf{A})$ with multiplicity consists of $\pm\sigma_i(\mathbf{A})$ for $1 \leq i \leq p = \min(m, n)$, together with $m + n - 2p$ zeros. Exactly r of the $\sigma_i(\mathbf{A})$ are nonzero (Theorem 13.9.3), so the positive entries are the r numbers $\sigma_1, \dots, \sigma_r$ and the negative entries are their negatives, giving $n_+ = n_- = r$. The rest are zero: $(m+n) - 2r$ of them. Hence $\text{In}(\mathcal{H}(\mathbf{A})) = (r, r, m + n - 2r)$.
- (b) $\text{rank } \mathbf{A} = n_+(\mathcal{H}(\mathbf{A})) = \frac{1}{2}(\text{rank } \mathcal{H}(\mathbf{A}))$, using $n_+ + n_- = \text{rank}$. For $\mathbf{A} = \mathbf{O}$ the dilation is the zero matrix of size $m + n$, whose inertia is $(0, 0, m + n)$, so the formula returns 0.
- (c) Here $m = 2$, $n = 3$, and $\text{rank } \mathbf{A} = 2$, since the first two columns are $\mathbf{e}_1, \mathbf{e}_2$. Also

$$\mathbf{A}\mathbf{A}^\top = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix},$$

of trace 7 and determinant 6, hence eigenvalues 6 and 1; so $\sigma_1(\mathbf{A}) = \sqrt{6}$ and $\sigma_2(\mathbf{A}) = 1$. By Proposition 17.9.3 the eigenvalues of the 5×5 matrix $\mathcal{H}(\mathbf{A})$ are $\pm\sqrt{6}, \pm 1$ and one 0, so $\text{In}(\mathcal{H}(\mathbf{A})) = (2, 2, 1)$, which is $(r, r, m + n - 2r) = (2, 2, 5 - 4)$.

Exercise. C3: Inertia from the leading principal minors

Let $\mathbf{M} \in M_n(F)$ be Hermitian and suppose that every leading principal submatrix $\mathbf{M}_k \in M_k(F)$ (rows and columns $1, \dots, k$) is **invertible**, for $k = 1, \dots, n$. Write $d_k = \det \mathbf{M}_k$ and $d_0 = 1$.

- (a) Prove that for $2 \leq k \leq n$ the Schur complement $\mathbf{M}_k/\mathbf{M}_{k-1}$ is the scalar d_k/d_{k-1} .
- (b) Deduce that $\text{In}(\mathbf{M}) = (n_+, n_-, 0)$, where n_- is the number of indices $k \in \{1, \dots, n\}$ with $d_k/d_{k-1} < 0$ — that is, the number of **sign changes** in the sequence $d_0, d_1, \dots, d_n$.
- (c) Apply this to the matrix $\mathbf{M}$ of [Example \(The Chapter 14 example, through a Schur complement\)](#) and compare.

Hint: for (b), induct on k using [Theorem 17.10.4](#).

Solution.

- (a) Write $\mathbf{M}_k = \begin{pmatrix} \mathbf{M}_{k-1} & \mathbf{b} \\ \mathbf{b}^* & c \end{pmatrix}$. Since $\mathbf{M}_{k-1}$ is invertible, $\mathbf{M}_k/\mathbf{M}_{k-1}$ is defined and is a 1×1 matrix, and Theorem 8.3.2 (a) gives $d_k = \det \mathbf{M}_k = \det \mathbf{M}_{k-1} \cdot (\mathbf{M}_k/\mathbf{M}_{k-1}) = d_{k-1}(\mathbf{M}_k/\mathbf{M}_{k-1})$. As $d_{k-1} \neq 0$, dividing gives $\mathbf{M}_k/\mathbf{M}_{k-1} = d_k/d_{k-1}$.
- (b) Induct on k . For $k = 1$, $\mathbf{M}_1 = (d_1)$ is a nonzero real scalar (real because $\mathbf{M}$ is Hermitian), and $\text{In}(\mathbf{M}_1)$ is $(1, 0, 0)$ or $(0, 1, 0)$ according to the sign of $d_1/d_0 = d_1$. For the step, $\mathbf{M}_{k-1}$ is invertible, so [Theorem 17.10.4](#) applies to $\mathbf{M}_k$ with $\mathbf{A} = \mathbf{M}_{k-1}$:

$$\text{In}(\mathbf{M}_k) = \text{In}(\mathbf{M}_{k-1}) + \text{In}(d_k/d_{k-1}),$$

and the last term is $(1, 0, 0)$ or $(0, 1, 0)$ according to the sign of the ratio, which is nonzero. Summing from $k = 1$ to n ,

$$\text{In}(\mathbf{M}) = \text{In}(\mathbf{M}_n) = (\#\{k : d_k/d_{k-1} > 0\}, \#\{k : d_k/d_{k-1} < 0\}, 0),$$

and $d_k/d_{k-1} < 0$ says exactly that d_{k-1} and d_k have opposite signs. (Taking $n_- = 0$

here recovers the leading-minor criterion of Theorem 13.1.5: all $d_k > 0$ if and only if $\mathbf{M} \succ \mathbf{O}$.)

- (c) For $\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ the leading minors are $d_1 = 1$, $d_2 = 2 - 1 = 1$, and $d_3 = -4$, all nonzero, so (b) applies. The sequence $d_0, d_1, d_2, d_3 = 1, 1, 1, -4$ has exactly one sign change, at the last step. Hence $n_- = 1$, $n_+ = 2$, and $\text{In}(\mathbf{M}) = (2, 1, 0)$, agreeing with [Example \(The Chapter 14 example, through a Schur complement\)](#) and with Example (A signature by symmetric elimination).