

15.8 Exterior Powers

The symmetric power made the slots of a tensor interchangeable. This section makes them *hostile*: we divide $V^{\otimes k}$ by a subspace chosen so that a tensor dies as soon as two of its slots hold the same vector. The result, the exterior power $\Lambda^k V$, has a basis indexed by **strictly increasing** index tuples, so $\dim \Lambda^k V = \binom{n}{k}$ and $\Lambda^k V = \{0\}$ once k exceeds n . That collapse is the whole reason the determinant exists, and Section 09 will read the determinant off the top exterior power.

One choice in the construction is delicate, and it is the choice Chapter 14 prepared us for. There are two natural ways to say “the value flips when two slots are exchanged”, and they are not the same condition: Theorem 14.3.3 showed that *alternating* implies *skew-symmetric* over every field, while the converse needs $\text{char } F \neq 2$. We must therefore decide which relation to divide by, and the answer is the one that works over every field.

This is Section 02’s method for the third time: universal property, uniqueness, then a model built as a quotient of the tensor power. Throughout, F is a field, V is a finite-dimensional vector space over F with $n = \dim V$, and $k \geq 1$ unless stated otherwise. **No hypothesis on the characteristic is made anywhere in this section**, and the construction is arranged so that none is needed.

15.8.1 Alternating multilinear maps

Chapter 7 defined alternating k -linear **forms**, valued in F (Definition 7.2.2), because that is all the determinant needed. The target plays no role in the definition, so we extend it, as Section 01 extended multilinearity.

An alternating map dies the moment two of its arguments coincide.

Definition 15.8.1. Alternating Multilinear Map

Let W be a vector space over F . A k -linear map $f: V^k \rightarrow W$ (Definition 15.1.1) is **alternating** if

$$f(u_1, \dots, u_k) = 0 \quad \text{whenever } u_i = u_j \text{ for some pair of positions } i \neq j.$$

The condition is on **some** pair of positions, adjacent or not, and it says nothing about inputs whose entries are all distinct. For $W = F$ this is Definition 7.2.2 verbatim. The examples of Chapter 7 are all available: the determinant $\det(\mathbf{a}_1, \dots, \mathbf{a}_n)$ as a function of the columns; the signed area $(\mathbf{u}, \mathbf{w}) \mapsto u_1 w_2 - u_2 w_1$ on F^2 ; any linear functional, vacuously, at $k = 1$. A nonexample by minimal change: the dot product on F^2 is bilinear but $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1 \neq 0$, so the alternating clause fails at that one input while bilinearity survives.

Everything Chapter 7 deduced from the definition used only the vector-space structure of the target, so it carries over.

Lemma 15.8.1. Properties of Alternating Maps

Let $f: V^k \rightarrow W$ be alternating k -linear, let $u_1, \dots, u_k \in V$, and let $\sigma \in S_k$.

- (a) Exchanging two arguments multiplies the value by -1 ; in particular f is skew-symmetric.

- (b) $f(u_{\sigma(1)}, \dots, u_{\sigma(k)}) = \text{sgn}(\sigma) f(u_1, \dots, u_k)$.
 (c) If $(u_1, \dots, u_k)$ is linearly dependent, then $f(u_1, \dots, u_k) = 0$.

Proof.

- (a) Fix positions $i < j$, freeze the other arguments, and let $g(x, y) \in W$ be the value of f with x in position i and y in position j . Then g is bilinear and $g(x, x) = 0$, so expanding

$$0 = g(u_i + u_j, u_i + u_j) = g(u_i, u_j) + g(u_j, u_i)$$

gives the swap rule. This is the computation of Theorem 7.2.1 (a), which used nothing about F as a target beyond its being a vector space.

- (b) For $k = 1$ the only permutation is the identity and there is nothing to prove. For $k \geq 2$, use Theorem 0.10.4 to write $\sigma = \tau_1 \cdots \tau_r$ as a product of transpositions. Applying (a) once per transposition multiplies the value by $(-1)^r$, and $(-1)^r = \text{sgn}(\sigma)$ by Theorem 7.3.1 and Corollary 7.3.2. This is Lemma 7.4.1, with the same proof.
 (c) By the Linear Dependence Lemma (Theorem 1.4.1) some u_j lies in $\text{span}(u_1, \dots, u_{j-1})$. Expanding f in slot j writes the value as a combination of values in which some u_m with $m < j$ sits both in slot m and in slot j , and each such value is 0. This is Theorem 7.2.1 (d). If $j = 1$ then $u_1 = 0$ and linearity in slot 1 gives 0 at once.

■

15.8.2 Which relation to divide by

Now the delicate point, stated before the definition so that the definition looks inevitable. We want a quotient of $V^{\otimes k}$ in which the image of $u_1 \otimes \cdots \otimes u_k$ vanishes exactly when it should. There are two candidate subspaces to divide by:

$$A_k = \text{span}\{u_1 \otimes \cdots \otimes u_k : u_i = u_j \text{ for some } i \neq j\},$$

$$N_k = \text{span}\{x + P_\tau x : x \text{ a simple tensor, } \tau \text{ a transposition}\},$$

where P_τ permutes the slots as in Lemma 15.7.7. Dividing by A_k imposes “repeat a vector and you get zero”; dividing by N_k imposes “swap two slots and the sign flips”. The two are not the same subspace.

Proposition 15.8.2. The Repetition Relations Are the Stronger Ones

For every field F and every $k \geq 2$ we have $N_k \subseteq A_k$. If $\text{char } F \neq 2$, then $N_k = A_k$. If $\text{char } F = 2$, $k = 2$, and $n \geq 1$, the inclusion is **strict**.

Idea. For the inclusion, feed the sum $u_i + u_j$ into **both** slots i and j : the resulting tensor is in A_k , and expanding it produces the two “square” terms, also in A_k , plus the two cross terms, whose sum is therefore in A_k too. That is Theorem 14.3.3 (a) transcribed into tensors. For the reverse, a tensor with a repeat is its own swap, so $x + P_\tau x = 2x$, and recovering x is a division by 2 — the same division, in the same place, as in Theorem 14.3.3 (b).

Proof. ($\subseteq$) Let $x = u_1 \otimes \cdots \otimes u_k$ be simple and τ the transposition of positions $i < j$. Let y be the tensor obtained from x by putting $u_i + u_j$ in **both** positions i and j , which lies in A_k because those two slots now agree. Expanding y by multilinearity in those two slots gives four terms:

$$y = [u_i, u_i] + [u_i, u_j] + [u_j, u_i] + [u_j, u_j],$$

where $[p, q]$ denotes x with p in position i and q in position j . The first and last terms lie in A_k , again by a repeat, and $[u_i, u_j] = x$ while $[u_j, u_i] = P_\tau x$. Hence $x + P_\tau x = y - [u_i, u_i] - [u_j, u_j] \in A_k$. Generators of N_k lie in A_k , so $N_k \subseteq A_k$.

($\supseteq$), assuming $\text{char } F \neq 2$. Let $x = u_1 \otimes \cdots \otimes u_k$ with $u_i = u_j$, $i \neq j$, and let τ swap i and j . Then $P_\tau x = x$, so $x + P_\tau x = (1 + 1)x$ lies in N_k . Since $\text{char } F \neq 2$, the scalar $1 + 1$ is invertible (Definition 0.7.2), so $x \in N_k$. Generators of A_k lie in N_k , so $A_k \subseteq N_k$.

Strictness in characteristic 2, $k = 2$. Suppose $\text{char } F = 2$ and $n \geq 1$, and fix a basis $(v_1, \dots, v_n)$. Expand a generator of N_2 in the basis of Theorem 15.6.3: writing $u = \sum_i a_i v_i$ and $w = \sum_j b_j v_j$,

$$u \otimes w + w \otimes u = \sum_{i,j} (a_i b_j + a_j b_i) v_i \otimes v_j,$$

and the coefficient of a **diagonal** basis tensor $v_i \otimes v_i$ is $2a_i b_i = 0$. Every element of N_2 therefore has coefficient 0 on $v_1 \otimes v_1$. But $v_1 \otimes v_1$ is itself a generator of A_2 , and it is a basis vector of $V^{\otimes 2}$, hence nonzero. So $v_1 \otimes v_1 \in A_2 \setminus N_2$ and the inclusion is strict. This proves the proposition. ■

So the two relations differ, and we must choose. **We divide by A_k .** Three reasons, and they are the same three that made Chapter 7 build the determinant on alternation rather than skewness:

- The quotient by A_k is the object whose maps are the **alternating** ones, and alternating is the condition the determinant satisfies over every field.
- A_k is the larger subspace, so the quotient by it is the smaller space; and by [Proposition 15.8.2](#) the two quotients agree whenever $\text{char } F \neq 2$. So nothing is lost outside characteristic 2, and something is gained inside it.
- The dimension $\binom{n}{k}$ comes out right over every field. Dividing by N_2 instead would give, for $V = \mathbb{F}_2^2$, a space of dimension 3 where $\binom{2}{2} = 1$ is wanted; Exercise C1 carries out that count.

15.8.3 The exterior power

The k -th exterior power is the space in which alternating k -linear maps become linear.

Definition 15.8.2. k -th Exterior Power

A **k -th exterior power of V** is a pair $(P, \wedge)$, where P is a vector space over F and $\wedge: V^k \rightarrow P$ is an **alternating** k -linear map, such that

- (E1) (**universal property**) for **every** vector space W over F and **every alternating** k -linear map $f: V^k \rightarrow W$, there is a **unique** linear map $\bar{f}: P \rightarrow W$ with $\bar{f} \circ \wedge = f$.

We write $\Lambda^k V$ for P and

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k := \wedge(v_1, \dots, v_k),$$

calling it a **wedge** of the list. By convention $\Lambda^0 V = F$.

Clause by clause, and the small words matter. The structure map $\wedge$ must itself be **alternating**, which is what makes $v \wedge v = 0$ a fact rather than a wish. Only **alternating** maps are required to factor, and that is forced: $\bar{f} \circ \wedge$ is alternating for every linear $\bar{f}$, so no space could linearize a nonalternating map. **Unique** does the usual two jobs: existence of $\bar{f}$ says $\Lambda^k V$ is large enough, uniqueness says it is no larger.

At $k = 1$ the condition is vacuous and (V, id_V) works, so $\Lambda^1 V = V$. At $k = 0$ the convention $\Lambda^0 V = F$ is the one that makes $\binom{n}{k}$ correct at $k = 0$, where it reads $\binom{n}{0} = 1$.

Theorem 15.8.3. Uniqueness of the Exterior Power

Let $(P, \wedge)$ and $(P', \wedge')$ both be k -th exterior powers of V . Then there is a unique linear isomorphism $\varphi: P \rightarrow P'$ with $\varphi \circ \wedge = \wedge'$.

Proof. The proof of Theorem 15.6.1, with “alternating k -linear” in place of “ k -linear”: (E1) for P applied to $\wedge'$ gives φ , (E1) for P' applied to $\wedge$ gives ψ , and $\psi\varphi$ is a linear self-map of P with $\psi\varphi\wedge = \wedge$, so the uniqueness clause of (E1) applied to $\wedge$ forces $\psi\varphi = \text{id}_P$; symmetrically $\varphi\psi = \text{id}_{P'}$. This proves the theorem. ■

Theorem 15.8.4. Existence of the Exterior Power

Let $A_k \subseteq V^{\otimes k}$ be as above and let $q: V^{\otimes k} \rightarrow V^{\otimes k}/A_k$ be the quotient map (Definition 4.4.3). Then $V^{\otimes k}/A_k$, together with

$$v_1 \wedge \cdots \wedge v_k = q(v_1 \otimes \cdots \otimes v_k),$$

is a k -th exterior power of V .

Proof. A_k is a subspace, being a span, and $\wedge$ is k -linear, being μ_k followed by the linear map q . It is alternating: if $u_i = u_j$ with $i \neq j$, then $u_1 \otimes \cdots \otimes u_k$ is by definition a generator of A_k , so its image under q is 0.

Let $f: V^k \rightarrow W$ be alternating k -linear. By (T1) there is a unique linear $\tilde{f}: V^{\otimes k} \rightarrow W$ with $\tilde{f}(u_1 \otimes \cdots \otimes u_k) = f(u_1, \dots, u_k)$. On a generator of A_k , that is, on a simple tensor with $u_i = u_j$ for some $i \neq j$, the value is $f(u_1, \dots, u_k) = 0$ because f is alternating. A linear map vanishing on a spanning set of A_k vanishes on A_k , so $A_k \subseteq \ker \tilde{f}$, and Theorem 4.4.10 (a) gives exactly one linear $\bar{f}$ on the quotient with $\bar{f} \circ q = \tilde{f}$, whence $\bar{f} \circ \wedge = f$.

Uniqueness: q is surjective and $V^{\otimes k}$ is spanned by its simple tensors (Theorem 15.6.2), so the wedges $u_1 \wedge \cdots \wedge u_k$ span $V^{\otimes k}/A_k$, and two linear maps agreeing on them agree everywhere.

This proves the theorem. ■

From [Lemma 15.8.1](#) applied to the alternating map $\wedge$ itself we get the rules used in every computation below, for all $u_i \in V$ and $\sigma \in S_k$:

$$\begin{aligned} u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(k)} &= \operatorname{sgn}(\sigma) u_1 \wedge \cdots \wedge u_k, \\ u_1 \wedge \cdots \wedge u_k &= 0 \quad \text{if the list is dependent.} \end{aligned}$$

Four instances, smallest first, and one look-alike that is not one.

- **One slot.** $\Lambda^1 V = V$, and the wedge of a single vector is the vector itself. The alternating condition asks about pairs of positions, and with one position there are none.
- **The plane.** For $V = F^2$, expanding both slots and canceling the two repeats gives $\mathbf{u} \wedge \mathbf{w} = (u_1 w_2 - u_2 w_1) \mathbf{e}_1 \wedge \mathbf{e}_2$. The coefficient is the signed area of Chapter 7 §01; the wedge is that number, moved out of F and into a line inside $\Lambda^2 V$.
- **Polynomials.** On $V = F[x]_{\leq 2}$ with basis $(1, x, x^2)$, the same expansion gives $(1 + x) \wedge (x + x^2) = 1 \wedge x + 1 \wedge x^2 + x \wedge x^2$, the term $x \wedge x$ having died.
- **A degenerate case.** If $V = \{0\}$ then every wedge of $k \geq 1$ vectors is 0, so $\Lambda^k V = \{0\}$. It matters because it is the base of the count below: $\binom{0}{k} = 0$ for $k \geq 1$.
- **Not an example: the symmetric product.** In $\operatorname{Sym}^2 V$ the product uw is bilinear and satisfies $uw = wu$, but $e_1 e_1 \neq 0$ by Theorem 15.7.3, so the symmetric product map is **not** alternating. Exactly one clause differs between the two constructions, and it is this one.

Warning. **The relation is $v \wedge v = 0$, not $u \wedge w = -w \wedge u$.** The second follows from the first ([Lemma 15.8.1](#) (a)), but not conversely: over $\mathbb{F}_2$ the two say different things, since $-1 = 1$ there makes “ $u \wedge w = -w \wedge u$ ” mean “ $u \wedge w = w \wedge u$ ”, which is no condition at all on a repeated slot. This is exactly Theorem 14.3.3, transcribed from forms to tensors by [Proposition 15.8.2](#), and it is why the construction above divides by A_k . A reader who remembers that theorem should recognize the choice as the only characteristic-free one.

Quick check. In $\Lambda^2 V$, simplify $(u + w) \wedge (u - w)$.

Solution. Expand both slots: $u \wedge u - u \wedge w + w \wedge u - w \wedge w$. The first and last terms are 0 because $\wedge$ is alternating, and $w \wedge u = -u \wedge w$ by [Lemma 15.8.1](#) (a). So the value is $-2u \wedge w$. Over $\mathbb{F}_2$ that is 0, which is correct there and not a sign of anything going wrong: over $\mathbb{F}_2$, $u + w$ and $u - w$ are the same vector, so the wedge of the pair must vanish.

15.8.4 A basis, and the dimension

Sorting is again the move, but now it costs a sign, and tuples with a repeat are simply gone. What survives is the strictly increasing tuples.

Theorem 15.8.5. Basis of an Exterior Power

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V and $k \geq 1$. Then the wedges

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n,$$

one for each **strictly increasing** tuple, form a basis of $\Lambda^k V$. Consequently

$$\dim \Lambda^k V = \binom{n}{k}, \quad \text{and} \quad \Lambda^k V = \{0\} \text{ for } k > n,$$

over **every** field F .

Idea. Spanning: start from the basis of Theorem 15.6.3, throw away every tuple with a repeated index — those wedges are 0 — and sort the rest, paying a sign. Independence: one linear functional per increasing tuple I , and the functional must be **alternating**, so the natural candidate is a determinant: read the k arguments through the k dual functionals $\varphi_{i_1}, \dots, \varphi_{i_k}$, assemble the $k \times k$ matrix of values, and take its determinant. Determinants are alternating in their columns by construction, which is exactly what is needed, and the value at the basis wedge indexed by J is $\det \mathbf{I}_k = 1$ if $I = J$ and 0 otherwise, because some row is then zero.

Proof. Spanning. By Theorem 15.6.3 the tensors $v_{j_1} \otimes \cdots \otimes v_{j_k}$, over all tuples J , form a basis of $V^{\otimes k}$, and q is surjective, so the wedges $v_{j_1} \wedge \cdots \wedge v_{j_k}$ span $\Lambda^k V$. If J has two equal entries, the corresponding wedge is 0 because $\wedge$ is alternating. If the entries of J are distinct, let $\sigma \in S_k$ sort them into increasing order; then by Lemma 15.8.1 (b) the wedge equals $\pm$ the wedge of the sorted tuple. So the increasing tuples already span, and if $k > n$ there are no such tuples at all, whence $\Lambda^k V = \{0\}$.

Independence. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $\mathcal{B}$ (Theorem 5.1.3). For an increasing tuple $I = (i_1 < \cdots < i_k)$ define

$$g_I(u_1, \dots, u_k) = \det \left(\varphi_{i_a}(u_b) \right)_{a,b=1}^k \in F,$$

the determinant of the $k \times k$ matrix whose (a, b) entry is $\varphi_{i_a}(u_b)$. Column b of that matrix depends linearly on u_b , because each φ_{i_a} is linear, and $\det$ is linear in each column and vanishes when two columns coincide (Theorem 7.4.3). Hence g_I is k -linear, and alternating: if $u_a = u_b$ with $a \neq b$, two columns of the matrix are equal and the determinant is 0.

By (E1), g_I induces a linear $\bar{g}_I: \Lambda^k V \rightarrow F$. Evaluate it at the wedge of an increasing tuple J : the matrix has (a, b) entry $\varphi_{i_a}(v_{j_b}) = \delta_{i_a j_b}$. If $I = J$ this matrix is $\mathbf{I}_k$ and the determinant is 1. If $I \neq J$, then since I and J each list k distinct indices and are not the same increasing tuple, some i_a is not among $j_1, \dots, j_k$; row a is then zero, so the rows are dependent, and by Theorem 7.4.5 together with Theorem 7.2.1 (d) the determinant is 0. Hence $\bar{g}_I(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \delta_{IJ}$.

Now let $\sum_I a_I v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$, the sum over increasing tuples. Applying $\bar{g}_J$ leaves $a_J = 0$, and J was arbitrary. So the list is independent, hence a basis.

The count. An increasing k -tuple in $\{1, \dots, n\}$ is the same thing as a k -element subset of

$\{1, \dots, n\}$, listed in its only increasing order. There are $\binom{n}{k}$ such subsets, and none at all when $k > n$. No step divided by an integer, so the conclusion holds over every field. This proves the theorem. ■

Corollary 15.8.6. The Top Exterior Power Is a Line

If $\dim V = n \geq 1$, then $\dim \Lambda^n V = 1$, with basis the single wedge $v_1 \wedge \cdots \wedge v_n$ of any basis of V . For $k > n$, $\Lambda^k V = \{0\}$.

Proof. There is exactly one increasing n -tuple in $\{1, \dots, n\}$, namely $(1, 2, \dots, n)$, so Theorem 15.8.5 gives a basis with one member, and $\binom{n}{n} = 1$. The second statement is the last clause of that theorem. ■

That one-dimensionality is the source of the determinant, and Section 09 is about it. For now, notice the shape of the numbers: $\dim \Lambda^k V$ rises from 1 at $k = 0$ to a maximum in the middle and back to 1 at $k = n$, and then stops. Compare $\dim \operatorname{Sym}^k V = \binom{n+k-1}{k}$, which grows forever. The symmetric power forgets the order; the exterior power forgets the order **and** forbids repetition, and forbidding repetition is what makes the tower finite.

Remark. The two counts fit together at $k = 2$: $\binom{n+1}{2} + \binom{n}{2} = n^2 = \dim V^{\otimes 2}$ over every field. In characteristic $\neq 2$ this reflects an actual decomposition of $V^{\otimes 2}$ into symmetric and alternating tensors, in the manner of Theorem 14.3.4. Over $\mathbb{F}_2$ the identity between the numbers survives while the decomposition does not, exactly as in Theorem 15.7.8.

Warning. $\Lambda^k V$ is a quotient of $V^{\otimes k}$, not a subspace. Its elements are cosets of A_k , and $u \wedge w = -w \wedge u$ is an identity between cosets, not between tensors. It is also a mistake to replace it by a subspace of $V^{\otimes k}$ — say “the tensors that change sign under every swap of two slots”. Over $\mathbb{F}_2$ that subspace is the space of symmetric tensors of Definition 15.7.5, because $-1 = 1$ there; for $V = \mathbb{F}_2^2$ and $k = 2$ it has dimension $\binom{3}{2} = 3$ by Theorem 15.7.8 (a), while $\Lambda^2 V$ has dimension $\binom{2}{2} = 1$.

Example. A wedge in $\Lambda^2(F^3)$

Let $V = F^3$, $\mathbf{u} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, and $\mathbf{w} = 4\mathbf{e}_1 + 5\mathbf{e}_2 + 6\mathbf{e}_3$. Write out a basis of $\Lambda^2 V$ and expand $\mathbf{u} \wedge \mathbf{w}$ in it.

Solution. By Theorem 15.8.5 a basis is $(\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3)$, of size $\binom{3}{2} = 3$. Expanding both slots gives nine terms; the three with a repeated index vanish, and each remaining pair

combines by $\mathbf{e}_j \wedge \mathbf{e}_i = -\mathbf{e}_i \wedge \mathbf{e}_j$. The coefficient of $\mathbf{e}_i \wedge \mathbf{e}_j$ is therefore $u_i w_j - u_j w_i$:

$$\begin{aligned} u \wedge w &= (1 \cdot 5 - 2 \cdot 4) \mathbf{e}_1 \wedge \mathbf{e}_2 + (1 \cdot 6 - 3 \cdot 4) \mathbf{e}_1 \wedge \mathbf{e}_3 \\ &\quad + (2 \cdot 6 - 3 \cdot 5) \mathbf{e}_2 \wedge \mathbf{e}_3 \\ &= -3 \mathbf{e}_1 \wedge \mathbf{e}_2 - 6 \mathbf{e}_1 \wedge \mathbf{e}_3 - 3 \mathbf{e}_2 \wedge \mathbf{e}_3. \end{aligned}$$

The three coefficients are the 2×2 minors of the 3×2 matrix with columns $\mathbf{u}$ and $\mathbf{w}$. That is not an accident, and Section 09 proves it in general.

15.8.5 Independence detects wedges

The last result of the section is small to state and is the tool everything after it uses: **a wedge is nonzero exactly when its factors are independent**. One direction is [Lemma 15.8.1 \(c\)](#); the other needs [Theorem 15.8.5](#), which is why it comes last.

Theorem 15.8.7. Wedges Detect Independence

Let $u_1, \dots, u_k \in V$ with $1 \leq k \leq n$. Then

$$\begin{aligned} u_1 \wedge u_2 \wedge \dots \wedge u_k &\neq 0 \\ \text{if and only if } (u_1, \dots, u_k) &\text{ is independent.} \end{aligned}$$

For $k > n$ the wedge is always 0.

Idea. $(\Rightarrow)$ is the contrapositive of “alternating maps kill dependent lists”. $(\Leftarrow)$ is the reason [Theorem 15.8.5](#) was worth proving: an independent list extends to a basis of V , and that theorem then says its wedge is one of the basis vectors of $\Lambda^k V$, hence nonzero.

Proof. $(\Rightarrow)$ We prove the contrapositive. If $(u_1, \dots, u_k)$ is dependent, then $u_1 \wedge \dots \wedge u_k = 0$ by [Lemma 15.8.1 \(c\)](#), applied to the alternating map $\wedge$.

$(\Leftarrow)$ Suppose the list is independent. By the Basis Extension Theorem (Theorem 1.6.5) it extends to a basis $\mathcal{B} = (u_1, \dots, u_k, u_{k+1}, \dots, u_n)$ of V , which is possible because $k \leq n$. Apply [Theorem 15.8.5](#) to $\mathcal{B}$: the wedges indexed by increasing k -tuples form a basis of $\Lambda^k V$, and $u_1 \wedge \dots \wedge u_k$ is the one indexed by $(1, 2, \dots, k)$. A member of a basis is nonzero, so $u_1 \wedge \dots \wedge u_k \neq 0$.

For $k > n$, $\Lambda^k V = \{0\}$ by [Theorem 15.8.5](#), so every wedge vanishes; consistently, any $k > n$ vectors in V are dependent. This proves the theorem. ■

Remark. The theorem turns a question about a list into a question about a single vector, and that is its value. Section 09 uses it to prove that $\Lambda^n T$ is multiplication by $\det T$ and that T is invertible exactly when that scalar is nonzero; Section 10 uses it to decide which elements of $\Lambda^2 V$ are wedges at all. Note the hypothesis it does **not** need: no assumption on F , because neither half divided by anything.

15.8.6 Exercises

15.8.6.1 A. Check your understanding

Exercise. A1

- (a) State the universal property (E1), with every quantifier, and say which maps are required to factor.
- (b) Which subspace of $V^{\otimes k}$ is divided out, and why is the other candidate N_k rejected?
- (c) Let $\dim V = 5$. Compute $\dim \Lambda^k V$ for $k = 0, 1, 2, 3, 4, 5, 6$.
- (d) Determine whether $u \wedge w \wedge u$ can be nonzero. Justify your answer.
- (e) State the test for $u_1 \wedge \cdots \wedge u_k \neq 0$, and say which hypothesis on the field it needs.

Solution.

- (a) For every vector space W over F and every **alternating** k -linear map $f: V^k \rightarrow W$, there exists exactly one linear $\bar{f}: \Lambda^k V \rightarrow W$ with $\bar{f} \circ \wedge = f$. Only alternating maps factor; a nonalternating one cannot, since $\bar{f} \circ \wedge$ is alternating for every linear $\bar{f}$.
- (b) The subspace A_k spanned by the simple tensors with a repeated entry ([Theorem 15.8.4](#)). The candidate N_k , spanned by the elements $x + P_\tau x$, is rejected because $N_k \subsetneq A_k$ in characteristic 2 ([Proposition 15.8.2](#)), so the quotient by N_k would have the wrong dimension there and its maps would be the skew-symmetric ones rather than the alternating ones.
- (c) $\binom{5}{k}$, giving 1, 5, 10, 10, 5, 1, 0 for $k = 0, \dots, 6$.
- (d) No: the list (u, w, u) repeats u in positions 1 and 3, so the wedge is 0 because $\wedge$ is alternating. (Equivalently, the list is dependent, so [Theorem 15.8.7](#) applies.)
- (e) $u_1 \wedge \cdots \wedge u_k \neq 0$ if and only if the list is linearly independent ([Theorem 15.8.7](#)). It needs no hypothesis on F .

15.8.6.2 B. Practice

Exercise. B1: Computing wedges

Let $V = F^3$ with $F = \mathbb{Q}$.

- (a) Expand $(2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$ in the basis $(\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3)$.
- (b) Expand $\mathbf{e}_1 \wedge (\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ in the basis of $\Lambda^3 V$.

Solution.

- (a) With $\mathbf{u} = (2, -1, 1)$ and $\mathbf{w} = (1, 1, -1)$, the coefficient of $\mathbf{e}_i \wedge \mathbf{e}_j$ is $u_i w_j - u_j w_i$, as in [Example \(A wedge in \$\Lambda^2\(F^3\)\$ \)](#):

$$\begin{aligned} 2 \cdot 1 - (-1) \cdot 1 &= 3, & 2 \cdot (-1) - 1 \cdot 1 &= -3, \\ (-1) \cdot (-1) - 1 \cdot 1 &= 0, \end{aligned}$$

so the wedge is $3\mathbf{e}_1 \wedge \mathbf{e}_2 - 3\mathbf{e}_1 \wedge \mathbf{e}_3$.

- (b) By [Corollary 15.8.6](#) the basis is the single wedge $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Expand from the left: in the second slot, $\mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \cdots = 0$, so only $\mathbf{e}_2$ survives; then in the third slot, both $\mathbf{e}_1$ and $\mathbf{e}_2$ already occur, so only $\mathbf{e}_3$ survives. Hence the wedge is $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. (Equivalently, the three vectors are the columns of a unit upper triangular matrix, of determinant 1; Exercise C3 explains why the answer had to be that determinant.)

Exercise. B2: Independence by wedging

Determine whether each list is linearly independent, by computing a wedge. Justify your answer.

- (a) $((1, 2), (3, 6))$ in $\mathbb{Q}^2$.
- (b) $((1, 0, 1), (0, 1, 1), (1, 1, 2))$ in $\mathbb{Q}^3$.
- (c) $((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1))$ in $\mathbb{Q}^3$.

Solution.

- (a) $(1, 2) \wedge (3, 6) = (1 \cdot 6 - 2 \cdot 3) \mathbf{e}_1 \wedge \mathbf{e}_2 = 0$. By [Theorem 15.8.7](#) the list is dependent, as it visibly is: the second vector is three times the first.
- (b) The minor computation $(1 \cdot 1 - 0 \cdot 0, 1 \cdot 1 - 1 \cdot 0, 0 \cdot 1 - 1 \cdot 1) = (1, 1, -1)$ gives, for the first two vectors, $\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_3$. Wedging with the third vector $(1, 1, 2)$ and keeping only the terms with all three indices distinct,

$$\begin{aligned} & (\mathbf{e}_1 \wedge \mathbf{e}_2) \wedge 2\mathbf{e}_3 + (\mathbf{e}_1 \wedge \mathbf{e}_3) \wedge \mathbf{e}_2 - (\mathbf{e}_2 \wedge \mathbf{e}_3) \wedge \mathbf{e}_1 \\ &= (2 - 1 - 1) \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = 0, \end{aligned}$$

using $\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 = -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ and $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ from [Lemma 15.8.1](#)

- (b). So the list is dependent; indeed the third vector is the sum of the first two.
- (c) Here $k = 4 > 3 = n$, so $\Lambda^4(\mathbb{Q}^3) = \{0\}$ by [Theorem 15.8.5](#) and the wedge is 0 without any computation. The list is dependent, as any 4 vectors in a 3-dimensional space must be.

Exercise. B3: A functional built from the universal property

Let $V = F^3$ and let $f: V^2 \rightarrow F$ be $f(\mathbf{u}, \mathbf{w}) = u_1 w_3 - u_3 w_1$.

- (a) Prove that f is alternating and bilinear, and deduce that there is a unique linear $\bar{f}: \Lambda^2 V \rightarrow F$ with $\bar{f}(\mathbf{u} \wedge \mathbf{w}) = f(\mathbf{u}, \mathbf{w})$.
- (b) Compute $\bar{f}$ on the basis $(\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3)$, and evaluate $\bar{f}$ on the answer to [Example \(A wedge in \$\Lambda^2\(F^3\)\$ \)](#).

Solution.

- (a) Each slot is a linear functional with the other fixed, so f is bilinear, and $f(\mathbf{u}, \mathbf{u}) = u_1 u_3 - u_3 u_1 = 0$, so f is alternating ([Definition 15.8.1](#) with $k = 2$). By (E1) there is exactly one linear $\bar{f}: \Lambda^2 V \rightarrow F$ with $\bar{f} \circ \wedge = f$.
- (b) $\bar{f}(\mathbf{e}_1 \wedge \mathbf{e}_2) = f(\mathbf{e}_1, \mathbf{e}_2) = 0$; $\bar{f}(\mathbf{e}_1 \wedge \mathbf{e}_3) = 1$; $\bar{f}(\mathbf{e}_2 \wedge \mathbf{e}_3) = 0$. So $\bar{f}$ reads off the $\mathbf{e}_1 \wedge \mathbf{e}_3$

coordinate. On $\mathbf{u} \wedge \mathbf{w} = -3\mathbf{e}_1 \wedge \mathbf{e}_2 - 6\mathbf{e}_1 \wedge \mathbf{e}_3 - 3\mathbf{e}_2 \wedge \mathbf{e}_3$ it gives -6 , which agrees with $f(\mathbf{u}, \mathbf{w}) = 1 \cdot 6 - 3 \cdot 4 = -6$.

15.8.6.3 C. Going deeper

Exercise. C1: The characteristic-two witness

Let $F = \mathbb{F}_2$, $V = \mathbb{F}_2^2$, and $k = 2$.

- List the elements of A_2 and of N_2 , and compute both dimensions.
- Compute $\dim \Lambda^2 V$ from (a) and check it against [Theorem 15.8.5](#).
- Compute $\dim(V^{\otimes 2}/N_2)$ and explain, in one sentence, what would go wrong if the exterior power had been defined as that quotient.

Solution.

- By Theorem 15.6.3, $V^{\otimes 2}$ has basis $\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2$. Now A_2 is spanned by the tensors $\mathbf{u} \otimes \mathbf{u}$, and the three choices $\mathbf{u} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$ give

$$\mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2,$$

whose span also contains $t = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$. Those three vectors $\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, t$ are independent, being distinct disjoint sums of basis vectors, and they span all three listed generators, so $\dim A_2 = 3$; there are no further generators, since $\mathbf{0}$ is the only remaining vector of $\mathbb{F}_2^2$. For N_2 , the coefficient formula in the proof of [Proposition 15.8.2](#) gives

$$\mathbf{u} \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{u} = (a_1 b_2 + a_2 b_1) t,$$

since the diagonal coefficients vanish and the two off-diagonal ones are equal. Hence $N_2 = \text{span}(t)$ and $\dim N_2 = 1$.

- $\dim \Lambda^2 V = 4 - 3 = 1$ by Theorem 4.4.5, matching $\binom{2}{2} = 1$.
- $\dim(V^{\otimes 2}/N_2) = 4 - 1 = 3$. Defining $\Lambda^2 V$ as that quotient would make the image of $\mathbf{e}_1 \otimes \mathbf{e}_1$ nonzero, so $\mathbf{v} \wedge \mathbf{v} \neq 0$ for some $\mathbf{v}$, the structure map would be skew-symmetric but not alternating, and the dimension would be 3 instead of $\binom{2}{2} = 1$.

Exercise. C2: Alternating forms, recounted

- Prove that the alternating k -linear forms on V are in bijection with the linear functionals on $\Lambda^k V$, and that the bijection is an isomorphism of vector spaces.
- Deduce that the space of alternating k -linear forms on V has dimension $\binom{n}{k}$.
- Hence recover Corollary 7.4.4: the alternating n -linear forms on F^n form a line.

Solution.

- Write $\mathcal{A}$ for the space of alternating k -linear forms on V , a vector space by Theorem 7.2.3. The map $\Theta: (\Lambda^k V)^* \rightarrow \mathcal{A}$, $\Theta(\lambda) = \lambda \circ \wedge$, is well defined, since a composite of the alternating $\wedge$ with a linear map is alternating and k -linear, and it is linear in λ

because composition with a fixed map is. It is injective: if $\lambda \circ \wedge = 0$ then λ vanishes on the wedges, which span $\Lambda^k V$ (Theorem 15.8.4), so $\lambda = 0$. It is surjective: given $f \in \mathcal{A}$, (E1) supplies $\bar{f}$ with $\bar{f} \circ \wedge = f$. So Θ is an isomorphism.

- (b) By (a) and Corollary 5.1.4, $\dim \mathcal{A} = \dim(\Lambda^k V)^* = \dim \Lambda^k V = \binom{n}{k}$, using Theorem 15.8.5.
- (c) Take $V = F^n$ and $k = n$: the dimension is $\binom{n}{n} = 1$. Since $\det$ is a nonzero alternating n -linear form (Theorem 7.4.3, with $\det \mathbf{I}_n = 1$), it spans that line, which is Corollary 7.4.4.

Exercise. C3: A wedge sees only the subspace

Let $(u_1, \dots, u_k)$ and $(w_1, \dots, w_k)$ be two lists in V with $w_b = \sum_{a=1}^k p_{ab} u_a$ for a matrix $\mathbf{P} = (p_{ab}) \in M_k(F)$.

- (a) Prove that $w_1 \wedge \cdots \wedge w_k = (\det \mathbf{P}) u_1 \wedge \cdots \wedge u_k$.
- (b) Deduce that if $(u_1, \dots, u_k)$ and $(w_1, \dots, w_k)$ are two bases of the **same** k -dimensional subspace $U \subseteq V$, then their wedges differ by a nonzero scalar factor.
- (c) Deduce that the line $\text{span}(u_1 \wedge \cdots \wedge u_k) \subseteq \Lambda^k V$ depends only on U , not on the basis chosen for it.

Hint: for (a), expand every slot and use Lemma 15.8.1 (b) together with Definition 7.4.1.

Solution.

- (a) Expanding slot by slot,

$$w_1 \wedge \cdots \wedge w_k = \sum_{a_1, \dots, a_k} p_{a_1 1} \cdots p_{a_k k} u_{a_1} \wedge \cdots \wedge u_{a_k},$$

the sum over all k -tuples $(a_1, \dots, a_k)$ from $\{1, \dots, k\}$. A term with two equal indices has a repeated argument, so its wedge is 0. A term with all indices distinct has $(a_1, \dots, a_k) = (\sigma(1), \dots, \sigma(k))$ for a unique $\sigma \in S_k$, and then $u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(k)} = \text{sgn}(\sigma) u_1 \wedge \cdots \wedge u_k$ by Lemma 15.8.1 (b). Hence

$$w_1 \wedge \cdots \wedge w_k = \left(\sum_{\sigma \in S_k} \text{sgn}(\sigma) p_{\sigma(1)1} \cdots p_{\sigma(k)k} \right) u_1 \wedge \cdots \wedge u_k,$$

and the bracket is $\det \mathbf{P}$ by Definition 7.4.1.

- (b) Both lists are bases of U , so each w_b is a combination of the u_a , giving $\mathbf{P}$ as above, and $\mathbf{P}$ is the change-of-coordinates matrix between two bases of U , hence invertible (Proposition 4.2.2). By Theorem 7.5.2, $\det \mathbf{P} \neq 0$, and (a) gives the claim. Both wedges are nonzero, by Theorem 15.8.7.
- (c) By (b) the two wedges are nonzero scalar multiples of each other, so they span the same one-dimensional subspace of $\Lambda^k V$. Any basis of U therefore determines the same line, which consequently depends on U alone. (This line is the object that Section 09's coordinates — the $k \times k$ minors — describe, and it is how a k -dimensional subspace is turned into a single point.)