

15.10 The Exterior Algebra, and Clifford at Last

The exterior powers have so far been used one degree at a time, and the wedge of two of them has had nowhere to live: $\Lambda^2 V$ and $\Lambda^3 V$ are different spaces, and until now nothing said what $(u \wedge w) \wedge x$ means as an operation. Stacking them fixes that, exactly as stacking the tensor powers produced the tensor algebra (Definition 15.6.2) and stacking the symmetric powers produced the symmetric algebra (Definition 15.7.3). The result is a 2^n -dimensional algebra, and it is the last object this chapter needs. With it we can finally pay the debt Chapter 14 recorded in writing: a Clifford algebra of (V, q) exists, and its dimension is 2^n .

Throughout, F is a field and V is a vector space over F with $\dim V = n$. From the subsection on decomposability on, $\text{char } F \neq 2$, because Definition 14.12.2 assumes it. We use the exterior powers of Definition 15.8.2 with their universal property for alternating maps, their bases of increasing wedges (Theorem 15.8.5), the test Theorem 15.8.7, and the reordering rule Lemma 15.8.1 (b) of the previous section.

15.10.1 Stacking the exterior powers

The pieces are already in place; only the multiplication is missing. The pattern is the one Proposition 15.7.4 used for symmetric powers: a product of a k -fold wedge and an l -fold wedge ought to be the $(k + l)$ -fold wedge of all the vectors, and the only thing to check is that this depends on the two factors and not on the way each was written.

Proposition 15.10.1. The Wedge of Two Exterior Powers

Let $k, l \geq 1$. There is a **unique** bilinear map

$$\Lambda^k V \times \Lambda^l V \rightarrow \Lambda^{k+l} V, \quad (\omega, \eta) \mapsto \omega \wedge \eta,$$

such that for all $u_1, \dots, u_k, w_1, \dots, w_l \in V$,

$$(u_1 \wedge \dots \wedge u_k) \wedge (w_1 \wedge \dots \wedge w_l) = u_1 \wedge \dots \wedge u_k \wedge w_1 \wedge \dots \wedge w_l.$$

Idea. Freeze the second factor's vectors and use the universal property once, in the first k slots; that turns the left-hand factor into an element of $\Lambda^k V$. Then let the second factor's vectors vary: what we have built depends on them l -linearly, and vanishes as soon as two of them coincide, with values in a space of linear maps, so the universal property applies a second time.

Proof. Fix $w_1, \dots, w_l \in V$. The map $V^k \rightarrow \Lambda^{k+l} V$ given by

$$(u_1, \dots, u_k) \mapsto u_1 \wedge \dots \wedge u_k \wedge w_1 \wedge \dots \wedge w_l$$

is k -linear, since the wedge of $k + l$ vectors is linear in each slot and the last l are frozen, and alternating, since a repeat among the first k arguments is a repeat among all $k + l$. By

Definition 15.8.2 it induces a unique linear map

$$R(w_1, \dots, w_l): \Lambda^k V \rightarrow \Lambda^{k+l} V$$

with $R(w_1, \dots, w_l)(u_1 \wedge \dots \wedge u_k)$ equal to the displayed wedge.

Now let the w 's vary. The assignment $(w_1, \dots, w_l) \mapsto R(w_1, \dots, w_l)$, with values in the vector space $\mathcal{L}(\Lambda^k V, \Lambda^{k+l} V)$, is l -linear and alternating: two linear maps on $\Lambda^k V$ are equal as soon as they agree on the wedges $u_1 \wedge \dots \wedge u_k$, which span $\Lambda^k V$ (Theorem 15.8.5), and on those the required identities are the linearity and the alternation of the wedge of $k + l$ vectors in its last l slots. So Definition 15.8.2 applies again and gives a unique linear map

$$\Lambda^l V \rightarrow \mathcal{L}(\Lambda^k V, \Lambda^{k+l} V), \quad w_1 \wedge \dots \wedge w_l \mapsto R(w_1, \dots, w_l).$$

Reading a linear map into a space of linear maps as a bilinear map gives the required product, with the stated values. It is unique because the wedges span both factors and a bilinear map is determined on a pair of spanning sets. This proves the proposition. ■

With the product in hand, the algebra assembles itself.

The exterior algebra is all the exterior powers at once, multiplied by wedging.

Definition 15.10.1. Exterior Algebra

Let V be a vector space over F , with $\Lambda^0 V = F$ and $\Lambda^1 V = V$ as in Definition 15.8.2. The **exterior algebra**, or **Grassmann algebra**, of V is the vector space

$$\Lambda V := \bigoplus_{k \geq 0} \Lambda^k V,$$

whose elements are the sequences $x = (x_0, x_1, \dots)$ with $x_k \in \Lambda^k V$ and only finitely many x_k nonzero, with componentwise operations, and whose multiplication is

$$(x \wedge y)_m = \sum_{k+l=m} x_k \wedge y_l,$$

where $\wedge$ on the right is [Proposition 15.10.1](#) for $k, l \geq 1$, and is scalar multiplication when $k = 0$ or $l = 0$. We identify each $\Lambda^k V$ with the sequences supported in position k , so that F and V are subspaces of ΛV , and call an element of $\Lambda^k V \subseteq \Lambda V$ **homogeneous of degree k** .

In words: an element of ΛV is a finite sum $x_0 + x_1 + \dots + x_N$ of pieces of different degrees, the scalars sit inside as the degree-0 piece, and **the product adds degrees**. The finiteness clause is automatic here, since $\Lambda^k V = \{0\}$ once $k > n$ (Theorem 15.8.5).

Theorem 15.10.2. Structure of the Exterior Algebra

Let $\dim V = n$ with basis $(e_1, \dots, e_n)$. For a subset $S = \{i_1 < \dots < i_k\} \subseteq [n]$ write $e_S = e_{i_1} \wedge \dots \wedge e_{i_k}$, with $e_\emptyset = 1$. Then

- (a) ΛV is an associative unital F -algebra with identity $1 \in \Lambda^0 V = F$, and $\Lambda^k V \wedge \Lambda^l V \subseteq \Lambda^{k+l} V$.

- (b) The 2^n elements e_S , one for each subset $S \subseteq [n]$, form a basis of ΛV . In particular $\dim_F \Lambda V = 2^n$.
- (c) ΛV is generated as an algebra by 1 and V .

Proof.

- (a) Bilinearity of the product holds componentwise, by [Proposition 15.10.1](#). For associativity, both $(x \wedge y) \wedge z$ and $x \wedge (y \wedge z)$ are trilinear in (x, y, z) , so it is enough to compare them on homogeneous elements, and then on the wedges of vectors, which span each $\Lambda^k V$ for $k \geq 1$ (Theorem 15.8.5) while $\Lambda^0 V$ is spanned by 1. On those both sides are the single wedge of all the listed vectors in order, by the defining identity of [Proposition 15.10.1](#) applied twice. The scalar 1 acts on each $\Lambda^k V$ as multiplication by 1, hence as the identity. The grading statement is built into the definition of the product.
- (b) By Theorem 15.8.5 the e_S with $|S| = k$ form a basis of $\Lambda^k V$, and $\Lambda^k V = \{0\}$ for $k > n$. A basis of a direct sum is the union of bases of the summands, so the e_S over all $S \subseteq [n]$ form a basis of ΛV . Their number is $\sum_{k=0}^n \binom{n}{k} = 2^n$, by the binomial theorem applied to $(1+1)^n$.
- (c) Each e_S is a product of elements of V by the defining identity of the product, and the e_S span by (b).

■

The one structural fact that is not inherited from the tensor algebra is how the order of a product matters.

Theorem 15.10.3. Graded Commutativity

Let $\omega \in \Lambda^k V$ and $\eta \in \Lambda^l V$. Then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

In particular $\omega \wedge \omega = 0$ whenever k is **odd**, over every field.

Idea. Count the swaps. To turn $w_1 \wedge \cdots \wedge w_l \wedge u_1 \wedge \cdots \wedge u_k$ into $u_1 \wedge \cdots \wedge u_k \wedge w_1 \wedge \cdots \wedge w_l$, each of the k vectors u_i must cross each of the l vectors w_j , and every crossing costs a sign.

Proof. Both sides are bilinear in (ω, η) , so by Theorem 15.8.5 it suffices to take $\omega = u_1 \wedge \cdots \wedge u_k$ and $\eta = w_1 \wedge \cdots \wedge w_l$, and the cases $k = 0$ or $l = 0$ are scalar multiplication and are clear. The permutation of $\{1, \dots, k+l\}$ carrying the list $(w_1, \dots, w_l, u_1, \dots, u_k)$ to $(u_1, \dots, u_k, w_1, \dots, w_l)$ has exactly kl inversions, one for each pair consisting of a u and a w , so its sign is $(-1)^{kl}$ (Definition 7.3.2). By Lemma 15.8.1 (b) the two wedges differ by that factor, which is the claim.

For the last sentence let k be odd and $\omega \in \Lambda^k V$. Write $\omega = \sum_S c_S e_S$ in the basis of [Theorem 15.10.2](#), the sum over the S with $|S| = k$. Then

$$\omega \wedge \omega = \sum_S \sum_T c_S c_T e_S \wedge e_T.$$

A term with $S = T$ has a repeated vector, so $e_S \wedge e_S = 0$. A pair $S \neq T$ occurs twice, and the two occurrences are $c_{SCT} e_S \wedge e_T$ and $c_{TCS} e_T \wedge e_S = -c_{SCT} e_S \wedge e_T$, since $(-1)^{kl} = (-1)^{k^2} = -1$ for odd k . So the two cancel, and nothing divides by 2 anywhere. ■

Warning. Graded commutative is not commutative. In ΛF^2 we have $\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1$, and these are different elements unless $\text{char } F = 2$. What is true in all degrees is only the signed rule. And the sign is $+1$ as soon as one of the degrees is even, so an element of $\Lambda^2 V$ commutes with everything, while two elements of $\Lambda^1 V = V$ anticommute.

Warning. $\omega \wedge \omega$ need not be zero. The identity $v \wedge v = 0$ is a statement about degree 1. For $\omega = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4$ in ΛF^4 , a direct expansion gives $\omega \wedge \omega = 2 \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$, which is nonzero whenever $\text{char } F \neq 2$. Squaring to zero is the privilege of odd degree.

Example. The exterior algebra of a three-dimensional space

Describe ΛF^3 completely, and compute $(\mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3) \wedge (\mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_3)$.

Solution. By [Theorem 15.10.2](#) it has dimension $2^3 = 8$, with basis

$$1; \quad \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3; \quad e_{12}, e_{13}, e_{23}; \quad e_{123},$$

writing e_{ij} for $\mathbf{e}_i \wedge \mathbf{e}_j$ and e_{123} for $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. The degrees have dimensions 1, 3, 3, 1. Every product of two elements of degree ≥ 2 lands in degree ≥ 4 , hence is 0.

For the computation, expand by bilinearity into four products:

$$\begin{aligned} \mathbf{e}_1 \wedge \mathbf{e}_2 &= e_{12}, \\ \mathbf{e}_1 \wedge (\mathbf{e}_1 \wedge \mathbf{e}_3) &= 0 \quad (\text{repeated } \mathbf{e}_1), \\ (\mathbf{e}_2 \wedge \mathbf{e}_3) \wedge \mathbf{e}_2 &= 0 \quad (\text{repeated } \mathbf{e}_2), \\ (\mathbf{e}_2 \wedge \mathbf{e}_3) \wedge (\mathbf{e}_1 \wedge \mathbf{e}_3) &= 0 \quad (\text{degree } 4). \end{aligned}$$

So the product is e_{12} , while the reversed product is $-e_{12}$ plus the same three zeros.

Quick check. In ΛF^4 , what is $(\mathbf{e}_1 \wedge \mathbf{e}_2) \wedge (\mathbf{e}_2 \wedge \mathbf{e}_3)$, and what is $(\mathbf{e}_1 \wedge \mathbf{e}_2) \wedge (\mathbf{e}_3 \wedge \mathbf{e}_4)$?

Solution. The first is $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = 0$, since $\mathbf{e}_2$ is repeated. The second is $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$, a basis vector of $\Lambda^4 F^4$, so it is nonzero. Wedging two 2-vectors gives zero exactly when the four vectors involved are dependent.

15.10.2 Decomposable elements

Section 2 warned that not every element of $V \otimes W$ is a simple tensor $v \otimes w$, and proved it with $\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2$ (Example (A tensor that is not simple)). The same thing happens one floor up, and for the same reason: the simple objects form a low-dimensional cone inside a large space, and a sum of two of them usually escapes it.

Definition 15.10.2. Decomposable Element

Let $k \geq 1$. An element $\omega \in \Lambda^k V$ is **decomposable**, or a **k -blade**, if there exist $v_1, \dots, v_k \in V$ with

$$\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_k.$$

Examples. Every element of $\Lambda^1 V = V$ is decomposable, with $k = 1$; so is 0, being $v \wedge v \wedge v_3 \wedge \cdots \wedge v_k$; and so is every element of the line $\Lambda^n V$, since $ce_1 \wedge \cdots \wedge e_n = (ce_1) \wedge e_2 \wedge \cdots \wedge e_n$.

Why the word. A decomposable $\omega \neq 0$ carries a k -dimensional subspace: its factors are independent by Theorem 15.8.7, and they span one. Nondecomposable elements are exactly those that come from no single subspace, which is why the failure below is a phenomenon and not a notational accident.

Theorem 15.10.4. Not Every 2-Vector Is Decomposable

Let $\dim V = n \geq 4$, let $(e_1, \dots, e_n)$ be a basis of V , and set

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2 V.$$

Then ω is **not** decomposable.

Idea. A decomposable 2-vector $u \wedge w$ squares to zero, because the four-fold wedge $u \wedge w \wedge u \wedge w$ repeats a vector. So compute $\omega \wedge \omega$ and find it nonzero. That test needs $\text{char } F \neq 2$; the proof below takes a second route that does not, by asking which vectors annihilate ω .

Proof. Suppose $\omega = u \wedge w$ for some $u, w \in V$. Since $e_1 \wedge e_2$ and $e_3 \wedge e_4$ are distinct basis vectors of $\Lambda^2 V$ (Theorem 15.8.5), $\omega \neq 0$, so (u, w) is linearly independent by Theorem 15.8.7; in particular $u \neq 0$.

Consider $u \wedge \omega = u \wedge u \wedge w = 0$, the wedge having a repeated argument. We show this is impossible. Let $v = \sum_{i=1}^n x_i e_i$ be any vector and compute $v \wedge \omega$, using bilinearity and dropping every term with a repeated basis vector:

$$\begin{aligned} v \wedge \omega &= x_3 e_3 \wedge e_1 \wedge e_2 + x_4 e_4 \wedge e_1 \wedge e_2 \\ &\quad + x_1 e_1 \wedge e_3 \wedge e_4 + x_2 e_2 \wedge e_3 \wedge e_4 \\ &\quad + \sum_{i \geq 5} x_i (e_i \wedge e_1 \wedge e_2 + e_i \wedge e_3 \wedge e_4). \end{aligned}$$

Sorting each wedge by Lemma 15.8.1 (b) turns this into a combination of the distinct basis vectors

$$e_{123}, e_{124}, e_{134}, e_{234}, e_{12i}, e_{34i} \quad (i \geq 5)$$

of $\Lambda^3 V$, with coefficients x_3, x_4, x_1, x_2 and x_i, x_i respectively, up to signs. Since these basis vectors are distinct and the e_S are independent (Theorem 15.8.5), $v \wedge \omega = 0$ forces $x_1 = x_2 = \cdots = x_n = 0$, that is, $v = 0$. This contradicts $u \neq 0$ with $u \wedge \omega = 0$. Hence ω is not decomposable. ■

Remark. When $\text{char } F \neq 2$ there is a one-line version: $\omega \wedge \omega = 2 e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$, whereas $(u \wedge w) \wedge (u \wedge w) = 0$ always. The proof above was written to avoid that division, because the test fails over $\mathbb{F}_2$ — see the warning next.

Warning. **The square test is blind in characteristic 2.** Over $\mathbb{F}_2$ every $\omega \in \Lambda^2 V$ satisfies $\omega \wedge \omega = 0$: in the expansion $\sum_{S,T} c_S c_T e_S \wedge e_T$ the diagonal terms vanish and each off-diagonal pair contributes $2 c_S c_T e_S \wedge e_T = 0$. So over $\mathbb{F}_2$ the vanishing of $\omega \wedge \omega$ says nothing at all, while $e_1 \wedge e_2 + e_3 \wedge e_4$ is still not decomposable, by the proof above. A criterion and a phenomenon are different things.

Quick check. Why does Theorem 15.10.4 need $n \geq 4$? Show that every element of $\Lambda^2 V$ is decomposable when $\dim V = 3$.

Solution. With $n \leq 3$ there is no room to write $e_1 \wedge e_2 + e_3 \wedge e_4$ at all. For $n = 3$, take $\omega = a e_{12} + b e_{13} + c e_{23}$. If $a = b = c = 0$ then $\omega = 0$, which is decomposable. If $a \neq 0$, then

$$\left(e_1 - \frac{c}{a} e_3\right) \wedge (a e_2 + b e_3) = a e_{12} + b e_{13} + c e_{23} = \omega,$$

using $e_3 \wedge e_2 = -e_{23}$. If $a = 0$, relabel the basis so that the surviving nonzero coefficient sits in the first slot and repeat. So the phenomenon starts exactly at $n = 4$.

Together with Example (A tensor that is not simple) this is the chapter's negative result, stated in its two forms: a general tensor is not a simple tensor, and a general k -vector is not a blade. The optimistic reading — that every element of a space built from V is built from finitely many vectors of V in the obvious way — is false in both places, and Section 11 will find the first form of it wearing a physical name.

15.10.3 Algebras modulo relations

One tool is still missing. The Clifford algebra is to be built by imposing relations on $T(V)$, and imposing relations on an **algebra** means dividing by more than a subspace: the relations must stay relations after multiplication on either side. Chapter 6 met the commutative version of the idea for $F[x]$ (Definition 6.2.1).

Definition 15.10.3. Two-Sided Ideal, and the Ideal Generated by a Set

Let A be an F -algebra (Definition 14.12.1). A **two-sided ideal** of A is a subspace $\mathcal{I} \subseteq A$ such that $yx \in \mathcal{I}$ and $xy \in \mathcal{I}$ for **every** $x \in \mathcal{I}$ and **every** $y \in A$. For a subset $S \subseteq A$, the **ideal generated by S** is

$$\langle S \rangle := \text{span}\{ ysz : y, z \in A, s \in S \}.$$

Examples. In $F[x]$ the ideals are the sets $\langle p \rangle$ of Definition 6.2.1; in any A , both $\{0\}$ and A are ideals. **Nonexample by minimal change.** The diagonal matrices in $M_2(F)$ form a subspace closed under multiplication, but not an ideal: it contains $\mathbf{E}_{11}$, while $\mathbf{E}_{21}\mathbf{E}_{11} = \mathbf{E}_{21}$ is not diagonal. Being closed among one's own elements is not the condition; absorbing products with the **whole** algebra is.

The span above is a subspace by construction, and it is a two-sided ideal because $a(ysz) = (ay)sz$ and $(ysz)a = ys(za)$ are again of the listed form. It contains S , taking $y = z = 1_A$, and every two-sided ideal containing S contains each ysz ; so $\langle S \rangle$ is the smallest two-sided ideal containing S , which is what “generated” means.

Lemma 15.10.5. Quotient of an Algebra by an Ideal

Let A be an F -algebra and $\mathcal{I}$ a two-sided ideal of A .

- (a) The quotient space $A/\mathcal{I}$ carries exactly one multiplication making the quotient map $\pi: A \rightarrow A/\mathcal{I}$ an algebra homomorphism, namely $(x + \mathcal{I})(y + \mathcal{I}) = xy + \mathcal{I}$; with it $A/\mathcal{I}$ is an F -algebra with identity $1_A + \mathcal{I}$, and $\ker \pi = \mathcal{I}$.
- (b) If $\varphi: A \rightarrow B$ is an algebra homomorphism with $\mathcal{I} \subseteq \ker \varphi$, then there is exactly one algebra homomorphism $\bar{\varphi}: A/\mathcal{I} \rightarrow B$ with $\bar{\varphi} \circ \pi = \varphi$.
- (c) The kernel of any algebra homomorphism is a two-sided ideal.

Proof.

- (a) The rule is well defined: if $x' = x + u$ and $y' = y + w$ with $u, w \in \mathcal{I}$, then

$$x'y' = xy + uy + xw + uw \in xy + \mathcal{I},$$

since $\mathcal{I}$ absorbs multiplication on both sides and is closed under addition. Bilinearity, associativity, and the identity are then inherited from A , because each is an identity among cosets that follows by applying π to the corresponding identity in A . Any multiplication making π multiplicative must satisfy the displayed rule, and π is surjective, so the multiplication is unique. Finally $\ker \pi = \mathcal{I}$ by Lemma 4.4.3.

- (b) Since φ is linear with $\mathcal{I} \subseteq \ker \varphi$, Theorem 4.4.10 (a) gives exactly one **linear** $\bar{\varphi}$ with $\bar{\varphi}\pi = \varphi$. It is multiplicative: every element of $A/\mathcal{I}$ is $\pi(x)$ for some x , and

$$\bar{\varphi}(\pi(x)\pi(y)) = \bar{\varphi}(\pi(xy)) = \varphi(xy) = \varphi(x)\varphi(y),$$

which is $\bar{\varphi}(\pi(x))\bar{\varphi}(\pi(y))$. Also $\bar{\varphi}(\pi(1_A)) = \varphi(1_A) = 1_B$. Uniqueness is already contained in the uniqueness of the linear map.

- (c) If $\varphi(x) = 0$ and $y \in A$, then $\varphi(xy) = \varphi(x)\varphi(y) = 0$ and $\varphi(yx) = 0$; and $\ker \varphi$ is a subspace because φ is linear.

■

15.10.4 The Clifford algebra, constructed

Chapter 14 §12 defined a Clifford algebra of (V, q) as a pair (C, ι) with $\iota: V \rightarrow C$ linear satisfying

- (C1) $\iota(v)^2 = q(v)1_C$ for every $v \in V$, and
- (C2) the universal property: every linear $f: V \rightarrow A$ into an F -algebra with $f(v)^2 = q(v)1_A$ factors as $f = \bar{f} \circ \iota$ for a unique algebra homomorphism $\bar{f}$,

and then said in as many words that it could not produce one: “The definition says what a Clifford algebra would be; it does not produce one.” What it needed was an algebra of formal strings of vectors, which is now Definition 15.6.2, and a way to impose the relation $v \otimes v = q(v)1$, which is now Lemma 15.10.5.

Theorem 15.10.6. Existence of the Clifford Algebra

Let F be a field with $\text{char } F \neq 2$, let V be a finite-dimensional F -vector space, let β be a symmetric bilinear form on V and $q(v) = \beta(v, v)$. Let

$$\mathcal{I}_q := \langle v \otimes v - q(v)1 : v \in V \rangle \subseteq T(V)$$

be the two-sided ideal generated by these elements, put $\text{Cl}(q) := T(V)/\mathcal{I}_q$, and let $\iota: V \rightarrow \text{Cl}(q)$ be the restriction to $V \subseteq T(V)$ of the quotient map π . Then $(\text{Cl}(q), \iota)$ is a Clifford algebra of (V, q) in the sense of Definition 14.12.2.

In particular a Clifford algebra exists for every (V, q) , and by Proposition 14.12.1 any two are isomorphic by a unique isomorphism compatible with the structure maps.

Idea. The tensor algebra is the free associative algebra on V : a linear map out of V extends uniquely to it (Theorem 15.6.6), with no relations in the way. Dividing by the smallest ideal containing $v \otimes v - q(v)1$ imposes exactly the relation (C1) and nothing else, and “nothing else” is precisely what the universal property (C2) asserts.

Proof. By Lemma 15.10.5 (a), $\text{Cl}(q)$ is an F -algebra and π is a surjective algebra homomorphism with $\ker \pi = \mathcal{I}_q$. The map $\iota = \pi|_V$ is linear, being a restriction of a linear map.

(C1). Let $v \in V$. In $T(V)$ the product of v with itself is $v \otimes v$ (Definition 15.6.2), and $v \otimes v - q(v)1$ is a generator of $\mathcal{I}_q = \ker \pi$. Since π is multiplicative and linear,

$$\iota(v)^2 = \pi(v)\pi(v) = \pi(v \otimes v) = \pi(q(v)1) = q(v)1_{\text{Cl}(q)}.$$

(C2). Let A be an F -algebra and let $f: V \rightarrow A$ be linear with $f(v)^2 = q(v)1_A$ for every $v \in V$. By Theorem 15.6.6 there is a unique algebra homomorphism $\tilde{f}: T(V) \rightarrow A$ with $\tilde{f}|_V = f$. For every $v \in V$,

$$\tilde{f}(v \otimes v - q(v)1) = f(v)f(v) - q(v)1_A = 0,$$

so $\ker \tilde{f}$ contains every generator of $\mathcal{I}_q$. By Lemma 15.10.5 (c), $\ker \tilde{f}$ is a two-sided ideal, and $\mathcal{I}_q$ is the **smallest** one containing those generators (Definition 15.10.3), so $\mathcal{I}_q \subseteq \ker \tilde{f}$. By Lemma 15.10.5 (b) there is exactly one algebra homomorphism $\bar{f}: \text{Cl}(q) \rightarrow A$ with $\bar{f} \circ \pi = \tilde{f}$, and restricting to V gives $\bar{f} \circ \iota = \tilde{f}|_V = f$.

For the uniqueness clause of (C2), let $g: \text{Cl}(q) \rightarrow A$ be any algebra homomorphism with $g \circ \iota = f$. Then $g \circ \pi$ is an algebra homomorphism $T(V) \rightarrow A$ whose restriction to V is $g\iota = f$, so $g\pi = \bar{f}$ by the uniqueness clause of Theorem 15.6.6. Hence $g\pi = \bar{f}\pi$, and π is surjective, so $g = \bar{f}$. This proves the theorem. ■

Remark. Neither $\text{char } F \neq 2$ nor finite dimension was used in this proof; the construction and both checks work for any V and any quadratic form. The hypotheses are carried because Definition 14.12.2 states them, and because the dimension count below needs an orthogonal basis.

15.10.5 The dimension, at last

Chapter 14 proved half of the count. Its Proposition 14.12.3 showed that for an orthogonal basis $(e_1, \dots, e_n)$ of β the 2^n **sorted products**

$$\iota(e_{i_1})\iota(e_{i_2}) \cdots \iota(e_{i_k}), \quad i_1 < i_2 < \cdots < i_k,$$

span C , so that $\dim_F C \leq 2^n$. The other half — their independence — is what this section owes. The idea is to make ΛV , which has exactly the right dimension, into a place where the vectors of V act with square $q(v)$; the universal property then sends C into the operators on ΛV , and evaluating at 1 turns the sorted products into distinct basis vectors.

Two families of operators on ΛV do the work: wedging a basis vector on, and deleting it again.

Lemma 15.10.7. Creation and Deletion Operators

Let $(e_1, \dots, e_n)$ be a basis of V , and for $S \subseteq [n]$ let e_S be as in Theorem 15.10.2. For $i \in [n]$ write $\sigma(i, S) = |\{j \in S : j < i\}|$. Let $E_i \in \mathcal{L}(\Lambda V)$ be left multiplication by e_i , that is, $E_i(\omega) = e_i \wedge \omega$, and let $D_i \in \mathcal{L}(\Lambda V)$ be the unique linear map with

$$D_i(e_S) = \begin{cases} (-1)^{\sigma(i, S)} e_{S \setminus \{i\}}, & i \in S, \\ 0, & i \notin S, \end{cases}$$

which exists and is unique because the e_S form a basis (Theorem 15.10.2). Then for all $i, j \in [n]$,

$$\begin{aligned} E_i E_j + E_j E_i &= 0, \\ D_i D_j + D_j D_i &= 0, \\ D_i E_j + E_j D_i &= \delta_{ij} \text{id}_{\Lambda V}. \end{aligned}$$

Idea. E_i inserts the index i into the set S and D_i removes it, each at the cost of the sign needed to move e_i past the smaller members of S . All three identities are then a comparison of two signs, and in the third the two cases “ $i \in S$ ” and “ $i \notin S$ ” are exactly the two ways the identity can be satisfied: one term is zero and the other is e_S .

Proof. First record what E_i does on the basis. If $i \in S$ then $e_i \wedge e_S$ repeats e_i and is 0. If $i \notin S$, then sorting $e_i \wedge e_S$ moves e_i past the $\sigma(i, S)$ members of S smaller than i , so by Lemma 15.8.1 (b)

$$E_i(e_S) = (-1)^{\sigma(i, S)} e_{S \cup \{i\}}.$$

First identity. For every $\omega \in \Lambda V$, $E_i E_j(\omega) = e_i \wedge e_j \wedge \omega$ and $E_j E_i(\omega) = e_j \wedge e_i \wedge \omega$, and $e_j \wedge e_i = -e_i \wedge e_j$ by Theorem 15.10.3. Adding gives 0. (For $i = j$ each term is separately 0.)

Third identity, case $i = j$. If $i \notin S$, then $D_i(e_S) = 0$ and $E_i(e_S) = (-1)^{\sigma(i, S)} e_{S \cup \{i\}}$; since $\sigma(i, S \cup \{i\}) = \sigma(i, S)$, applying D_i returns $(-1)^{2\sigma(i, S)} e_S = e_S$. If $i \in S$, then $E_i(e_S) = 0$ and $D_i(e_S) = (-1)^{\sigma(i, S)} e_{S \setminus \{i\}}$; since $\sigma(i, S \setminus \{i\}) = \sigma(i, S)$, applying E_i returns e_S . Either way the sum is e_S .

Third identity, case $i \neq j$. If $j \in S$ then $E_j(e_S) = 0$, and $D_i(e_S)$ is either 0 or a multiple of $e_{S \setminus \{i\}}$, which still contains j , so E_j kills it; both terms vanish. If $j \notin S$ and $i \notin S$ then $D_i(e_S) = 0$ and $D_i(e_{S \cup \{j\}}) = 0$ as well, since $i \neq j$; again both terms vanish. So let $j \notin S$ and $i \in S$. Then

$$\begin{aligned} D_i E_j(e_S) &= (-1)^{\sigma(j, S) + \sigma(i, S \cup \{j\})} e_{(S \cup \{j\}) \setminus \{i\}}, \\ E_j D_i(e_S) &= (-1)^{\sigma(i, S) + \sigma(j, S \setminus \{i\})} e_{(S \setminus \{i\}) \cup \{j\}}, \end{aligned}$$

and the two index sets agree. Now $\sigma(i, S \cup \{j\}) = \sigma(i, S) + [j < i]$ and $\sigma(j, S \setminus \{i\}) = \sigma(j, S) - [i < j]$, writing $[P]$ for 1 if P holds and 0 otherwise. So the two exponents differ by $[j < i] + [i < j] = 1$, which is odd because $i \neq j$. The two terms therefore cancel.

Second identity. Both $D_i D_j(e_S)$ and $D_j D_i(e_S)$ vanish unless $i, j \in S$ and $i \neq j$, and in that case they are $(-1)^{\sigma(j, S) + \sigma(i, S \setminus \{j\})}$ and $(-1)^{\sigma(i, S) + \sigma(j, S \setminus \{i\})}$ times the same $e_{S \setminus \{i, j\}}$. The same two identities for σ show the exponents differ by an odd number, so the terms cancel. For $i = j$, $D_i^2(e_S) = 0$ because $i \notin S \setminus \{i\}$. This proves the lemma. ■

Theorem 15.10.8. Dimension of a Clifford Algebra

Let F be a field with $\text{char } F \neq 2$, let $\dim V = n$, let β be a symmetric bilinear form on V with quadratic form q , and let (C, ι) be a Clifford algebra of (V, q) . Let $(e_1, \dots, e_n)$ be an orthogonal basis of V for β . Then the 2^n sorted products of Proposition 14.12.3 form a **basis** of C . Consequently

$$\dim_F C = 2^n,$$

the map ι is injective, and C is isomorphic to ΛV as a **vector space**. This is Theorem 14.12.4.

Idea. Build a representation. ① Turn each basis vector e_i into the operator $c_i = E_i + d_i D_i$ on ΛV , where $d_i = q(e_i)$; Lemma 15.10.7 makes these anticommute and gives $c_i^2 = d_i \text{id}$, which is exactly (C1). ② The universal property then hands us an algebra homomorphism $C \rightarrow \mathcal{L}(\Lambda V)$. ③ Evaluate at the element $1 \in \Lambda^0 V$: the sorted product for the index set S is sent to e_S , because at each step the new index is smaller than all the ones already there, so the deletion half contributes nothing. Distinct basis vectors come out, so the sorted products were independent.

Proof. An orthogonal basis exists by Theorem 14.4.2, since $\text{char } F \neq 2$. Write $d_i = q(e_i)$ and $c_i = E_i + d_i D_i \in \mathcal{L}(\Lambda V)$, with E_i, D_i as in Lemma 15.10.7. By Proposition 14.12.3 the sorted products span C and $\dim_F C \leq 2^n$, so only independence is left.

Step 1: the c_i satisfy the Clifford relations. Expanding the product and collecting the four brackets,

$$\begin{aligned} c_i c_j + c_j c_i &= (E_i E_j + E_j E_i) + d_j (E_i D_j + D_j E_i) \\ &\quad + d_i (D_i E_j + E_j D_i) + d_i d_j (D_i D_j + D_j D_i). \end{aligned}$$

By Lemma 15.10.7 the first and last brackets are 0 and each middle bracket is $\delta_{ij} \text{id}$. So $c_i c_j = -c_j c_i$ for $i \neq j$. Setting $i = j$ in the display would give $2c_i^2$, so expand c_i^2 directly instead; with $E_i^2 = D_i^2 = 0$, which were checked separately in the lemma,

$$c_i^2 = E_i^2 + d_i (E_i D_i + D_i E_i) + d_i^2 D_i^2 = d_i \text{id},$$

with no division by 2 anywhere. Now $\mathcal{L}(\Lambda V)$ is an F -algebra under composition (Definition 14.12.1). Let $f: V \rightarrow \mathcal{L}(\Lambda V)$ be the linear map with $f(e_i) = c_i$, which exists and is unique by Theorem 2.1.6. For $v = \sum_i x_i e_i$,

$$f(v)^2 = \sum_i x_i^2 c_i^2 + \sum_{i < j} x_i x_j (c_i c_j + c_j c_i) = \left(\sum_i d_i x_i^2 \right) \text{id},$$

the cross terms vanishing because $i \neq j$ there. Since the basis is orthogonal, $\beta(e_i, e_j) = 0$ for $i \neq j$, so $q(v) = \beta(v, v) = \sum_i d_i x_i^2$. Hence $f(v)^2 = q(v) \text{id}$, which is (C1) for f .

Step 2: the induced homomorphism. By the universal property (C2) of Definition 14.12.2 there is an algebra homomorphism $\Psi: C \rightarrow \mathcal{L}(\Lambda V)$ with $\Psi\iota = f$. Define the linear map

$$\Phi: C \rightarrow \Lambda V, \quad \Phi(x) = \Psi(x)(1),$$

where $1 \in \Lambda^0 V = F \subseteq \Lambda V$. It is linear because Ψ is linear and evaluation at a fixed element is linear.

Step 3: Φ sends sorted products to basis vectors. Let $S = \{i_1 < \dots < i_k\} \subseteq [n]$. Since Ψ is multiplicative and $\Psi\iota = f$,

$$\Psi(\iota(e_{i_1}) \cdots \iota(e_{i_k})) = c_{i_1} c_{i_2} \cdots c_{i_k}.$$

We claim $c_{i_r} c_{i_{r+1}} \cdots c_{i_k}(1) = e_{T_r}$ with $T_r = \{i_r, \dots, i_k\}$, by downward induction on r . For $r = k + 1$ the product is empty and the value is $1 = e_\emptyset$. Assume the claim for $r + 1$. Because $i_r < i_{r+1}$ and the indices increase, $i_r \notin T_{r+1}$, so $D_{i_r}(e_{T_{r+1}}) = 0$; and i_r is smaller than every element of T_{r+1} , so $\sigma(i_r, T_{r+1}) = 0$ and $E_{i_r}(e_{T_{r+1}}) = e_{T_r}$. Hence $c_{i_r}(e_{T_{r+1}}) = e_{T_r}$, completing the induction. Taking $r = 1$,

$$\Phi(\iota(e_{i_1}) \cdots \iota(e_{i_k})) = e_S,$$

and for $S = \emptyset$, $\Phi(1_C) = \text{id}(1) = 1 = e_\emptyset$.

Step 4: conclusion. Suppose a linear combination of the sorted products is 0. Applying Φ gives the same combination of the e_S , which are linearly independent by Theorem 15.10.2 (b); so all coefficients vanish. The sorted products are therefore linearly independent, and being also a spanning list they form a basis of C , of size 2^n . Thus $\dim_F C = 2^n$.

Since 1_C and $\iota(e_1), \dots, \iota(e_n)$ are among the basis elements just found, the list $(\iota(e_1), \dots, \iota(e_n))$ is independent, so ι carries a basis of V to an independent list and is injective. Finally Φ carries the basis of C bijectively onto the basis (e_S) of ΛV , so Φ is a vector space isomorphism. This proves the theorem. ■

The debt, and how it was paid. Chapter 14 §12 stated Theorem 14.12.4 and then wrote, under the heading “What is proved here, and what is not”, that the inequality $\dim_F C \leq 2^n$ was proved in full, while “the reverse inequality — that the 2^n sorted products are linearly independent — is **not proved in this section**”, and that the two halves would have to be established together by exhibiting one algebra of dimension exactly 2^n satisfying (C1) and (C2). Both halves are now supplied. The missing one was independence, and [Theorem 15.10.8](#) supplies it; the prior question, whether any Clifford algebra exists at all, is [Theorem 15.10.6](#). Chapter 14 read the statement for $n \geq 3$ “as quoted from Chapter 15 and not as proved here”; it may now be read as proved.

The prediction was right in substance and slightly off in shape. The algebra of dimension 2^n that settled the matter is the exterior algebra, but it is not the Clifford algebra: it is only a space that $\text{Cl}(q)$ acts on, and independence came from the action rather than from an identification. Chapter 14 also guessed that “ $\text{Cl}(q)$ is assembled from the exterior powers $\Lambda^k V$, one for each k ”, and Φ is that assembly, matching the sorted product on S with the wedge e_S .

Warning. $\text{Cl}(q)$ and ΛV are the same size, not the same algebra. The map Φ above is an isomorphism of **vector spaces** only. It is not multiplicative, and no other map is either, unless $q = 0$: in ΛV the square of a vector is 0, while in $\text{Cl}(q)$ it is $q(v)1$. Writing an element of $\text{Cl}(q)$ as a sum of wedges is a choice of coordinates, not a description of the multiplication.

That warning deserves a proof rather than a slogan.

Proposition 15.10.9. Comparing the Two Algebras

Let $\text{char } F \neq 2$ and $\dim V = n \geq 1$, and let q be the quadratic form of a symmetric bilinear form on V .

- (a) If $q = 0$, then $\text{Cl}(0)$ and ΛV are isomorphic as F -algebras.
- (b) If $q \neq 0$, they are **not** isomorphic as F -algebras, although they are isomorphic as vector spaces.

Idea. For (b), find a property of ΛV that $\text{Cl}(q)$ fails. In ΛV everything of positive degree is nilpotent, so an element whose square is a **nonzero scalar** can have no positive-degree part at all; but in $\text{Cl}(q)$ a vector v with $q(v) \neq 0$ squares to a nonzero scalar and is not a scalar itself.

Proof.

- (a) Let $q = 0$ and let $j: V \rightarrow \Lambda V$ be the inclusion $V = \Lambda^1 V \subseteq \Lambda V$. Then $j(v)^2 = v \wedge v = 0 = q(v)1$, so (C1) holds for j . By the universal property (C2) of $\text{Cl}(0)$

there is an algebra homomorphism $\psi: \text{Cl}(0) \rightarrow \Lambda V$ with $\psi\iota = j$. Its image is a subalgebra containing 1 and V , hence all of ΛV by [Theorem 15.10.2](#) (c); so ψ is surjective. Both spaces have dimension 2^n , by [Theorem 15.10.8](#) and [Theorem 15.10.2](#) (b), so ψ is bijective and is an algebra isomorphism.

(b) *Claim.* If $x \in \Lambda V$ and $x \wedge x = a \cdot 1$ with $a \in F$, $a \neq 0$, then $x \in F \cdot 1$.

Proof of claim. Throughout, y^t means the t -fold wedge product of y with itself. Write $x = x_0 + m$ with $x_0 \in \Lambda^0 V = F$ and m the sum of the components of positive degree. Then

$$x \wedge x = x_0^2 + 2x_0m + m \wedge m,$$

and the only degree-0 term is x_0^2 , so $x_0^2 = a \neq 0$ and hence $x_0 \neq 0$. Canceling that term leaves

$$m \wedge (2x_0 \cdot 1 + m) = 0.$$

Every product of $n+1$ elements of positive degree lies in $\bigoplus_{k>n} \Lambda^k V = \{0\}$, so $m^{n+1} = 0$. Put $c = 2x_0$, which is nonzero because $\text{char } F \neq 2$ and $x_0 \neq 0$, and $u = -c^{-1}m$, so that $u^{n+1} = 0$ and $c \cdot 1 + m = c(1 - u)$. Then

$$(1 - u) \wedge (1 + u + u^2 + \cdots + u^n) = 1 - u^{n+1} = 1,$$

and likewise in the other order, so $c \cdot 1 + m$ is invertible. Multiplying $m \wedge (c \cdot 1 + m) = 0$ by its inverse gives $m = 0$, so $x = x_0 \in F \cdot 1$, proving the claim.

Now suppose $q \neq 0$ and let $\varphi: \text{Cl}(q) \rightarrow \Lambda V$ be an algebra isomorphism. Choose $v \in V$ with $q(v) \neq 0$ and put $x = \varphi(\iota(v))$. Then

$$x \wedge x = \varphi(\iota(v)^2) = \varphi(q(v)1) = q(v) \cdot 1,$$

a nonzero scalar, so $x \in F \cdot 1$ by the claim, say $x = c \cdot 1 = \varphi(c1_{\text{Cl}(q)})$. Since φ is injective, $\iota(v) = c1_{\text{Cl}(q)}$. But by [Theorem 15.10.8](#) the elements $1_{\text{Cl}(q)}, \iota(e_1), \dots, \iota(e_n)$ are part of a basis of $\text{Cl}(q)$, and $\iota(v)$ is a combination of $\iota(e_1), \dots, \iota(e_n)$ alone; comparing coordinates forces $c = 0$ and $\iota(v) = 0$, contradicting $\iota(v)^2 = q(v)1 \neq 0$. So no such φ exists. The vector space isomorphism is [Theorem 15.10.8](#). ■

The dimension count also makes Clifford algebras easy to recognize, which is how one identifies them in practice.

Corollary 15.10.10. Recognizing a Clifford Algebra

Let $\text{char } F \neq 2$, $\dim V = n$, and let q be as above. Let A be an F -algebra with $\dim_F A = 2^n$, and let $\iota_A: V \rightarrow A$ be linear with $\iota_A(v)^2 = q(v)1_A$ for every $v \in V$. If A is generated as an algebra by 1_A and $\iota_A(V)$, then (A, ι_A) is a Clifford algebra of (V, q) .

Proof. Let (C, ι) be the Clifford algebra of [Theorem 15.10.6](#). By its universal property there is an algebra homomorphism $\psi: C \rightarrow A$ with $\psi\iota = \iota_A$. Its image is a subalgebra of A containing 1_A and $\iota_A(V)$, hence is all of A by the generation hypothesis. So ψ is surjective, and $\dim_F C = 2^n = \dim_F A$ by [Theorem 15.10.8](#), so ψ is an isomorphism.

It remains to transport (C2). Let $f: V \rightarrow B$ be linear with $f(v)^2 = q(v)1_B$. Then $\bar{f}\psi^{-1}: A \rightarrow$

B is an algebra homomorphism with $\bar{f}\psi^{-1}\iota_A = \bar{f}\iota = f$, where $\bar{f}: C \rightarrow B$ comes from (C2) for C ; and if $g: A \rightarrow B$ is another one, then $g\psi: C \rightarrow B$ satisfies $g\psi\iota = g\iota_A = f$, so $g\psi = \bar{f}$ by uniqueness for C , whence $g = \bar{f}\psi^{-1}$. ■

Example. The Clifford algebra of the Euclidean plane

Let $V = \mathbb{R}^2$ with $q(\mathbf{x}) = x_1^2 + x_2^2$, the form of signature $(2, 0)$. Show that $\text{Cl}_{2,0}(\mathbb{R}) \cong M_2(\mathbb{R})$.

Solution. Put

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in $M_2(\mathbb{R})$, and let $\iota(\mathbf{x}) = x_1\mathbf{A} + x_2\mathbf{B}$, a linear map $\mathbb{R}^2 \rightarrow M_2(\mathbb{R})$. Multiplying out, $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{I}_2$ and $\mathbf{AB} = -\mathbf{BA}$, the products being $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and its negative. Hence

$$\iota(\mathbf{x})^2 = x_1^2\mathbf{A}^2 + x_1x_2(\mathbf{AB} + \mathbf{BA}) + x_2^2\mathbf{B}^2 = q(\mathbf{x})\mathbf{I}_2,$$

which is (C1). Next, $\dim_{\mathbb{R}} M_2(\mathbb{R}) = 4 = 2^2$, and the four matrices $\mathbf{I}_2, \mathbf{A}, \mathbf{B}, \mathbf{AB}$ are linearly independent: the first two span the diagonal matrices and the last two span the off-diagonal ones, and the diagonal and off-diagonal matrices meet only in 0. So they are a basis, and $M_2(\mathbb{R})$ is generated by $\mathbf{I}_2$ and $\iota(\mathbb{R}^2)$. By [Corollary 15.10.10](#), $(M_2(\mathbb{R}), \iota)$ is a Clifford algebra of $(\mathbb{R}^2, q)$, so $\text{Cl}_{2,0}(\mathbb{R}) \cong M_2(\mathbb{R})$.

Compare Chapter 14's $\text{Cl}_{0,2}(\mathbb{R}) = \mathbb{H}$ (Example (The quaternions)). All three of $\mathbb{H}$, $M_2(\mathbb{R})$, and $\Lambda\mathbb{R}^2$ have dimension 4, and no two are isomorphic: $\mathbb{H}$ has no zero divisors, $M_2(\mathbb{R})$ has plenty, and $\Lambda\mathbb{R}^2$ differs from both by [Proposition 15.10.9](#) (b). Changing a sign in q changes the algebra completely and never changes its size.

Quick check. Where in the proof of [Theorem 15.10.8](#) is the hypothesis $\text{char } F \neq 2$ used?

Solution. Twice, both times through the orthogonal basis. It is needed to produce one at all: Theorem 14.4.2 assumes $\text{char } F \neq 2$. And it is needed for Chapter 14's Proposition 14.12.3, which supplies the spanning half, since that proof uses Lemma 14.12.2, which comes from polarization. The operator identities of [Lemma 15.10.7](#) and the induction of Step 3 use no such hypothesis.

With that, Chapter 14's last open statement is closed and the construction side of this chapter is complete. The final section spends the machinery on one application, where the failure of decomposability proved above reappears under another name.

15.10.6 Exercises

15.10.6.1 A. Check your understanding

Exercise. A1

- (a) Define the exterior algebra ΛV and state its dimension when $\dim V = n$.
- (b) State the graded commutativity rule, and say what it gives when both degrees are odd.
- (c) Define what it means for $\omega \in \Lambda^k V$ to be decomposable, and give one decomposable and one nondecomposable element of $\Lambda^2 F^4$.
- (d) Which half of Theorem 14.12.4 was left unproved in Chapter 14, and which result of this section supplies it?
- (e) True or false: $\text{Cl}(q)$ and ΛV are isomorphic algebras. Justify your answer.

Solution.

- (a) $\Lambda V = \bigoplus_{k \geq 0} \Lambda^k V$ with the product of [Proposition 15.10.1](#) extended by bilinearity ([Definition 15.10.1](#)). Its dimension is $\sum_k \binom{n}{k} = 2^n$ ([Theorem 15.10.2](#)).
- (b) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ for $\omega \in \Lambda^k V$, $\eta \in \Lambda^l V$ ([Theorem 15.10.3](#)). Both odd gives $\omega \wedge \eta = -\eta \wedge \omega$: odd-degree elements anticommute.
- (c) ω is decomposable if $\omega = v_1 \wedge \cdots \wedge v_k$ for some $v_i \in V$ ([Definition 15.10.2](#)). Decomposable: $e_1 \wedge e_2$. Nondecomposable: $e_1 \wedge e_2 + e_3 \wedge e_4$ ([Theorem 15.10.4](#)).
- (d) The independence of the 2^n sorted products, that is, the inequality $\dim_F C \geq 2^n$. It is supplied by [Theorem 15.10.8](#), which needs [Theorem 15.10.6](#) first so that the object exists.
- (e) False in general. They are isomorphic as vector spaces, and as algebras exactly when $q = 0$ ([Proposition 15.10.9](#)).

15.10.6.2 B. Practice**Exercise. B1: Multiplying in the exterior algebra**

In ΛF^4 with $F = \mathbb{Q}$, put $x = \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3$ and $y = \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_4$. Compute the following, expressed in the basis of [Theorem 15.10.2](#).

- (a) $x \wedge y$.
- (b) $y \wedge x$.
- (c) $x \wedge x$.

Solution. Write e_S for the sorted wedge on S .

- (a) Expanding into four products:

$$\begin{aligned} \mathbf{e}_1 \wedge \mathbf{e}_2 &= e_{12}, & \mathbf{e}_1 \wedge (\mathbf{e}_1 \wedge \mathbf{e}_4) &= 0, \\ (\mathbf{e}_2 \wedge \mathbf{e}_3) \wedge \mathbf{e}_2 &= 0, & (\mathbf{e}_2 \wedge \mathbf{e}_3) \wedge (\mathbf{e}_1 \wedge \mathbf{e}_4) &= e_{1234}. \end{aligned}$$

The last one sorts as $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4 = e_{1234}$, since moving $\mathbf{e}_1$ two places to the left costs $(-1)^2$. So $x \wedge y = e_{12} + e_{1234}$.

- (b) The surviving products are $\mathbf{e}_2 \wedge \mathbf{e}_1 = -e_{12}$ and $(\mathbf{e}_1 \wedge \mathbf{e}_4) \wedge (\mathbf{e}_2 \wedge \mathbf{e}_3) = e_{1234}$, the latter because $\mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ sorts with two transpositions. So $y \wedge x = -e_{12} + e_{1234}$. This agrees with [Theorem 15.10.3](#) applied degree by degree: the $(1, 1)$ part changes sign,

the $(1, 2)$ and $(2, 2)$ parts do not.

- (c) $x \wedge x = \mathbf{e}_1 \wedge \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 + 0 = 2e_{123}$, since $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 = e_{123}$. An inhomogeneous element need not square to zero.

Exercise. B2: Decomposable or not

In $\Lambda^2 \mathbb{Q}^4$, writing $e_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j$, determine whether each of the following is decomposable. Justify your answer.

$$\omega_1 = e_{12} + e_{13} + 3e_{14} - 2e_{23} - e_{24} + 5e_{34}, \quad \omega_2 = e_{12} + e_{34} + e_{14}.$$

Solution. For $\omega = \sum_{i < j} c_{ij} e_{ij}$ in $\Lambda^2 \mathbb{Q}^4$, expanding gives

$$\omega \wedge \omega = 2(c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}) e_{1234},$$

since the only pairs of index sets with empty intersection are $\{12\}, \{34\}$ and $\{13\}, \{24\}$ and $\{14\}, \{23\}$, each occurring in two orders, and $e_{13} \wedge e_{24} = -e_{1234}$ while the other two products are $+e_{1234}$.

For ω_1 : $1 \cdot 5 - 1 \cdot (-1) + 3 \cdot (-2) = 5 + 1 - 6 = 0$, so $\omega_1 \wedge \omega_1 = 0$ and the test is inconclusive. In fact ω_1 is decomposable: it is $\mathbf{v}_1 \wedge \mathbf{v}_2$ for $\mathbf{v}_1 = (1, 0, 2, 1)$ and $\mathbf{v}_2 = (0, 1, 1, 3)$, by Example (A wedge of two vectors in four-dimensional space).

For ω_2 : $1 \cdot 1 - 0 \cdot 0 + 1 \cdot 0 = 1$, so $\omega_2 \wedge \omega_2 = 2e_{1234} \neq 0$. A decomposable $u \wedge w$ satisfies $(u \wedge w) \wedge (u \wedge w) = 0$, so ω_2 is **not** decomposable.

Exercise. B3: A Clifford algebra of a line

Let $V = \mathbb{R}$ with $q(x) = x^2$, the form of signature $(1, 0)$. Let $A = \mathbb{R} \times \mathbb{R}$ with componentwise operations, and $\iota_A(x) = (x, -x)$. Prove that (A, ι_A) is a Clifford algebra of (V, q) , so that $\text{Cl}_{1,0}(\mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$. Contrast this with $\text{Cl}_{0,1}(\mathbb{R})$.

Solution. A is an associative unital $\mathbb{R}$ -algebra with identity $(1, 1)$ and $\dim_{\mathbb{R}} A = 2 = 2^1$. The map ι_A is linear, and

$$\iota_A(x)^2 = (x, -x)(x, -x) = (x^2, x^2) = q(x)(1, 1),$$

which is (C1). The subalgebra generated by $(1, 1)$ and $\iota_A(\mathbb{R}) = \text{span}\{(1, -1)\}$ contains the independent pair $(1, 1), (1, -1)$, hence is all of A . By [Corollary 15.10.10](#), (A, ι_A) is a Clifford algebra of (V, q) .

The contrast: Chapter 14 showed $\text{Cl}_{0,1}(\mathbb{R}) = \mathbb{C}$ (Example (The complex numbers)), where $q(x) = -x^2$. Both algebras have dimension 2, but $\mathbb{C}$ is a field and $\mathbb{R} \times \mathbb{R}$ has zero divisors, $(1, 0)(0, 1) = (0, 0)$. A single sign in q decides which.

15.10.6.3 C. Going deeper

Exercise. C1: The Plücker relation

Let $\dim V = 4$ with basis $(e_1, \dots, e_4)$, let $\text{char } F \neq 2$, and let $\omega = \sum_{i < j} c_{ij} e_{ij} \in \Lambda^2 V$.

- Prove that $\omega \wedge \omega = 2(c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}) e_{1234}$.
- Deduce that if $c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} \neq 0$ then ω is not decomposable.
- Prove the converse: if the expression vanishes, then ω is decomposable. *Hint: if $\omega \neq 0$, some $c_{ij} \neq 0$; after relabeling the basis assume $c_{12} \neq 0$ and look for ω as a wedge of two vectors whose first two coordinates you may normalize.*

Solution.

- By bilinearity, $\omega \wedge \omega = \sum_{S,T} c_S c_T e_S \wedge e_T$ over 2-element sets S, T . A term vanishes unless $S \cap T = \emptyset$, so the surviving pairs are $(\{1, 2\}, \{3, 4\})$, $(\{1, 3\}, \{2, 4\})$, $(\{1, 4\}, \{2, 3\})$, and their reverses. By [Theorem 15.10.3](#) with $k = l = 2$, reversing a pair does not change the product, so each contributes twice. Sorting, $e_{12} \wedge e_{34} = e_{1234}$; $e_{13} \wedge e_{24} = e_1 \wedge e_3 \wedge e_2 \wedge e_4 = -e_{1234}$, one transposition; and $e_{14} \wedge e_{23} = e_1 \wedge e_4 \wedge e_2 \wedge e_3 = e_{1234}$, two transpositions. Collecting gives the stated formula.
- If $\omega = u \wedge w$ then $\omega \wedge \omega = u \wedge w \wedge u \wedge w = 0$, a wedge with a repeated vector. Since $\text{char } F \neq 2$, (a) then forces the bracket to vanish. Contrapositively, a nonzero bracket rules out decomposability.
- If $\omega = 0$ it is decomposable. Otherwise relabel the basis so that $c_{12} \neq 0$; this is legitimate because a permutation of the basis permutes the six coefficients up to sign and changes the bracket by at most a sign, so both the hypothesis and the conclusion are preserved. Put

$$u = c_{12}e_1 - c_{23}e_3 - c_{24}e_4, \quad w = e_2 + \frac{c_{13}}{c_{12}}e_3 + \frac{c_{14}}{c_{12}}e_4.$$

Expanding and sorting each wedge,

$$\begin{aligned} u \wedge w &= c_{12}e_{12} + c_{13}e_{13} + c_{14}e_{14} \\ &\quad + c_{23}e_{23} + c_{24}e_{24} + \frac{1}{c_{12}}(c_{13}c_{24} - c_{14}c_{23})e_{34}. \end{aligned}$$

Each of the first five coefficients is read off a single product: for instance the e_{23} term comes from $(-c_{23}e_3) \wedge e_2 = c_{23}e_{23}$, and the e_{13} term from $c_{12}e_1 \wedge \frac{c_{13}}{c_{12}}e_3$. The last comes from

$$(-c_{23}e_3 - c_{24}e_4) \wedge \left(\frac{c_{13}}{c_{12}}e_3 + \frac{c_{14}}{c_{12}}e_4 \right) = \frac{c_{13}c_{24} - c_{14}c_{23}}{c_{12}} e_{34}.$$

By hypothesis $c_{13}c_{24} - c_{14}c_{23} = c_{12}c_{34}$, so that coefficient is c_{34} and $u \wedge w = \omega$. Hence ω is decomposable.

Exercise. C2: Characteristic two

Let $F = \mathbb{F}_2$ and $\dim V = 4$ with basis $(e_1, \dots, e_4)$.

- Prove that $\omega \wedge \omega = 0$ for **every** $\omega \in \Lambda^2 V$.
- Explain why (a) does not make every element of $\Lambda^2 V$ decomposable, and name the element and the argument that settle it.
- Is the statement “ $\omega \wedge \omega = 0$ for every ω of odd degree” also special to characteristic 2?

Justify your answer.

Solution.

- (a) By the computation of [Exercise \(C1: The Plücker relation\)](#) (a), $\omega \wedge \omega$ is 2 times an element of $\Lambda^4 V$, and $2 = 0$ in $\mathbb{F}_2$. (Equivalently: in $\sum_{S,T} c_{SCT} e_S \wedge e_T$ the diagonal terms vanish and the off-diagonal ones pair up, each pair summing to $2c_{SCT} e_S \wedge e_T = 0$.)
- (b) Because $\omega \wedge \omega = 0$ is only a **necessary** condition for decomposability, and over $\mathbb{F}_2$ everything satisfies it, so it carries no information at all. The element $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$ is still not decomposable, by [Theorem 15.10.4](#), whose proof computes $v \wedge \omega$ for a general v and never divides by 2.
- (c) No, that one is characteristic-free: [Theorem 15.10.3](#) proves $\omega \wedge \omega = 0$ for odd degree over every field, by pairing off the terms with $S \neq T$ rather than by dividing. Over $\mathbb{F}_2$ the statement is true but vacuous as a test, for the same reason as in (a).

Exercise. C3: The even part of a Clifford algebra

Let $\text{char } F \neq 2$, $\dim V = n \geq 1$, and let (C, ι) be a Clifford algebra of (V, q) , and let $(e_1, \dots, e_n)$ be an orthogonal basis of V for β . Let $C^+ \subseteq C$ be the span of the sorted products with k **even**, and C^- the span of those with k odd. Prove that $C = C^+ \oplus C^-$, that C^+ is a subalgebra of C , and that $\dim_F C^+ = \dim_F C^- = 2^{n-1}$. Is C^- a subalgebra?

Hint: what does the sorting procedure of Proposition 14.12.3 do to the parity of the length of a product?

Solution. The sorted products form a basis of C by [Theorem 15.10.8](#), and they are partitioned by the parity of $|S|$, so $C = C^+ \oplus C^-$ and the two dimensions are $\sum_{k \text{ even}} \binom{n}{k}$ and $\sum_{k \text{ odd}} \binom{n}{k}$. These are equal and sum to 2^n , since $\sum_k (-1)^k \binom{n}{k} = (1 - 1)^n = 0$ for $n \geq 1$; so each is 2^{n-1} .

For the subalgebra claim, note that the two moves in the proof of Proposition 14.12.3 — swapping two neighboring distinct factors, and replacing $\iota(e_i)^2$ by the scalar d_i — change the length of a product by 0 and by 2 respectively. Both preserve parity. So the product of two sorted products of lengths k and l is a combination of sorted products of length congruent to $k + l$ modulo 2. Hence $C^+ C^+ \subseteq C^+$, and since $1_C \in C^+$ (the empty product, of length 0), C^+ is a subalgebra.

C^- is not: $1_C \notin C^-$, and $C^- C^- \subseteq C^+$ by the same parity count, so C^- fails to be closed as soon as it holds an element with nonzero square, such as $\iota(e_i)$ with $q(e_i) \neq 0$.