

15.7 Symmetric Powers

The tensor algebra is free, and being free it is useless on its own: it records the order of the slots even when nothing in the problem cares about the order. A symmetric bilinear form does not care (Definition 14.3.1); nor does a quadratic form; nor does the monomial x_1x_2 , which is the same polynomial as x_2x_1 . This section forces $T(V)$ to forget the order, by dividing the tensor power $V^{\otimes k}$ by the span of the differences $u \otimes w - w \otimes u$. The result, the symmetric power $\text{Sym}^k V$, turns out to have a basis indexed by nondecreasing index tuples, and the algebra assembled from all the $\text{Sym}^k V$ turns out to be the polynomial algebra of Chapter 0, one variable per basis vector.

The construction repeats Section 02's three moves for the second time: universal property, uniqueness, then a model built as a quotient. If the shape of the argument below feels familiar, that is the intention.

Throughout, F is a field, V is a finite-dimensional vector space over F with $n = \dim V \geq 1$, and $k \geq 1$ unless stated otherwise. **The field is arbitrary**, and the single place where the characteristic matters is isolated in the last part of the section.

15.7.1 Symmetric multilinear maps

Before asking for a space, we name the maps it is supposed to linearize. Chapter 14 called a *bilinear form* symmetric when swapping its two arguments changed nothing (Definition 14.3.1). With k arguments there are more ways to rearrange them, and we demand invariance under all of them.

A symmetric map cannot tell in which order its arguments were handed to it.

Definition 15.7.1. Symmetric Multilinear Map

Let W be a vector space over F . A k -linear map $f: V^k \rightarrow W$ (Definition 15.1.1) is **symmetric** if

$$f(u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(k)}) = f(u_1, u_2, \dots, u_k)$$

for every permutation $\sigma \in S_k$ and **for all** $u_1, \dots, u_k \in V$.

At $k = 2$ the only nonidentity permutation is the swap, so this is exactly Definition 14.3.1 with a general target W in place of F . At $k = 1$ the condition is empty and every linear map is symmetric.

Three examples and one nonexample.

- **A symmetric bilinear form**, $\beta: V \times V \rightarrow F$ with $\beta(u, w) = \beta(w, u)$. This is the case $k = 2$, $W = F$.
- **Products of one fixed functional**. For $\varphi \in V^*$, the map $(u_1, \dots, u_k) \mapsto \varphi(u_1) \cdots \varphi(u_k)$ is k -linear, by the third example after Definition 7.2.1, and symmetric because multiplication in F is commutative: rearranging the factors does not change the product.
- **Multiplication in a commutative algebra**. If A is a commutative F -algebra and $f: V \rightarrow A$ is linear, then $(u_1, \dots, u_k) \mapsto f(u_1) \cdots f(u_k)$ is symmetric, by commutativity of A .
- **A nonexample by minimal change**. On $V = F^2$, the map $(\mathbf{u}, \mathbf{w}) \mapsto u_1w_2$ is bilinear but not symmetric: it sends $(\mathbf{e}_1, \mathbf{e}_2)$ to 1 and $(\mathbf{e}_2, \mathbf{e}_1)$ to 0. Only the symmetry clause fails; bilinearity survives untouched.

Note which permutations are allowed: **all** of S_k , not only the swaps of adjacent slots. Nothing is lost by asking for all of them, and asking for all of them spares us a lemma about generating S_k .

15.7.2 The symmetric power

Now the object. The demand is Definition 15.6.1's, restricted to the symmetric maps.

The k -th symmetric power is the space in which symmetric k -linear maps become linear.

Definition 15.7.2. k -th Symmetric Power

A **k -th symmetric power of V** is a pair (P, π) , where P is a vector space over F and $\pi: V^k \rightarrow P$ is a **symmetric k -linear** map, such that

(S1) (**universal property**) for **every** vector space W over F and **every symmetric k -linear** map $f: V^k \rightarrow W$, there is a **unique** linear map $\bar{f}: P \rightarrow W$ with $\bar{f} \circ \pi = f$.

We write $\text{Sym}^k V$ for P and denote the value $\pi(u_1, \dots, u_k)$ by juxtaposition,

$$u_1 u_2 \cdots u_k := \pi(u_1, \dots, u_k),$$

calling it a **symmetric product**. By convention $\text{Sym}^0 V = F$.

The one difference from (T1) is the word **symmetric**, in two places: the structure map π is required to be symmetric, and only symmetric maps f are required to factor. Both changes are needed. If π were not symmetric, the notation $uw = wu$ would be false; if nonsymmetric maps had to factor, no space could do the job, since $\bar{f} \circ \pi$ is symmetric whenever π is.

Uniqueness is the template of Theorem 15.2.1 for the third time, and we no longer write it out.

Theorem 15.7.1. Uniqueness of the Symmetric Power

Let (P, π) and (P', π') both be k -th symmetric powers of V . Then there is a unique linear isomorphism $\varphi: P \rightarrow P'$ with $\varphi \circ \pi = \pi'$.

Proof. Word for word the proof of Theorem 15.6.1, with “symmetric k -linear” in place of “ k -linear” throughout: (S1) for P applied to the symmetric map π' gives φ , (S1) for P' applied to π gives ψ , the composite $\psi\varphi$ is a linear self-map of P with $\psi\varphi\pi = \pi$, and the uniqueness clause of (S1) applied to π forces $\psi\varphi = \text{id}_P$. Symmetrically $\varphi\psi = \text{id}_{P'}$. This proves the theorem. ■

And now the model: the tensor power, divided by the relations we want to impose.

Theorem 15.7.2. Existence of the Symmetric Power

Let $C_k \subseteq V^{\otimes k}$ be the span of all the elements

$$u_1 \otimes \cdots \otimes u_k - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}, \quad u_i \in V, \sigma \in S_k,$$

and let $q: V^{\otimes k} \rightarrow V^{\otimes k}/C_k$ be the quotient map (Definition 4.4.3). Then $V^{\otimes k}/C_k$, together with

$$\pi(u_1, \dots, u_k) = q(u_1 \otimes \cdots \otimes u_k),$$

is a k -th symmetric power of V . At $k = 2$ the subspace C_2 is the span of the elements $u \otimes w - w \otimes u$.

Idea. Two things to check, and both are one line each once the right tool is named. That π is symmetric is the definition of C_k : permuting the arguments changes the tensor by an element of C_k , which q kills. That symmetric maps factor is Theorem 4.4.10: turn f into a linear map on $V^{\otimes k}$ by (T1), observe that symmetry of f makes it vanish on the generators of C_k , and let the quotient's universal property do the rest. This is the move the chapter uses three times: **a quotient kills exactly the relations you want.**

Proof. C_k is a subspace, being a span. The map π is k -linear, being the k -linear μ_k of Theorem 15.6.2 followed by the linear map q . It is symmetric: for $\sigma \in S_k$, the difference

$$u_1 \otimes \cdots \otimes u_k - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}$$

lies in C_k by definition, so the two tensors have the same image under q : $\pi(u_{\sigma(1)}, \dots, u_{\sigma(k)}) = \pi(u_1, \dots, u_k)$.

Now let $f: V^k \rightarrow W$ be symmetric k -linear. By (T1) there is a unique linear $\tilde{f}: V^{\otimes k} \rightarrow W$ with $\tilde{f}(u_1 \otimes \cdots \otimes u_k) = f(u_1, \dots, u_k)$. On a generator of C_k ,

$$\tilde{f}(u_1 \otimes \cdots \otimes u_k - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}) = f(u_1, \dots, u_k) - f(u_{\sigma(1)}, \dots, u_{\sigma(k)}),$$

which is 0 because f is symmetric. A linear map vanishing on a spanning set of C_k vanishes on C_k , so $C_k \subseteq \ker \tilde{f}$. By Theorem 4.4.10 (a) there is exactly one linear $\bar{f}: V^{\otimes k}/C_k \rightarrow W$ with $\bar{f} \circ q = \tilde{f}$, and then

$$\bar{f} \circ \pi = \bar{f} \circ q \circ \mu_k = \tilde{f} \circ \mu_k = f.$$

For uniqueness, suppose h is linear with $h \circ \pi = f$. Since q is surjective and $V^{\otimes k}$ is spanned by its simple tensors (Theorem 15.6.2), the elements $\pi(u_1, \dots, u_k)$ span $V^{\otimes k}/C_k$. So h and $\bar{f}$ agree on a spanning set, hence everywhere. This proves the theorem. ■

Warning. $\text{Sym}^k V$ is a quotient of $V^{\otimes k}$, not a subspace of it. Its elements are cosets, and the identity $uw = wu$ in $\text{Sym}^2 V$ says that $u \otimes w$ and $w \otimes u$ lie in the **same** coset. It does **not** say $u \otimes w = w \otimes u$ in $V^{\otimes 2}$, which is false for independent u, w by Theorem 15.6.3. Confusing the two is the commonest error here; the safe habit is never to write an element of $\text{Sym}^k V$ as a tensor.

15.7.3 A basis, and the dimension

Sorting an index tuple is now legal, so the basis of Theorem 15.6.3 collapses: only the sorted tuples survive. Counting them is the substance of the section.

Theorem 15.7.3. Basis of a Symmetric Power

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V and $k \geq 1$. Then the symmetric products

$$v_{i_1} v_{i_2} \cdots v_{i_k}, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n,$$

one for each **nondecreasing** tuple, form a basis of $\text{Sym}^k V$. Consequently

$$\dim \text{Sym}^k V = \binom{n+k-1}{k},$$

over **every** field F .

Idea. Spanning is sorting: the image of any index tuple equals the image of its sorted version, so the sorted ones already span. Independence needs one linear functional per nondecreasing tuple, and it must be built **without dividing by anything**, since we want the theorem over every field. The functional that works is a sum over all ways of dealing out the n dual basis functionals to the k slots with prescribed multiplicities: every coefficient in it is 1. Counting the tuples is then the stars-and-bars bijection, which turns a nondecreasing tuple into a strictly increasing one by adding $0, 1, 2, \dots$ to its entries.

Proof. Spanning. By Theorem 15.6.3 the tensors $v_{j_1} \otimes \cdots \otimes v_{j_k}$, over all tuples J , form a basis of $V^{\otimes k}$, and q is surjective, so their images $v_{j_1} \cdots v_{j_k}$ span $\text{Sym}^k V$. Given any J , let $\sigma \in S_k$ sort it, so that $(j_{\sigma(1)}, \dots, j_{\sigma(k)})$ is nondecreasing. Since π is symmetric, $v_{j_1} \cdots v_{j_k} = v_{j_{\sigma(1)}} \cdots v_{j_{\sigma(k)}}$. So the nondecreasing tuples already give a spanning set.

Independence. Let $(\varphi_1, \dots, \varphi_n)$ be the dual basis of $\mathcal{B}$ (Theorem 5.1.3). For a nondecreasing tuple $I = (i_1, \dots, i_k)$, let

$$m_j = |\{t : i_t = j\}|, \quad j = 1, \dots, n,$$

be its **multiplicity vector**, so $m_1 + \cdots + m_n = k$, and let $\Phi(I)$ be the set of functions $\tau: \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ taking the value j exactly m_j times, for each j . Define

$$g_I(u_1, \dots, u_k) = \sum_{\tau \in \Phi(I)} \varphi_{\tau(1)}(u_1) \varphi_{\tau(2)}(u_2) \cdots \varphi_{\tau(k)}(u_k).$$

Each summand is k -linear, being a product of one functional per slot, so g_I is k -linear.

It is symmetric. Let $\rho \in S_k$. Substituting and then reindexing the product by $s = \rho(t)$,

$$g_I(u_{\rho(1)}, \dots, u_{\rho(k)}) = \sum_{\tau \in \Phi(I)} \prod_{s=1}^k \varphi_{(\tau \rho^{-1})(s)}(u_s).$$

The map $\tau \mapsto \tau\rho^{-1}$ is a bijection of $\Phi(I)$ onto itself, since composing with a permutation of the domain does not change how often a value is taken. So the sum is unchanged, and g_I is symmetric.

By (S1), g_I induces a linear $\bar{g}_I: \text{Sym}^k V \rightarrow F$. Evaluate it on a basis candidate: for any tuple J ,

$$\bar{g}_I(v_{j_1} \cdots v_{j_k}) = \sum_{\tau \in \Phi(I)} \prod_{t=1}^k \delta_{\tau(t)j_t}.$$

A summand is 1 exactly when $\tau(t) = j_t$ for every t , and 0 otherwise; so the whole sum is 1 if the function $t \mapsto j_t$ belongs to $\Phi(I)$, and 0 if it does not. For nondecreasing I and J , that function lies in $\Phi(I)$ exactly when J has the same multiplicity vector as I , which for nondecreasing tuples means $J = I$. Hence $\bar{g}_I(v_{j_1} \cdots v_{j_k}) = \delta_{IJ}$.

Now suppose $\sum_I a_I v_{i_1} \cdots v_{i_k} = 0$, the sum over nondecreasing tuples. Applying $\bar{g}_J$ leaves $a_J = 0$. Since J was arbitrary, every coefficient is 0, and the list is independent. So it is a basis.

The count. Send a nondecreasing tuple $(i_1, \dots, i_k)$ to

$$(i_1, i_2 + 1, i_3 + 2, \dots, i_k + k - 1).$$

Consecutive entries satisfy $i_t + t - 1 < i_{t+1} + t$ because $i_t \leq i_{t+1}$, so the image is **strictly** increasing, and its entries lie between 1 and $n + k - 1$. Conversely, subtracting $0, 1, \dots, k - 1$ from a strictly increasing tuple in $\{1, \dots, n + k - 1\}$ returns a nondecreasing tuple in $\{1, \dots, n\}$: the differences $j_{t+1} - j_t \geq 1$ become $i_{t+1} - i_t \geq 0$, and $1 \leq j_1, j_k \leq n + k - 1$ give $1 \leq i_1$ and $i_k \leq n$. The two assignments are mutually inverse, so the nondecreasing k -tuples from $\{1, \dots, n\}$ are in bijection with the k -element subsets of $\{1, \dots, n + k - 1\}$, and there are $\binom{n+k-1}{k}$ of those. This proves the theorem. ■

Nothing in the proof divided by an integer, which is why the conclusion holds over $\mathbb{F}_2$ as it does over $\mathbb{C}$. Keep that in view: the last part of this section shows that a *different* and very common description of $\text{Sym}^k V$ does not survive over $\mathbb{F}_2$, and the difference between the two is exactly one division.

Quick check. Let $\dim V = 3$. Compute $\dim \text{Sym}^2 V$ and $\dim V^{\otimes 2}$, and say which basis tensors of $V^{\otimes 2}$ become equal in $\text{Sym}^2 V$.

Solution. $\dim \text{Sym}^2 V = \binom{4}{2} = 6$ and $\dim V^{\otimes 2} = 3^2 = 9$. The six surviving basis elements are $v_1^2, v_1 v_2, v_1 v_3, v_2^2, v_2 v_3, v_3^2$. The three pairs $v_i \otimes v_j$ and $v_j \otimes v_i$ with $i < j$ become equal, which removes 3 from the count: $9 - 3 = 6$.

Example. A symmetric square, in coordinates

Let $V = F^2$. Write down a basis of $\text{Sym}^2 V$ and expand $(\mathbf{e}_1 + 2\mathbf{e}_2)(3\mathbf{e}_1 - \mathbf{e}_2)$ in it.

Solution. By Theorem 15.7.3 a basis is $(\mathbf{e}_1^2, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2^2)$, of size $\binom{3}{2} = 3$. Expanding one slot at a time and then using $\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_2$,

$$\begin{aligned} (\mathbf{e}_1 + 2\mathbf{e}_2)(3\mathbf{e}_1 - \mathbf{e}_2) &= 3\mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2 + 6\mathbf{e}_2\mathbf{e}_1 - 2\mathbf{e}_2\mathbf{e}_2 \\ &= 3\mathbf{e}_1^2 + 5\mathbf{e}_1\mathbf{e}_2 - 2\mathbf{e}_2^2. \end{aligned}$$

Compare with expanding $(x + 2y)(3x - y) = 3x^2 + 5xy - 2y^2$ in ordinary algebra. That is not a coincidence, and the next part of this section says exactly what it is.

Warning. Not every element of $\text{Sym}^k V$ is a product of k vectors. Take $F = \mathbb{R}$, $V = \mathbb{R}^2$, and $\omega = \mathbf{e}_1^2 + \mathbf{e}_2^2 \in \text{Sym}^2 V$. If $\omega = \mathbf{u}\mathbf{w}$ with $\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2$ and $\mathbf{w} = c\mathbf{e}_1 + d\mathbf{e}_2$, then comparing coordinates in the basis $(\mathbf{e}_1^2, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2^2)$ gives $ac = 1$, $ad + bc = 0$, and $bd = 1$. The first and last force $a, b, c, d \neq 0$, and multiplying $ad = -bc$ by bc gives $-(bc)^2 = (ac)(bd) = 1$, impossible in $\mathbb{R}$. This is the same phenomenon as the nonsimple tensors of Section 02, and it is why $\text{Sym}^k V$ is much larger than the set of products.

15.7.4 The symmetric algebra is a polynomial algebra

Stacking the symmetric powers gives an algebra, exactly as stacking the tensor powers did, and the multiplication descends from concatenation.

Proposition 15.7.4. The Symmetric Product of Two Symmetric Tensors

Let $k, l \geq 1$. There is a unique bilinear map $\text{Sym}^k V \times \text{Sym}^l V \rightarrow \text{Sym}^{k+l} V$ with

$$(u_1 \cdots u_k)(w_1 \cdots w_l) = u_1 \cdots u_k w_1 \cdots w_l.$$

Proof. Concatenation carries the relations into the relations. Let $x \in C_k$ and $y \in V^{\otimes l}$. For a generator $x = u_1 \otimes \cdots \otimes u_k - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}$ and a simple $y = w_1 \otimes \cdots \otimes w_l$, the product $x \cdot y$ of Proposition 15.6.4 is again a generator of C_{k+l} , for the permutation acting as σ on the first k slots and fixing the last l . Since generators span C_k , since $V^{\otimes l}$ is spanned by its simple tensors, and since $\cdot$ is bilinear, $C_k \cdot V^{\otimes l} \subseteq C_{k+l}$. Swapping the roles of the two blocks gives $V^{\otimes k} \cdot C_l \subseteq C_{k+l}$.

Descending, one argument at a time. Let $q_m: V^{\otimes m} \rightarrow \text{Sym}^m V$ denote the quotient maps. Fix $y \in V^{\otimes l}$. The linear map $x \mapsto q_{k+l}(x \cdot y)$ kills C_k by the first inclusion, so Theorem 4.4.10 gives a unique linear $\beta_y: \text{Sym}^k V \rightarrow \text{Sym}^{k+l} V$ with $\beta_y(q_k x) = q_{k+l}(x \cdot y)$. The assignment $y \mapsto \beta_y$ is linear, since $\beta_{ay+y'}$ and $a\beta_y + \beta_{y'}$ agree on the spanning set $q_k(V^{\otimes k}) = \text{Sym}^k V$; and it kills C_l , since $y \in C_l$ gives $x \cdot y \in C_{k+l}$ and hence $\beta_y = 0$. By Theorem 4.4.10 again it descends to a linear map $\text{Sym}^l V \rightarrow \mathcal{L}(\text{Sym}^k V, \text{Sym}^{k+l} V)$, and evaluating gives the required bilinear map. It is unique because the products of vectors span each factor. When $k = 0$ or $l = 0$ the product is scalar multiplication, as in Section 06. This proves the proposition. ■

Definition 15.7.3. Symmetric Algebra

The **symmetric algebra** of V is the vector space

$$\text{Sym}(V) = \bigoplus_{k \geq 0} \text{Sym}^k V,$$

of sequences with finitely many nonzero components, with the multiplication assembled from [Proposition 15.7.4](#) by $(xy)_m = \sum_{k+l=m} x_k y_l$.

Proposition 15.7.5. $\text{Sym}(V)$ Is a Commutative Algebra

$\text{Sym}(V)$ is an associative, **commutative**, unital F -algebra with identity $1 \in \text{Sym}^0 V = F$, and $\text{Sym}^k V \cdot \text{Sym}^l V \subseteq \text{Sym}^{k+l} V$.

Proof. Bilinearity, associativity, and the identity are proved exactly as in Theorem 15.6.5: all three are identities between multilinear expressions, so it suffices to check them on products of vectors, which span each $\text{Sym}^k V$, and there all three read “concatenate the strings”.

Commutativity is the new point. Both $(u_1 \cdots u_k)(w_1 \cdots w_l)$ and $(w_1 \cdots w_l)(u_1 \cdots u_k)$ equal the symmetric product of the same $k + l$ vectors in two different orders, and π is symmetric, so they are equal. Two bilinear maps agreeing on spanning sets of both arguments agree everywhere, so $xy = yx$ for all x, y . This proves the proposition. ■

To name what $\text{Sym}(V)$ is, we need polynomials in more than one variable. Chapter 0 built $F[x]$; the several-variable version is the same construction with monomials indexed by exponent vectors.

Definition 15.7.4. The Polynomial Algebra $F[x_1, \dots, x_n]$

Let $n \geq 1$. The **polynomial algebra in n variables** over F is the vector space $F[x_1, \dots, x_n]$ with basis the **monomials**

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad a = (a_1, \dots, a_n) \in \mathbb{N}^n,$$

with the unique bilinear multiplication satisfying $x^a x^b = x^{a+b}$. The **degree** of x^a is $|a| = a_1 + \cdots + a_n$, and $F[x_1, \dots, x_n]_k$ denotes the span of the monomials of degree k .

The bilinear extension exists and is unique because a bilinear map may be prescribed freely on a pair of bases, by the argument used twice in the proof of Proposition 15.6.4. The multiplication is associative, commutative, and unital, with identity the monomial all of whose exponents are 0, because exponent addition in $\mathbb{N}^n$ is associative, commutative, and has an identity. For $n = 1$ the monomials are $1, x, x^2, \dots$ and the rule is $x^a x^b = x^{a+b}$, so $F[x_1]$ is the ring $F[x]$ of Definition 0.8.2, whose basis of monomials is Example (A Basis of $F[x]$). Everything Chapter 6 proved — division, greatest common divisors, unique factorization — lives in that one-variable case.

Theorem 15.7.6. The Symmetric Algebra Is a Polynomial Algebra

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V . Then there is an isomorphism of F -algebras

$$\Psi: \text{Sym}(V) \xrightarrow{\sim} F[x_1, \dots, x_n], \quad \Psi(v_i) = x_i,$$

and it carries $\text{Sym}^k V$ onto $F[x_1, \dots, x_n]_k$. In particular

$$\dim F[x_1, \dots, x_n]_k = \binom{n+k-1}{k}.$$

Idea. Both sides have a basis indexed by the same combinatorial object: a nondecreasing k -tuple on the left, a degree- k monomial on the right, and each is determined by its multiplicity vector $m \in \mathbb{N}^n$ with $|m| = k$. So match the bases, which gives a linear isomorphism for free, and then check multiplicativity on basis elements, where both products are “add the exponent vectors”.

Proof. Write $|m| = m_1 + \dots + m_n$ for $m \in \mathbb{N}^n$. Given m with $|m| = k$, let $I(m)$ be the nondecreasing tuple taking the value j exactly m_j times, and write v^m for the corresponding symmetric product $v_{i_1} \dots v_{i_k}$. The assignment $m \mapsto I(m)$ is a bijection onto the nondecreasing k -tuples, since a nondecreasing tuple is recovered from its multiplicity vector and conversely. So by [Theorem 15.7.3](#) the family $(v^m)_{|m|=k}$ is a basis of $\text{Sym}^k V$, and hence $(v^m)_{m \in \mathbb{N}^n}$ is a basis of $\text{Sym}(V)$.

Define Ψ to be the unique linear map with $\Psi(v^m) = x^m$, which exists by [Theorem 2.1.6](#). It sends a basis of $\text{Sym}(V)$ bijectively onto the basis of monomials, so it is a linear isomorphism, and it sends the degree- k basis onto the degree- k monomials, so $\Psi(\text{Sym}^k V) = F[x_1, \dots, x_n]_k$.

Multiplicativity: the maps $(y, z) \mapsto \Psi(yz)$ and $(y, z) \mapsto \Psi(y)\Psi(z)$ are bilinear, so it is enough to compare them on the basis. By [Proposition 15.7.4](#), $v^m v^{m'}$ is the symmetric product of the $|m| + |m'|$ vectors of the two strings, which after sorting is $v^{m+m'}$; and $x^m x^{m'} = x^{m+m'}$ by [Definition 15.7.4](#). Also Ψ sends the empty product 1 to the monomial with all exponents 0, which is 1. Hence Ψ is an algebra isomorphism. Finally v_i is the basis element v^m whose m has a single 1 in position i , so $\Psi(v_i) = x_i$.

The dimension formula follows from [Theorem 15.7.3](#), since Ψ restricts to an isomorphism in each degree. This proves the theorem. ■

So the symmetric algebra is nothing exotic: *it is the polynomial algebra, with one variable per basis vector of V* . The isomorphism depends on the basis, exactly as the coordinate isomorphism $V \cong F^n$ does; $\text{Sym}(V)$ is the basis-free version of “polynomials in n variables”, and $\text{Sym}^k V$ is the basis-free version of “homogeneous polynomials of degree k ”. Read backwards, the theorem also computes something Chapter 6 never did: the number of monomials of degree k in n variables is $\binom{n+k-1}{k}$.

Remark. The case $k = 2$ recovers a count from Chapter 14. A symmetric bilinear form on V is a symmetric 2-linear map into F , so by (S1) the symmetric forms on V are in bijection

with the linear functionals on $\text{Sym}^2 V$, and there are $\binom{n+1}{2} = n(n+1)/2$ dimensions of them — the number of entries on and above the diagonal of a symmetric matrix, which is what Proposition 14.3.1 would give by counting.

15.7.5 Symmetric powers versus symmetric tensors

Here is the trap, and it is worth the space. Many treatments define the k -th symmetric power as a **subspace** of $V^{\otimes k}$: the tensors left fixed by every permutation of the slots. That is a different construction from the quotient above, and over a field of characteristic 0 the two agree. Over $\mathbb{F}_2$ they do not, and the reason is a single division by $k!$.

We first need the permutation operators.

Lemma 15.7.7. Permuting the Slots of a Tensor

For each $\sigma \in S_k$ there is a unique linear operator P_σ on $V^{\otimes k}$ with

$$P_\sigma(u_1 \otimes \cdots \otimes u_k) = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)},$$

and $P_\sigma P_\tau = P_{\tau\sigma}$ for all $\sigma, \tau \in S_k$, with $P_{\text{id}} = \text{id}$.

Proof. The map $(u_1, \dots, u_k) \mapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}$ is k -linear, since in slot t it is μ_k in slot $\sigma^{-1}(t)$. So (T1) gives a unique linear P_σ with the stated values, and uniqueness holds because those tensors span $V^{\otimes k}$. For the composition rule, put $w_t = u_{\tau(t)}$; then

$$P_\sigma P_\tau(u_1 \otimes \cdots \otimes u_k) = P_\sigma(w_1 \otimes \cdots \otimes w_k) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)},$$

and $w_{\sigma(t)} = u_{\tau(\sigma(t))}$, which is the t -th argument of $P_{\tau\sigma}$. Both sides are linear and agree on simple tensors, hence are equal. This proves the lemma. ■

Definition 15.7.5. Symmetric Tensor

A tensor $x \in V^{\otimes k}$ is **symmetric** if $P_\sigma x = x$ for **every** $\sigma \in S_k$. The symmetric tensors form a subspace of $V^{\otimes k}$, which we write $(V^{\otimes k})^{S_k}$; it is the intersection of the kernels of the linear maps $P_\sigma - \text{id}$.

Theorem 15.7.8. Symmetric Tensors and the Symmetric Power

Let $n = \dim V \geq 1$ and $k \geq 1$, and let $\iota: (V^{\otimes k})^{S_k} \rightarrow \text{Sym}^k V$ be the restriction of the quotient map q .

- (a) Over **every** field, $\dim(V^{\otimes k})^{S_k} = \binom{n+k-1}{k} = \dim \text{Sym}^k V$.
- (b) If $\text{char } F = 0$ or $\text{char } F > k$, then ι is an isomorphism.
- (c) If $\text{char } F = 2$, $k = 2$, and $n \geq 2$, then ι is **neither** injective nor surjective.

Idea. For (a), a symmetric tensor must have equal coordinates along each orbit of basis tensors, so the **orbit sums** are a basis; orbits correspond to multisets, which [Theorem 15.7.3](#) already counted. For (b), average: $s = \sum_{\sigma} P_{\sigma}$ maps everything into the symmetric tensors and kills C_k , so $\frac{1}{k!}s$ is an inverse for ι — provided $k!$ is invertible, which is exactly the hypothesis. For (c), the tensor $v_1 \otimes v_2 + v_2 \otimes v_1$ is symmetric and nonzero, but its image is $2v_1v_2 = 0$.

Proof.

- (a) Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V . For a tuple J write $v_J = v_{j_1} \otimes \cdots \otimes v_{j_k}$, so that $(v_J)_J$ is a basis of $V^{\otimes k}$ ([Theorem 15.6.3](#)), and note $P_{\sigma}v_J = v_{J\sigma}$, where $J\sigma$ is the tuple $t \mapsto j_{\sigma(t)}$. Two tuples lie in the same **orbit** when one is a rearrangement of the other, that is, when they have the same multiplicity vector. For each nondecreasing I let $s_I = \sum_{J \in O_I} v_J$, the sum over the orbit of I .

Each s_I is symmetric, because $J \mapsto J\sigma$ permutes O_I . The s_I are independent, because distinct orbits involve disjoint sets of basis vectors. They span: if $x = \sum_J a_J v_J$ satisfies $P_{\sigma}x = x$ for every σ , then comparing coordinates in $\sum_J a_J v_{J\sigma} = \sum_J a_J v_J$ gives $a_J = a_{J\sigma}$ for all J and σ , so $J \mapsto a_J$ is constant on orbits and $x = \sum_I a_I s_I$. Hence (s_I) is a basis, with one member per orbit, that is, per multiplicity vector of total k . By the bijection in the proof of [Theorem 15.7.3](#) there are $\binom{n+k-1}{k}$ of those. No division occurred, so this holds over every field.

- (b) Let $s = \sum_{\sigma \in S_k} P_{\sigma}$, a linear operator on $V^{\otimes k}$. By [Lemma 15.7.7](#), for any τ ,

$$P_{\tau}s = \sum_{\sigma} P_{\tau}P_{\sigma} = \sum_{\sigma} P_{\sigma\tau} = s, \quad sP_{\tau} = \sum_{\sigma} P_{\sigma}P_{\tau} = \sum_{\sigma} P_{\sigma\tau} = s,$$

since $\sigma \mapsto \sigma\tau$ and $\sigma \mapsto \tau\sigma$ are bijections of S_k . The first identity says $\text{im } s \subseteq (V^{\otimes k})^{S_k}$. The second gives $s(\text{id} - P_{\tau}) = 0$, and the generators of C_k are exactly the elements $(\text{id} - P_{\sigma})(u_1 \otimes \cdots \otimes u_k)$, so $C_k \subseteq \ker s$.

By hypothesis $k! \cdot 1 \neq 0$ in F . Indeed $k! \cdot 1 = (1 \cdot 1)(2 \cdot 1) \cdots (k \cdot 1)$ in F , a product of k factors; if $\text{char } F = 0$ none of them is 0, and if $\text{char } F = p > k$ then $i \cdot 1 \neq 0$ for $1 \leq i \leq k < p$ by [Definition 0.7.2](#). A field has no zero divisors, so the product is nonzero, hence invertible. Since $C_k \subseteq \ker s$, [Theorem 4.4.10](#) gives a linear map $\theta: \text{Sym}^k V \rightarrow (V^{\otimes k})^{S_k}$ with $\theta(q(x)) = \frac{1}{k!}s(x)$.

Now compute both composites. For $x \in V^{\otimes k}$, each $P_{\sigma}x$ differs from x by an element of C_k , so $q(s(x)) = \sum_{\sigma} q(P_{\sigma}x) = k!q(x)$, whence

$$\iota\theta(q(x)) = \frac{1}{k!}q(s(x)) = q(x),$$

and q is surjective, so $\iota\theta = \text{id}$. Conversely, if x is symmetric then $s(x) = \sum_{\sigma} P_{\sigma}x = k!x$, so $\theta\iota(x) = \theta(q(x)) = \frac{1}{k!}k!x = x$. Hence ι is an isomorphism with inverse θ .

- (c) Let $\text{char } F = 2$, $k = 2$, $n \geq 2$, and put $t = v_1 \otimes v_2 + v_2 \otimes v_1$. It is symmetric, since the only nonidentity element of S_2 swaps the two summands, and it is nonzero, being a sum of two distinct basis vectors of $V^{\otimes 2}$ ([Theorem 15.6.3](#)). But

$$\iota(t) = v_1v_2 + v_2v_1 = 2v_1v_2 = 0$$

because $1 + 1 = 0$ in F . So $\ker \iota \neq \{0\}$ and ι is not injective. By (a) the two spaces have the same finite dimension, so a surjective ι would be injective by [Theorem 2.3.1](#); hence ι is not surjective either. This proves the theorem.

Part (b) is where the division lives, and part (c) is what it costs. The operator s still exists over $\mathbb{F}_2$; what fails is the *rescaling* $\frac{1}{k!}s$, and with it the splitting of the quotient.

Warning. In characteristic $p \leq k$, “symmetric power” and “symmetric tensors” are different objects. They have the same dimension by Theorem 15.7.8 (a), so they are abstractly isomorphic, but the *natural* map between them is not an isomorphism, and no natural one is available. So: do not define $\text{Sym}^k V$ as $(V^{\otimes k})^{S_k}$, do not write $\frac{1}{k!} \sum_{\sigma} P_{\sigma}$ without saying that $k!$ is invertible, and do not transport a statement proved for one to the other. This is Chapter 14’s discipline again, where alternating and skew-symmetric forms coincided only away from characteristic 2 (Theorem 14.3.3): the same clause, the same division, and the same smallest witness, $\mathbb{F}_2^2$.

Everything else in this section is characteristic-free. In particular Theorem 15.7.3, Theorem 15.7.6, and the dimension $\binom{n+k-1}{k}$ hold over every field, because the quotient construction never divides. The next section takes the same care with the same reward: the exterior power is built from a relation chosen so that no division is ever needed.

15.7.6 Exercises

15.7.6.1 A. Check your understanding

Exercise. A1

- (a) State the universal property (S1), with every quantifier, and say which maps are required to factor.
- (b) Which subspace of $V^{\otimes k}$ is divided out to build $\text{Sym}^k V$? Write down its generators for $k = 3$.
- (c) Let $\dim V = 4$. Compute $\dim \text{Sym}^3 V$ and $\dim V^{\otimes 3}$.
- (d) Determine whether $\text{Sym}^2 V$ is a subspace of $V^{\otimes 2}$. Justify your answer.
- (e) What hypothesis on F makes the symmetric tensors of degree k naturally isomorphic to $\text{Sym}^k V$, and which step of the proof needs it?

Solution.

- (a) For every vector space W over F and every **symmetric** k -linear map $f: V^k \rightarrow W$, there exists exactly one linear $\bar{f}: \text{Sym}^k V \rightarrow W$ with $\bar{f} \circ \pi = f$. Only the symmetric maps factor; a nonsymmetric one cannot, since $\bar{f} \circ \pi$ is symmetric for every linear $\bar{f}$.
- (b) The subspace C_k of Theorem 15.7.2, spanned by the differences $u_1 \otimes \cdots \otimes u_k - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}$. For $k = 3$ the generators are the six differences obtained from the six $\sigma \in S_3$, for all $u_1, u_2, u_3 \in V$.
- (c) $\dim \text{Sym}^3 V = \binom{4+3-1}{3} = \binom{6}{3} = 20$ and $\dim V^{\otimes 3} = 4^3 = 64$.
- (d) No. It is a quotient of $V^{\otimes 2}$, and its elements are cosets of C_2 , not tensors. (Over a field of characteristic 0 there is a subspace of $V^{\otimes 2}$ naturally isomorphic to it, namely the symmetric tensors, but the isomorphism is not an equality, and it does not exist in characteristic 2.)

- (e) $\text{char } F = 0$ or $\text{char } F > k$, so that $k!$ is invertible in F ([Theorem 15.7.8](#) (b)). The step that needs it is the construction of the inverse map θ , which divides the symmetrization $s = \sum_{\sigma} P_{\sigma}$ by $k!$.

15.7.6.2 B. Practice

Exercise. B1: Expanding symmetric products

Let $V = F^2$.

- (a) Expand $(2\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}_1 + 3\mathbf{e}_2)$ in the basis $(\mathbf{e}_1^2, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2^2)$ of $\text{Sym}^2 V$.
 (b) Expand $(\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)\mathbf{e}_1$ in the basis of $\text{Sym}^3 V$ given by [Theorem 15.7.3](#).

Solution.

- (a) Expanding one slot at a time and then using $\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_2$,

$$(2\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}_1 + 3\mathbf{e}_2) = 2\mathbf{e}_1^2 + 6\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_2 - 3\mathbf{e}_2^2 = 2\mathbf{e}_1^2 + 5\mathbf{e}_1\mathbf{e}_2 - 3\mathbf{e}_2^2.$$

- (b) The basis is $(\mathbf{e}_1^3, \mathbf{e}_1^2\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2^2, \mathbf{e}_2^3)$, of size $\binom{4}{3} = 4$. Expanding the first two slots,

$$(\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)\mathbf{e}_1 = \mathbf{e}_1^2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2^2\mathbf{e}_1,$$

and the two middle terms are equal after sorting, so they cancel:

$$(\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)\mathbf{e}_1 = \mathbf{e}_1^3 - \mathbf{e}_1\mathbf{e}_2^2.$$

In the polynomial language of [Theorem 15.7.6](#) this is $(x + y)(x - y)x = x^3 - xy^2$.

Exercise. B2: Counting

- (a) Compute $\dim \text{Sym}^k V$ for $\dim V = 3$ and $k = 0, 1, 2, 3, 4$.
 (b) For $\dim V = n$, compute $\dim \text{Sym}^2 V$ in closed form and check it against the number of entries on or above the diagonal of an $n \times n$ matrix.
 (c) Determine all pairs (n, k) with $k \geq 1$ for which $\dim \text{Sym}^k V = \dim V^{\otimes k}$. Justify your answer.

Solution.

- (a) $\binom{k+2}{k}$ for $n = 3$, giving 1, 3, 6, 10, 15 for $k = 0, 1, 2, 3, 4$.
 (b) $\binom{n+1}{2} = n(n+1)/2$. An $n \times n$ matrix has n diagonal entries and $(n^2 - n)/2$ entries strictly above the diagonal, and $n + (n^2 - n)/2 = n(n+1)/2$. The agreement is the remark after [Theorem 15.7.6](#): a symmetric bilinear form is the same data as a functional on $\text{Sym}^2 V$, and also the same data as a symmetric matrix.
 (c) Exactly those with $n = 1$, or with $k = 1$. If $k = 1$ both sides are n . If $n = 1$ both sides are 1. Otherwise $n \geq 2$ and $k \geq 2$, and then $v_1 \otimes v_2 \neq v_2 \otimes v_1$ are two basis tensors with the same image, so $C_k \neq \{0\}$ and $\dim \text{Sym}^k V = n^k - \dim C_k < n^k$ by [Theorem 4.4.5](#).

Exercise. B3: A form factored through the symmetric square

Let β be a symmetric bilinear form on V . Prove that there is a unique linear functional $\bar{\beta} \in (\text{Sym}^2 V)^*$ with $\bar{\beta}(uw) = \beta(u, w)$ for all $u, w \in V$, and compute $\bar{\beta}$ on the basis of [Theorem 15.7.3](#) when $V = F^2$ and $\beta(\mathbf{x}, \mathbf{y}) = x_1 y_1 - x_2 y_2$.

Solution. β is a symmetric 2-linear map $V^2 \rightarrow F$ ([Definition 15.7.1](#) with $k = 2$, $W = F$), so (S1) gives exactly one linear $\bar{\beta}: \text{Sym}^2 V \rightarrow F$ with $\bar{\beta} \circ \pi = \beta$, which is the displayed identity.

For the given β : $\bar{\beta}(\mathbf{e}_1^2) = \beta(\mathbf{e}_1, \mathbf{e}_1) = 1$, $\bar{\beta}(\mathbf{e}_1 \mathbf{e}_2) = \beta(\mathbf{e}_1, \mathbf{e}_2) = 0$, and $\bar{\beta}(\mathbf{e}_2^2) = \beta(\mathbf{e}_2, \mathbf{e}_2) = -1$. So in the coordinates given by the basis $(\mathbf{e}_1^2, \mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_2^2)$ the functional reads $(1, 0, -1)$, and under [Theorem 15.7.6](#) it is the rule “read off the coefficients of x^2 and y^2 and subtract”.

15.7.6.3 C. Going deeper**Exercise. C1: The characteristic-two witness in full**

Let $F = \mathbb{F}_2$ and $V = \mathbb{F}_2^2$, and take $k = 2$.

- Write down a basis of $(V^{\otimes 2})^{S_2}$ and verify that its dimension is 3.
- Compute $\dim C_2$ and $\dim \text{Sym}^2 V$, and identify $\ker \iota$ and $\text{im } \iota$ for the map ι of [Theorem 15.7.8](#).
- Are $(V^{\otimes 2})^{S_2}$ and $\text{Sym}^2 V$ isomorphic as vector spaces? Explain why this does not repair the situation.

Solution.

- Here $S_2 = \{\text{id}, \sigma\}$ with σ the swap, and P_σ exchanges $\mathbf{e}_1 \otimes \mathbf{e}_2$ with $\mathbf{e}_2 \otimes \mathbf{e}_1$ and fixes $\mathbf{e}_1 \otimes \mathbf{e}_1$ and $\mathbf{e}_2 \otimes \mathbf{e}_2$. So $x = a \mathbf{e}_1 \otimes \mathbf{e}_1 + b \mathbf{e}_1 \otimes \mathbf{e}_2 + c \mathbf{e}_2 \otimes \mathbf{e}_1 + d \mathbf{e}_2 \otimes \mathbf{e}_2$ is symmetric if and only if $b = c$. A basis is

$$\mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1,$$

the orbit sums of [Theorem 15.7.8](#) (a), so the dimension is $3 = \binom{3}{2}$.

- C_2 is spanned by the elements $\mathbf{u} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{u}$. Writing $\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2$ and $\mathbf{w} = c\mathbf{e}_1 + d\mathbf{e}_2$ and expanding both slots, the $\mathbf{e}_1 \otimes \mathbf{e}_1$ and $\mathbf{e}_2 \otimes \mathbf{e}_2$ terms cancel and

$$\mathbf{u} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{u} = (ad - bc)(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1).$$

So C_2 is the line spanned by $\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$, the sign being irrelevant over $\mathbb{F}_2$, and $\dim C_2 = 1$ and $\dim \text{Sym}^2 V = 4 - 1 = 3$ by [Theorem 4.4.5](#). Now $\ker \iota = (V^{\otimes 2})^{S_2} \cap C_2 = C_2$, of dimension 1, and $\text{im } \iota$ is therefore 2-dimensional, spanned by $\mathbf{e}_1^2$ and $\mathbf{e}_2^2$; it misses $\mathbf{e}_1 \mathbf{e}_2$.

- Yes, they are isomorphic: both have dimension 3 over $\mathbb{F}_2$, and [Theorem 2.5.4](#) applies. It does not repair anything, because the isomorphism is not ι and does not respect the structure maps: an isomorphism chosen by hand would not satisfy $\varphi(\mathbf{u} \otimes \mathbf{w}) = \mathbf{u}\mathbf{w}$, so it could not be used to transport statements about products of vectors. A construction is only as good as its universal property, and $(V^{\otimes 2})^{S_2}$ has none here.

Exercise. C2: $\text{Sym}(V)$ is the free commutative algebra

- (a) Prove that the products $u_1 \cdots u_k$ with $u_i \in V$ span $\text{Sym}^k V$, and deduce that an algebra homomorphism out of $\text{Sym}(V)$ is determined by its restriction to V .
- (b) Prove that for every **commutative** associative unital F -algebra A and every linear map $f: V \rightarrow A$ there is a unique algebra homomorphism $\bar{f}: \text{Sym}(V) \rightarrow A$ with $\bar{f}|_V = f$.
- (c) Explain in one sentence why (b) fails if the word “commutative” is deleted, and name the algebra that has the corresponding property for all A .

Solution.

- (a) $V^{\otimes k}$ is spanned by its simple tensors (Theorem 15.6.2) and q is surjective and linear, so their images, the products $u_1 \cdots u_k$, span $\text{Sym}^k V$. If φ, ψ are algebra homomorphisms agreeing on V , then $\varphi(u_1 \cdots u_k) = \varphi(u_1) \cdots \varphi(u_k) = \psi(u_1 \cdots u_k)$, and both send 1 to 1_A ; since these elements span $\text{Sym}(V)$ and both maps are linear, $\varphi = \psi$.
- (b) For $k \geq 1$ the map $(u_1, \dots, u_k) \mapsto f(u_1) \cdots f(u_k)$ is k -linear, as in the proof of Theorem 15.6.6, and it is **symmetric** because A is commutative (the third example after Definition 15.7.1). By (S1) it induces a linear $\bar{f}_k: \text{Sym}^k V \rightarrow A$; set $\bar{f}_0(a) = a1_A$ and $\bar{f} = \sum_k \bar{f}_k$ on the direct sum. Multiplicativity is checked on products of vectors, which span, exactly as in Theorem 15.6.6, and $\bar{f}|_V = f$. Uniqueness is (a).
- (c) It fails because $\bar{f}$ would have to satisfy $f(u)f(w) = \bar{f}(uw) = \bar{f}(wu) = f(w)f(u)$, so the images of V would be forced to commute, which need not happen in a noncommutative A . The algebra with the property for **all** A is the tensor algebra $T(V)$, by Theorem 15.6.6.

Exercise. C3: One variable

Let $\dim V = 1$.

- (a) Prove that $C_k = \{0\}$ for every k , and deduce that the quotient map $T(V) \rightarrow \text{Sym}(V)$ assembled from the q 's is an isomorphism of algebras.
- (b) Hence deduce that $T(V) \cong \text{Sym}(V) \cong F[x]$, and reconcile this with the fact that $T(V)$ is noncommutative for $\dim V \geq 2$.

Solution.

- (a) Write $V = Fv$. By Theorem 15.6.3, $V^{\otimes k}$ is spanned by $v^{\otimes k}$. A generator of C_k is $u_1 \otimes \cdots \otimes u_k - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}$; writing $u_t = c_tv$ and expanding, both terms equal $c_1 c_2 \cdots c_k v^{\otimes k}$, since the scalars multiply in the same way whatever the order. So every generator is 0 and $C_k = \{0\}$. Hence each $q: V^{\otimes k} \rightarrow \text{Sym}^k V$ is an isomorphism (Theorem 4.4.5, or directly: its kernel is C_k), and the assembled map is a bijective algebra homomorphism, so an isomorphism.
- (b) By Example (The tensor algebra of a line), $T(V) \cong F[x]$, so (a) gives $\text{Sym}(V) \cong F[x]$ as well; the same conclusion comes from Theorem 15.7.6 with $n = 1$. There is no tension with noncommutativity: $T(V)$ is commutative precisely when $\dim V \leq 1$, because noncommutativity needs two independent vectors to put in the two slots (the warning

after Theorem 15.6.5). With one variable there is nothing to reorder, so imposing the symmetry relations imposes nothing.