

14.7 Where the Form Vanishes

Sections 4 and 5 diagonalized a symmetric form and counted the signs on the diagonal. Over $\mathbb{R}$ that is the whole story, but it hides the feature that makes indefinite forms geometrically different from inner products: there are nonzero vectors on which the form vanishes. This section takes those vectors seriously. They are not defects to be quarantined; they organize the space, because each one forces a two-dimensional piece of completely standard shape to split off. The result is a decomposition of any nondegenerate space into a pile of identical planes plus a remainder on which the form vanishes nowhere.

Throughout, F is a field with $\text{char } F \neq 2$ (Definition 0.7.2), V is a finite-dimensional vector space over F with $\dim V = n$, and β is a **symmetric** bilinear form on V (Definition 14.3.1) with quadratic form $q(v) = \beta(v, v)$ (Definition 14.3.2). Two vectors are orthogonal when β pairs them to 0 (Definition 14.4.1). For a subspace $U \subseteq V$ we write, as in Exercise (C2: Orthogonal complements for a form),

$$U^{\perp\beta} = \{v \in V : \beta(u, v) = 0 \text{ for every } u \in U\},$$

and we write $V = U \perp W$ to say that $V = U \oplus W$ with every vector of U orthogonal to every vector of W — an **orthogonal direct sum**. The subscript β is dropped when only one form is in play.

14.7.1 An intuition to give up

Thirteen chapters have trained one reflex above all others: $\langle v, v \rangle = 0$ forces $v = 0$. It is how we proved that an isometry is injective, that an orthogonal list is independent, that a Gram matrix determines its vectors. It is also the first thing that goes.

Take $V = \mathbb{R}^2$ with

$$q(x_1, x_2) = x_1^2 - x_2^2, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The vector $\mathbf{v} = (1, 1)$ has $q(\mathbf{v}) = 1 - 1 = 0$. It is not $\mathbf{0}$, and the form is not defective: $\det \mathbf{A} = -1 \neq 0$, so β is nondegenerate (Proposition 14.1.2). Indeed $\beta(\mathbf{v}, (1, -1)) = 1 - (1)(-1) = 2 \neq 0$, so $\mathbf{v}$ is perfectly visible to β — it is only invisible to *itself*. The line $\text{span}(\mathbf{v})$ is a line of “length zero” inside a space where length is otherwise perfectly well behaved.

It gets worse, and the worse case is the one worth internalizing. Take $V = \mathbb{R}^4$ with

$$q(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2 - x_4^2,$$

of signature $(2, 2)$ and therefore nondegenerate. Let $W = \text{span}(\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4)$, a plane. For $\mathbf{w} = a(\mathbf{e}_1 + \mathbf{e}_3) + b(\mathbf{e}_2 + \mathbf{e}_4)$,

$$q(\mathbf{w}) = a^2 + b^2 - a^2 - b^2 = 0,$$

and in fact β vanishes on every pair from W , since $\beta(\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4) = 0$ too. So β restricted to W is the **zero form** on a two-dimensional subspace of a four-dimensional space on which β is nondegenerate.

Say that again, because it is the sentence an inner-product-trained reader will refuse. A nondegenerate form can vanish identically on a subspace of half the dimension of the whole space. Nondegeneracy is a statement about V ; it says that no vector is orthogonal to *everything*. It says nothing whatsoever about what β does inside a subspace. The vectors of W above are

orthogonal to each other and to themselves; they are not orthogonal to $\mathbf{e}_1 - \mathbf{e}_3$, and that is all nondegeneracy ever promised.

Once that is accepted, the rest of the section is straightforward. The vanishing vectors get a name, and then a structure theorem.

14.7.2 Isotropic vectors and totally isotropic subspaces

An isotropic vector is a nonzero vector that the form cannot tell from zero when it looks at it twice.

Definition 14.7.1. Isotropic and Anisotropic

Let β be a symmetric bilinear form on V with quadratic form q .

- (a) A vector $v \in V$ is **isotropic** if $v \neq 0$ and $q(v) = 0$, and **anisotropic** if $q(v) \neq 0$.
- (b) The form β is **anisotropic** if it has no isotropic vectors: that is, if $q(v) = 0$ implies $v = 0$. Otherwise β is **isotropic**.

Both clauses hinge on small words. In (a) the vector is required to be **nonzero**, so 0 is neither isotropic nor anisotropic; the zero vector always has $q(0) = 0$ and carries no information. In (b), “anisotropic” is a property of the **form on a particular space**. Anisotropy passes down to subspaces, since a subspace has fewer vectors to test; isotropy does not, and a form isotropic on V is very often anisotropic on a subspace. The word comes from *iso* (equal) and *tropos* (direction): a vector along which the form does not distinguish one direction from another.

Examples, over several fields.

- **A positive definite form has no isotropic vectors.** If $q(v) > 0$ for $v \neq 0$ then $q(v) = 0$ forces $v = 0$. So every inner product is anisotropic, which is exactly the reflex described above. The same holds for negative definite forms.
- **Over $\mathbb{R}$, signature $(1, 1)$.** For $q = x_1^2 - x_2^2$ the isotropic vectors are the nonzero multiples of $(1, 1)$ and of $(1, -1)$, since $q = (x_1 - x_2)(x_1 + x_2)$. Two lines, and nothing else.
- **Over $\mathbb{C}$.** For $q = x_1^2 + x_2^2$ on $\mathbb{C}^2$ the factorization $q = (x_1 + ix_2)(x_1 - ix_2)$ is available, so $(1, i)$ and $(1, -i)$ are isotropic. Over $\mathbb{R}$ the same q is positive definite. **Isotropy depends on the field**, not only on the polynomial.
- **Over $\mathbb{F}_3$ and $\mathbb{F}_5$.** For $q = x_1^2 + x_2^2$ the squares in $\mathbb{F}_3$ are 0 and 1, so $q(\mathbf{x}) = 0$ needs $x_1 = x_2 = 0$ and the form is anisotropic. In $\mathbb{F}_5$, however, $1^2 + 2^2 = 5 = 0$, so $(1, 2)$ is isotropic. The same formula, two finite fields, two answers.
- **The degenerate case.** Every nonzero vector of the radical is isotropic, since $v \in \text{rad}(\beta)$ gives $\beta(u, v) = 0$ for every u , in particular for $u = v$. So a degenerate form is automatically isotropic. The converse fails, and the $\mathbb{R}^2$ example above is the witness.

When a whole subspace consists of isotropic vectors, more is true than the definition says, and it deserves its own name.

Definition 14.7.2. Totally Isotropic Subspace

A subspace $W \subseteq V$ is **totally isotropic** for β if $\beta(u, w) = 0$ for **all** $u, w \in W$; equivalently, if $W \subseteq W^{\perp\beta}$. The zero subspace is totally isotropic, trivially.

The two formulations agree because $W \subseteq W^{\perp\beta}$ says exactly that every $w \in W$ is orthogonal to every $u \in W$. A third formulation is available in characteristic $\neq 2$ and is the one to use in practice: W is totally isotropic if and only if $q(w) = 0$ for every $w \in W$. One direction is immediate; the other is Theorem 14.3.5 applied to the restricted form, which recovers β on W from q on W and so makes it zero. In the $\mathbb{R}^4$ example above we checked exactly this: q vanished on W , and therefore so did β .

Before the first theorem, one dimension count has to be on the table, because almost everything in this section and the next is a step away from it. Exercise (C2: Orthogonal complements for a form) set it as an exercise in Section 1; here it is with a proof, so that the results below rest on it rather than on the exercise.

Lemma 14.7.1. Dimension of an Orthogonal Complement

Let β be a **nondegenerate** bilinear form on V with $\dim V = n$, and let $U \subseteq V$ be a subspace. Then $U^{\perp\beta}$ is a subspace of V , and

$$\dim U^{\perp\beta} = n - \dim U.$$

Idea. Freezing the second slot of β turns it into a map $V \rightarrow V^*$, and restricting a functional to U turns V^* into U^* . The composite has kernel exactly $U^{\perp\beta}$, and nondegeneracy makes both maps surjective, so Rank–Nullity converts “onto U^* ” into the count.

Proof. Let $R: V \rightarrow V^*$ send v to the functional $\beta(\cdot, v)$ — the map called R_β in Section 1 — and let $\rho: V^* \rightarrow U^*$ be restriction to U . Both are linear, so $S := \rho \circ R: V \rightarrow U^*$ is linear, and $S(v) = \beta(\cdot, v)|_U$. By the definition of $U^{\perp\beta}$, a vector v lies in $U^{\perp\beta}$ exactly when $\beta(u, v) = 0$ for every $u \in U$, that is, exactly when $S(v) = 0$. Hence $U^{\perp\beta} = \ker S$, which is a subspace.

Since β is nondegenerate, R is an isomorphism (Proposition 14.1.2 (c)), and ρ is surjective (Theorem 5.5.1). A composite of two surjections is a surjection, so $\text{rank } S = \dim U^* = \dim U$ (Corollary 5.1.4). By Rank–Nullity (Theorem 2.3.1),

$$\dim U^{\perp\beta} = \dim \ker S = n - \text{rank } S = n - \dim U,$$

as claimed. ■

Totally isotropic subspaces cannot be too large.

Proposition 14.7.2. Half the Dimension, at Most

Let β be a nondegenerate symmetric form on V with $\dim V = n$, and let $W \subseteq V$ be totally isotropic. Then $\dim W \leq n/2$.

Proof. Since β is nondegenerate, $\dim W^{\perp\beta} = n - \dim W$ by Lemma 14.7.1. Since W is totally isotropic, $W \subseteq W^{\perp\beta}$, so $\dim W \leq n - \dim W$, which is the claim.



Warning. An isotropic vector is not a zero vector, and a form that vanishes on a subspace need not be degenerate. These are two different errors and both are common. The first confuses $q(v) = 0$ with $v = 0$; over $\mathbb{R}^2$ with $q = x_1^2 - x_2^2$ the vector $(1, 1)$ separates them. The second confuses “ β vanishes on $W \times W$ ” with “ β vanishes on $\{v\} \times V$ ”. Degeneracy is about the **second**: the radical consists of vectors killed by pairing with *everything*. The plane W of the $\mathbb{R}^4$ example carries the zero form and contains no radical vector at all.

Quick check. Is the plane $\text{span}(\mathbf{e}_1, \mathbf{e}_2) \subseteq \mathbb{R}^4$ totally isotropic for $q(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2 - x_4^2$? What about $\text{span}(\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_3)$?

Solution. No to both. On $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$ the form restricts to $x_1^2 + x_2^2$, which is positive definite, so that plane is as far from totally isotropic as possible: it is anisotropic. The second plane is $\text{span}(\mathbf{e}_1, \mathbf{e}_3)$, on which q restricts to $x_1^2 - x_3^2$; its two spanning vectors $\mathbf{e}_1 \pm \mathbf{e}_3$ are indeed isotropic, but $\beta(\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_3) = 1 - (-1) = 2 \neq 0$, so the plane is not totally isotropic. Containing isotropic vectors is not the same as being totally isotropic.

14.7.3 Splitting off a nondegenerate piece

Everything below rests on one mechanism: if a subspace carries a nondegenerate restriction, it splits off orthogonally and the rest of the space is again nondegenerate. That is the analogue of “restrict to an invariant complement”, and it is what makes induction possible.

Proposition 14.7.3. A Nondegenerate Subspace Splits Off

Let β be a nondegenerate symmetric form on V with $\dim V = n$, and let $U \subseteq V$ be a subspace such that $\beta|_{U \times U}$ is nondegenerate. Then

$$V = U \perp U^{\perp\beta},$$

and β restricted to $U^{\perp\beta}$ is nondegenerate.

Idea. Two things to check and one dimension count. The intersection $U \cap U^{\perp\beta}$ is precisely the radical of the restricted form, which the hypothesis kills; the dimension of $U^{\perp\beta}$ is $n - \dim U$ because the global form is nondegenerate; and a vector of $U^{\perp\beta}$ orthogonal to $U^{\perp\beta}$ is orthogonal to all of V , hence zero.

Proof. Write $U^\perp = U^{\perp\beta}$. A vector $v \in U \cap U^\perp$ lies in U and satisfies $\beta(u, v) = 0$ for every $u \in U$, so $v \in \text{rad}(\beta|_{U \times U})$, which is $\{0\}$ by hypothesis (Definition 14.1.4). Hence $U \cap U^\perp = \{0\}$ and the sum $U + U^\perp$ is direct (Theorem 1.7.4).

Since β is nondegenerate on V , Lemma 14.7.1 gives $\dim U^\perp = n - \dim U$. So $\dim(U \oplus U^\perp) = n$, and a subspace of V of dimension n is V (Theorem 1.6.8). The sum is orthogonal by the definition of $U^\perp$ together with the symmetry of β , so $V = U \perp U^\perp$.

Finally let $w \in U^\perp$ satisfy $\beta(x, w) = 0$ for every $x \in U^\perp$. For $u \in U$ we also have $\beta(u, w) = 0$, since $w \in U^\perp$. Every $v \in V$ is $u + x$ with $u \in U$ and $x \in U^\perp$, so $\beta(v, w) = 0$ for every $v \in V$, that is $w \in \text{rad}(\beta) = \{0\}$. This proves the proposition. ■

14.7.4 Hyperbolic planes

Which two-dimensional forms are isotropic? Over $\mathbb{R}$ with signature $(1, 1)$ we found two isotropic lines, and the example $q = x_1^2 - x_2^2$ can be rewritten by the substitution $u = x_1 + x_2$, $w = (x_1 - x_2)/2$ as $q = 2uw$ — the cross term alone, with no squares at all. That shape is the model, and it is the same over every field.

Definition 14.7.3. Hyperbolic Pair, Hyperbolic Plane

Let β be a symmetric bilinear form on V . A **hyperbolic pair** is an ordered pair (e, f) of vectors of V with

$$q(e) = 0, \quad q(f) = 0, \quad \beta(e, f) = 1.$$

The subspace $H = \text{span}(e, f)$ is then called a **hyperbolic plane**, and (e, f) is a **hyperbolic basis** of it.

Four things follow at once. First, e and f are linearly independent: if $f = ce$ then $\beta(e, f) = c q(e) = 0 \neq 1$. So H really is a plane, $\dim H = 2$. Second, in the basis (e, f) the Gram matrix of $\beta|_H$ is

$$[\beta|_H]_{(e,f)}^{(e,f)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose determinant is $-1 \neq 0$, so $\beta|_H$ is **nondegenerate** (Proposition 14.1.2) and Proposition 14.7.3 applies to it. Third, $q(ae + bf) = 2ab$, so on a hyperbolic plane q takes **every** value in F : given $c \in F$, take $a = c/2$ and $b = 1$. Fourth, over $\mathbb{R}$ the Gram matrix has eigenvalues 1 and -1 , so a hyperbolic plane has signature $(1, 1)$ and is the unique real example up to congruence.

Now the workhorse of this section and the next. It says that one isotropic vector is enough to manufacture a whole hyperbolic plane around it.

Lemma 14.7.4. Every Isotropic Vector Lies in a Hyperbolic Plane

Let $\text{char } F \neq 2$, let β be a **nondegenerate** symmetric bilinear form on V , and let $e \in V$ be isotropic. Then there is $f \in V$ such that (e, f) is a hyperbolic pair. In particular e lies in a hyperbolic plane $H = \text{span}(e, f)$, and $\beta|_H$ is nondegenerate.

Idea. Nondegeneracy hands us some u with $\beta(e, u) \neq 0$, and rescaling makes $\beta(e, u) = 1$. Only one condition is still missing, $q(u) = 0$, and u can be corrected without disturbing what has been achieved: adding a multiple of e changes $q(u)$ but not $\beta(e, u)$, because $q(e) = 0$. Solving for the multiple is one line, and it divides by 2 — which is where the characteristic

hypothesis is spent, for the third time in this chapter.

Proof. Since $e \neq 0$ and β is nondegenerate, there is $u_0 \in V$ with $c = \beta(e, u_0) \neq 0$ (Definition 14.1.4). Put $u = c^{-1}u_0$, so that

$$\beta(e, u) = 1.$$

Now set

$$f = u - \frac{1}{2}q(u)e,$$

which is legitimate because $\text{char } F \neq 2$ makes $2 = 1 + 1$ invertible. Then, using bilinearity, symmetry, and $q(e) = 0$,

$$\beta(e, f) = \beta(e, u) - \frac{1}{2}q(u)q(e) = 1 - 0 = 1,$$

and

$$\begin{aligned} q(f) &= q(u) - 2 \cdot \frac{1}{2}q(u)\beta(u, e) + \frac{1}{4}q(u)^2q(e) \\ &= q(u) - q(u) + 0 = 0. \end{aligned}$$

Together with $q(e) = 0$, this shows that (e, f) is a hyperbolic pair. As noted after Definition 14.7.3, e and f are independent, so $H = \text{span}(e, f)$ is a plane, and the Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is invertible, so $\beta|_H$ is nondegenerate. This proves the lemma. ■

Remark. Nondegeneracy of β is not decoration here. On $\mathbb{R}^2$ with $q(\mathbf{x}) = x_1^2$, the vector $\mathbf{e}_2$ is isotropic and lies in no hyperbolic plane, because $\beta(\mathbf{e}_2, \mathbf{v}) = 0$ for every $\mathbf{v}$ and a hyperbolic pair needs $\beta(\mathbf{e}, \mathbf{f}) = 1$. What fails is the very first line of the proof.

14.7.5 The Witt decomposition

Peeling off a hyperbolic plane and repeating is now a two-line induction, and it is the structural theorem of the section: it does for indefinite forms what “peel off an eigenvector” did for operators in Chapter 9.

Theorem 14.7.5. Witt Decomposition

Let $\text{char } F \neq 2$ and let β be a nondegenerate symmetric bilinear form on a finite-dimensional V . Then there are an integer $m \geq 0$, hyperbolic planes $H_1, \dots, H_m \subseteq V$, and a subspace $V_0 \subseteq V$ such that

$$V = H_1 \perp H_2 \perp \cdots \perp H_m \perp V_0$$

and β restricted to V_0 is **anisotropic**. In particular $\dim V = 2m + \dim V_0$.

Idea. Induction on $\dim V$. Either β is anisotropic, in which case take $m = 0$ and $V_0 = V$ and stop; or there is an isotropic vector, in which case Lemma 14.7.4 wraps a hyperbolic plane H_1 around it, Proposition 14.7.3 peels H_1 off with a nondegenerate remainder $H_1^\perp$ of dimension $\dim V - 2$, and the induction hypothesis finishes the job there.

Proof. Induct on $\dim V$. If $\dim V = 0$, take $m = 0$ and $V_0 = \{0\}$, on which the form is anisotropic vacuously. Let $\dim V \geq 1$ and suppose the statement holds for every nondegenerate symmetric form on a space of smaller dimension.

If β is anisotropic, take $m = 0$ and $V_0 = V$.

Otherwise β has an isotropic vector e . By [Lemma 14.7.4](#) there is f with (e, f) a hyperbolic pair, and $H_1 = \text{span}(e, f)$ is a hyperbolic plane with $\beta|_{H_1}$ nondegenerate. By [Proposition 14.7.3](#),

$$V = H_1 \perp H_1^{\perp\beta},$$

and β restricted to $V' = H_1^{\perp\beta}$ is nondegenerate, with $\dim V' = \dim V - 2 < \dim V$. The induction hypothesis applied to $(V', \beta|_{V'})$ gives hyperbolic planes $H_2, \dots, H_m \subseteq V'$ and an anisotropic $V_0 \subseteq V'$ with $V' = H_2 \perp \dots \perp H_m \perp V_0$. Each H_i with $i \geq 2$ and V_0 lie inside $V' = H_1^{\perp\beta}$, so all the summands are pairwise orthogonal, and

$$V = H_1 \perp H_2 \perp \dots \perp H_m \perp V_0.$$

Counting dimensions, $\dim V = 2m + \dim V_0$. This completes the induction. ■

Two questions are now unavoidable: is m the same for every such decomposition, and is V_0 determined? The first has an answer here, and it comes from a quantity that mentions no decomposition at all.

Definition 14.7.4. Witt Index

Let β be a nondegenerate symmetric bilinear form on a finite-dimensional V . The **Witt index** of β is

$$\text{ind}(\beta) := \max\{\dim W : W \subseteq V \text{ totally isotropic}\},$$

the largest dimension of a totally isotropic subspace. By [Proposition 14.7.2](#) it satisfies $\text{ind}(\beta) \leq n/2$, and it is 0 exactly when β is anisotropic.

The maximum exists because the dimensions in question form a nonempty finite set of integers: $W = \{0\}$ is always totally isotropic, and all dimensions are at most n . Nothing in the definition refers to a basis or a decomposition, so the Witt index is an invariant of (V, β) by construction. It is also exactly the number of planes.

Proposition 14.7.6. The Number of Hyperbolic Planes Is the Witt Index

In any decomposition $V = H_1 \perp \dots \perp H_m \perp V_0$ as in [Theorem 14.7.5](#), $m = \text{ind}(\beta)$. In particular m is the same for every such decomposition, and so is $\dim V_0 = n - 2m$.

Idea. One inequality is a construction: the first vectors $e_1, \dots, e_m$ of the m hyperbolic bases span a totally isotropic subspace of dimension m , since each is isotropic and different planes are orthogonal. The other inequality is an induction that trades one plane for one dimension: given a totally isotropic W , the vectors of W orthogonal to e_1 project into $H_1^\perp$ as a totally isotropic subspace of dimension at least $\dim W - 1$. Repeating m times lands inside the

anisotropic V_0 , where the only totally isotropic subspace is $\{0\}$.

Proof. Let (e_i, f_i) be a hyperbolic basis of H_i for each i .

Step 1: $\text{ind}(\beta) \geq m$. Put $W = \text{span}(e_1, \dots, e_m)$. The e_i lie in different summands of a direct sum and are nonzero, so they are independent and $\dim W = m$. For $i \neq j$ we have $\beta(e_i, e_j) = 0$ because $H_i \perp H_j$, and $\beta(e_i, e_i) = q(e_i) = 0$. By bilinearity β vanishes on $W \times W$, so W is totally isotropic of dimension m .

Step 2: a plane can be traded for a dimension.

Claim. Let β be nondegenerate on V , let $H = \text{span}(e, f) \subseteq V$ be a hyperbolic plane, let $V' = H^\perp$, and let $W \subseteq V$ be totally isotropic. Then V' contains a totally isotropic subspace of dimension at least $\dim W - 1$.

Proof. By Proposition 14.7.3, $V = H \perp V'$; let $P: V \rightarrow V'$ be the projection with kernel H , so that $v - P(v) \in H$ for every v . Put

$$W_1 = \{w \in W : \beta(e, w) = 0\},$$

the kernel of a linear functional on W , so $\dim W_1 \geq \dim W - 1$.

For $w \in V$ write $w - P(w) = ae + bf$. Since $e \in H$ is orthogonal to V' ,

$$\beta(e, w) = \beta(e, ae + bf) = aq(e) + b = b.$$

So $w \in W_1$ forces $w - P(w) \in \text{span}(e)$. Consequently, for $w, w' \in W_1$ the vectors $w - P(w)$ and $w' - P(w')$ both lie in $\text{span}(e)$, where β vanishes because $q(e) = 0$; hence

$$\beta(P(w), P(w')) = \beta(w, w') - \beta(w - P(w), w' - P(w')) = 0 - 0 = 0,$$

the first equality using the orthogonality of H and V' . So $P(W_1)$ is a totally isotropic subspace of V' .

It remains to bound its dimension. By Rank–Nullity (Theorem 2.3.1) applied to $P|_{W_1}$, whose kernel is $W_1 \cap H$,

$$\dim P(W_1) = \dim W_1 - \dim(W_1 \cap H).$$

If $W_1 \cap H = \{0\}$, then $\dim P(W_1) = \dim W_1 \geq \dim W - 1$ and we are done. Otherwise pick $x \neq 0$ in $W_1 \cap H$. Then $P(x) = 0$, so $x = x - P(x) \in \text{span}(e)$ by the paragraph above, whence $e \in W$. But then $\beta(e, w) = 0$ for every $w \in W$, because W is totally isotropic and $e \in W$; so $W_1 = W$. Moreover every $x \in W_1 \cap H$ lies in $\text{span}(e)$ by the same argument, so $W_1 \cap H = \text{span}(e)$ has dimension 1, and $\dim P(W_1) = \dim W - 1$. Either way the claim holds. ■

Step 3: $\text{ind}(\beta) \leq m$. Let $W \subseteq V$ be totally isotropic. Apply Step 2 to $H = H_1$: since $H_1^\perp = H_2 \perp \dots \perp H_m \perp V_0$ carries a nondegenerate form, we obtain a totally isotropic subspace of it of dimension at least $\dim W - 1$. Repeating with $H_2, \dots, H_m$ in turn produces a totally isotropic subspace of V_0 of dimension at least $\dim W - m$. But $\beta|_{V_0}$ is anisotropic, so a nonzero vector of V_0 has $q \neq 0$ and the only totally isotropic subspace of V_0 is $\{0\}$. Hence $\dim W - m \leq 0$.

Steps 1 and 3 give $\text{ind}(\beta) = m$, and $\dim V_0 = n - 2m$ follows by counting. This proves the proposition. ■

So the number of planes is pinned down. What is not yet pinned down is V_0 itself: different decompositions produce different subspaces, and the claim one wants is that they all carry *the same form* in a suitable sense. Making that precise needs the word “isometry” for forms, and proving it needs Witt’s theorems; both are the business of the next section.

14.7.6 The two familiar fields

Over $\mathbb{R}$ and $\mathbb{C}$ everything can be computed, because the classification is already known.

Proposition 14.7.7. The Witt Index over the Reals and the Complexes

- (a) Let β be a nondegenerate symmetric form on a real V with signature (n_+, n_-) (Definition 14.5.1), so $n_+ + n_- = n$. Then $\text{ind}(\beta) = \min(n_+, n_-)$, and the anisotropic part has dimension $|n_+ - n_-|$ and is definite.
- (b) Let β be a nondegenerate symmetric form on a complex V with $\dim V = n$. Then $\text{ind}(\beta) = \lfloor n/2 \rfloor$. In particular every such form with $n \geq 2$ has an isotropic vector, and the anisotropic part has dimension 0 or 1.

Proof.

- (a) Put $k = \min(n_+, n_-)$. By Theorem 14.4.2 and the rescaling that produced Eq. 14.5.1, together with $n_0 = 0$, there is an orthogonal basis $(v_1, \dots, v_n)$ with $q(v_i) = 1$ for $i \leq n_+$, $q(v_i) = -1$ for $n_+ < i \leq n$, and distinct basis vectors orthogonal. For $1 \leq i \leq k$ set

$$e_i = v_i + v_{n_++i}, \quad f_i = \frac{1}{2}(v_i - v_{n_++i}).$$

Then $q(e_i) = 1 + (-1) = 0$ and $q(f_i) = \frac{1}{4}(1 + (-1)) = 0$, while

$$\beta(e_i, f_i) = \frac{1}{2}(q(v_i) - q(v_{n_++i})) = \frac{1}{2}(1 + 1) = 1,$$

so (e_i, f_i) is a hyperbolic pair and $H_i = \text{span}(e_i, f_i)$ a hyperbolic plane. These planes are pairwise orthogonal, since for $i \neq j$ their spanning vectors involve disjoint sets of the v ’s. Let V_0 be the span of the remaining $|n_+ - n_-|$ basis vectors, all of the same sign; then $V = H_1 \perp \dots \perp H_k \perp V_0$, and $\beta|_{V_0}$ is definite, hence anisotropic. By Proposition 14.7.6, $\text{ind}(\beta) = k$.

- (b) By Corollary 14.4.4 and nondegeneracy, there is a basis $(v_1, \dots, v_n)$ with $q(v_j) = 1$ for every j and distinct basis vectors orthogonal. Write $n = 2k + r$ with $r \in \{0, 1\}$, and let i denote the imaginary unit. For $1 \leq j \leq k$ set

$$e_j = v_{2j-1} + i v_{2j}, \quad f_j = \frac{1}{2}(v_{2j-1} - i v_{2j}).$$

Then $q(e_j) = 1 + i^2 = 0$ and likewise $q(f_j) = \frac{1}{4}(1 + i^2) = 0$, while $\beta(e_j, f_j) = \frac{1}{2}(1 + 1) = 1$. As in (a) these span k pairwise orthogonal hyperbolic planes, and what is left is the span of v_n when $r = 1$, on which $q(cv_n) = c^2 \neq 0$ for $c \neq 0$, and $\{0\}$ when $r = 0$. So $\text{ind}(\beta) = k = \lfloor n/2 \rfloor$ by Proposition 14.7.6. This proves the proposition.



Two readings are worth recording. Over $\mathbb{R}$ the Witt index is the smaller of the two counts of signs: a definite form has index 0, and a form of signature (m, m) has the largest possible index $m = n/2$ — the $\mathbb{R}^4$ example that opened the section, with $n_+ = n_- = 2$, sits exactly there. Over $\mathbb{C}$ the index is as large as it can be for every nondegenerate form, because -1 is a square and so a sum of two squares always factors; the only anisotropic complex spaces are $\{0\}$ and the lines.

Quick check. A nondegenerate real form has signature $(3, 1)$. What is its Witt index, and what is the anisotropic part?

Solution. The index is $\min(3, 1) = 1$, so exactly one hyperbolic plane splits off and $\dim V_0 = 4 - 2 = 2$. By [Proposition 14.7.7](#) (a), V_0 carries a definite form of dimension $|3 - 1| = 2$, positive definite here since the positive signs are the ones in surplus. Concretely $\mathbb{R}^{3,1}$ with $q = x_1^2 + x_2^2 + x_3^2 - x_4^2$ splits as one hyperbolic plane, spanned by $\mathbf{e}_3 \pm \mathbf{e}_4$ after rescaling, orthogonal to the positive definite plane $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$. The largest totally isotropic subspace is a line, and the lines of this kind sweep out the light cone that Section 11 studies.

14.7.7 Exercises

14.7.7.1 A. Check your understanding

Exercise. A1

- Define **isotropic vector**, **totally isotropic subspace**, and **hyperbolic pair**.
- Determine whether the following statement is correct, and justify your answer: a nondegenerate form has no isotropic vectors.
- What is the Witt index of a positive definite form on a space of dimension n ? Of a form of signature $(4, 4)$?
- State the Gram matrix of a hyperbolic plane in a hyperbolic basis, and say why it shows the restricted form is nondegenerate.
- Which hypothesis of [Lemma 14.7.4](#) fails for the form $q(\mathbf{x}) = x_1^2$ on $\mathbb{R}^2$ at the isotropic vector $\mathbf{e}_2$?

Solution.

- A vector v is isotropic if $v \neq 0$ and $q(v) = 0$. A subspace W is totally isotropic if $\beta(u, w) = 0$ for all $u, w \in W$. A hyperbolic pair is a pair (e, f) with $q(e) = q(f) = 0$ and $\beta(e, f) = 1$.
- Incorrect. Nondegeneracy says $\text{rad}(\beta) = \{0\}$, a statement about pairing with **every** vector. On $\mathbb{R}^2$ with $q = x_1^2 - x_2^2$ the form is nondegenerate, since its Gram matrix has determinant -1 , and $(1, 1)$ is isotropic.
- 0 for a positive definite form, since it is anisotropic, so the only totally isotropic subspace is $\{0\}$. For signature $(4, 4)$ the index is $\min(4, 4) = 4$ by [Proposition 14.7.7](#) (a), the

largest value [Proposition 14.7.2](#) permits in dimension 8.

- (d) It is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, of determinant $-1 \neq 0$, and a form is nondegenerate exactly when its Gram matrix is invertible ([Proposition 14.1.2](#)).
- (e) Nondegeneracy of β . Here $\mathbf{e}_2 \in \text{rad}(\beta)$, so there is no $\mathbf{u}$ with $\beta(\mathbf{e}_2, \mathbf{u}) \neq 0$ and the first line of the proof has nothing to choose.

14.7.7.2 B. Practice

Exercise. B1: Find the isotropic vectors

For each of the following real quadratic forms, find all isotropic vectors, give the signature, and state the Witt index.

- (a) $q(x_1, x_2) = x_1^2 + 2x_1x_2$ on $\mathbb{R}^2$.
- (b) $q(x_1, x_2, x_3) = x_1x_2$ on $\mathbb{R}^3$. (*Is this form nondegenerate?*)
- (c) $q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ on $\mathbb{R}^3$.

Solution.

- (a) $q = x_1(x_1 + 2x_2)$, so $q = 0$ exactly on the two lines $x_1 = 0$ and $x_1 = -2x_2$. The isotropic vectors are the nonzero multiples of $(0, 1)$ and of $(-2, 1)$. The Gram matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, of determinant $-1 < 0$, so the eigenvalues have opposite signs and the signature is $(1, 1)$. The Witt index is $\min(1, 1) = 1$; indeed $(0, 1)$ is isotropic and by [Lemma 14.7.4](#) the whole space is one hyperbolic plane.
- (b) The Gram matrix is $\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, of rank 2, so the form is **degenerate** with $\text{rad}(\beta) = \text{span}(\mathbf{e}_3)$ and the notion of Witt index does not apply as defined. The isotropic vectors are those with $x_1x_2 = 0$, namely the union of the two planes $x_1 = 0$ and $x_2 = 0$, minus $\mathbf{0}$. On the nondegenerate part $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$ the signature is $(1, 1)$ and the index is 1.
- (c) There are none: q is positive definite, so $q(\mathbf{x}) = 0$ forces $\mathbf{x} = \mathbf{0}$. The signature is $(3, 0)$ and the Witt index is 0.

Exercise. B2: Build a hyperbolic basis

- (a) For $q(x_1, x_2) = x_1^2 + x_2^2$ on $\mathbb{C}^2$, find a hyperbolic pair.
- (b) For $q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$ on $\mathbb{R}^3$, find a hyperbolic pair $(\mathbf{e}, \mathbf{f})$ with $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_3$, and identify the anisotropic part $\text{span}(\mathbf{e}, \mathbf{f})^{\perp_\beta}$.

Solution.

- (a) Take $\mathbf{e} = (1, i)$, so $q(\mathbf{e}) = 1 + i^2 = 0$. The vector $\mathbf{u} = (1, -i)$ has $\beta(\mathbf{e}, \mathbf{u}) = 1 \cdot 1 + i \cdot (-i) = 2$, so rescale: $\mathbf{f} = \frac{1}{2}(1, -i)$ gives $\beta(\mathbf{e}, \mathbf{f}) = 1$ and $q(\mathbf{f}) = \frac{1}{4}(1 + i^2) = 0$. So $((1, i), \frac{1}{2}(1, -i))$ is a hyperbolic pair, and $\mathbb{C}^2$ is a single hyperbolic plane.
- (b) $q(\mathbf{e}) = 1 - 1 = 0$, so $\mathbf{e}$ is isotropic. Follow the proof of [Lemma 14.7.4](#): $\mathbf{u}_0 = \mathbf{e}_1$ gives $\beta(\mathbf{e}, \mathbf{e}_1) = 1$, so take $\mathbf{u} = \mathbf{e}_1$, and $q(\mathbf{u}) = 1$, whence

$$\mathbf{f} = \mathbf{e}_1 - \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_3) = \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3).$$

Check: $q(\mathbf{f}) = \frac{1}{4}(1 - 1) = 0$ and $\beta(\mathbf{e}, \mathbf{f}) = \frac{1}{2}(1 + 1) = 1$. For the complement, $\mathbf{v} = (v_1, v_2, v_3)$ is orthogonal to both $\mathbf{e}$ and $\mathbf{f}$ exactly when $v_1 - v_3 = 0$ and $v_1 + v_3 = 0$, that is $v_1 = v_3 = 0$. So $\text{span}(\mathbf{e}, \mathbf{f})^{\perp_\beta} = \text{span}(\mathbf{e}_2)$, on which $q = x_2^2$ is positive definite. This is the decomposition of [Theorem 14.7.5](#) with $m = 1$, matching $\min(2, 1) = 1$.

Exercise. B3: A totally isotropic subspace of the largest size

Let $q(\mathbf{x}) = x_1x_4 + x_2x_3$ on $\mathbb{R}^4$.

- (a) Write down the Gram matrix and verify that β is nondegenerate.
- (b) Exhibit a totally isotropic subspace of dimension 2, and explain why none of dimension 3 exists.
- (c) Determine the signature of q and confirm the Witt index by [Proposition 14.7.7](#) (a).

Solution.

- (a) Halving the cross-term coefficients (Eq. 14.3.1),

$$\mathbf{A} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

whose determinant is $\frac{1}{16}$ det of the reversal permutation matrix, namely $\frac{1}{16} \cdot 1 \neq 0$. So β is nondegenerate by [Proposition 14.1.2](#).

- (b) Take $W = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$. Every $\mathbf{w} = (a, b, 0, 0)$ has $q(\mathbf{w}) = 0$, and $\beta(\mathbf{e}_1, \mathbf{e}_2) = 0$ from the matrix, so W is totally isotropic of dimension 2. No totally isotropic subspace has dimension 3, because [Proposition 14.7.2](#) caps it at $4/2 = 2$.
- (c) The two planes $\text{span}(\mathbf{e}_1, \mathbf{e}_4)$ and $\text{span}(\mathbf{e}_2, \mathbf{e}_3)$ are orthogonal, and $(\mathbf{e}_1, 2\mathbf{e}_4)$ is a hyperbolic pair, since $q(\mathbf{e}_1) = q(2\mathbf{e}_4) = 0$ and $\beta(\mathbf{e}_1, 2\mathbf{e}_4) = 2 \cdot \frac{1}{2} = 1$; the same works for $(\mathbf{e}_2, 2\mathbf{e}_3)$. So each plane is hyperbolic, of signature $(1, 1)$. Hence q has signature $(2, 2)$ and, by [Proposition 14.7.7](#) (a), Witt index $\min(2, 2) = 2$, attained by the W of part (b).

14.7.7.3 C. Going deeper

Exercise. C1: Isotropy makes the form surjective

Let β be a nondegenerate symmetric form on V over a field F with $\text{char } F \neq 2$.

- (a) Prove that if β has an isotropic vector, then $q: V \rightarrow F$ is surjective.
- (b) Deduce that if some $c \in F$ is not of the form $q(v)$, then β is anisotropic.
- (c) Use (b) to show that the form $q(x_1, x_2) = x_1^2 + x_2^2$ on $\mathbb{F}_3^2$ is anisotropic, and give its Witt index.

Solution.

- (a) Let e be isotropic. By [Lemma 14.7.4](#) there is f with (e, f) a hyperbolic pair. For $c \in F$, put $v = \frac{c}{2}e + f$; this is legitimate since $\text{char } F \neq 2$. Then

$$q(v) = \frac{c^2}{4}q(e) + 2 \cdot \frac{c}{2}\beta(e, f) + q(f) = 0 + c + 0 = c.$$

So every $c \in F$ is a value of q .

- (b) Contrapositive of (a): if q is not surjective then β has no isotropic vector.
- (c) The squares in $\mathbb{F}_3$ are $0^2 = 0, 1^2 = 1, 2^2 = 1$, so the values of $x_1^2 + x_2^2$ are 0, 1, 2 — which is all of $\mathbb{F}_3$, so (b) gives nothing directly. Argue instead from the definition: $q(\mathbf{x}) = 0$ needs $x_1^2 = -x_2^2 = 2x_2^2$. If $x_2 \neq 0$ then $x_2^2 = 1$, so $x_1^2 = 2$, which is not a square in $\mathbb{F}_3$; hence $x_2 = 0$ and then $x_1 = 0$. The form is anisotropic, its Witt index is 0, and the Gram matrix $\mathbf{I}_2$ shows it is nondegenerate. Over $\mathbb{F}_5$ the same form is isotropic, since $1^2 + 2^2 = 0$.

Exercise. C2: A maximal totally isotropic subspace and its complement

Let β be nondegenerate symmetric on V with $\dim V = n$, and let $W \subseteq V$ be totally isotropic with $\dim W = \text{ind}(\beta) = m$.

- (a) Prove that $\dim W^{\perp\beta} = n - m$ and $W \subseteq W^{\perp\beta}$.
- (b) Prove that the form induced by β on the quotient $W^{\perp\beta}/W$ is well defined and anisotropic. *Hint: a coset with an isotropic representative would enlarge W .*
- (c) Deduce that $n - 2m = \dim(W^{\perp\beta}/W)$ agrees with the dimension of the anisotropic part in [Theorem 14.7.5](#).

Solution.

- (a) The dimension is [Lemma 14.7.1](#), using nondegeneracy, and the inclusion is [Definition 14.7.2](#).
- (b) For $u, v \in W^{\perp\beta}$ and $w, w' \in W$,

$$\beta(u + w, v + w') = \beta(u, v),$$

because each of the three remaining terms pairs a vector of W with a vector of $W^{\perp\beta} \supseteq W$, hence vanishes. So $\bar{\beta}(u + W, v + W) := \beta(u, v)$ is well defined, and it is bilinear and symmetric. Suppose $\bar{q}(u + W) = 0$ with $u \notin W$, that is $q(u) = 0$ with $u \in W^{\perp\beta} \setminus W$. Then $W + \text{span}(u)$ is totally isotropic: it is spanned by isotropic vectors, $\beta(u, w) = 0$ for $w \in W$ because $u \in W^{\perp\beta}$, and β vanishes on $W \times W$. Its dimension is $m + 1 > \text{ind}(\beta)$, a contradiction. So $\bar{\beta}$ is anisotropic.

- (c) By (a), $\dim(W^{\perp\beta}/W) = (n - m) - m = n - 2m$. By [Proposition 14.7.6](#) the anisotropic part V_0 of a Witt decomposition also has dimension $n - 2m$, since the index is m . (That the two anisotropic forms agree, and not merely their dimensions, is proved in the next section.)

Exercise. C3: Where characteristic 2 breaks

Work over $\mathbb{F}_2$ and let β be the dot product on $\mathbb{F}_2^2$, with Gram matrix $\mathbf{I}_2$.

- (a) Verify that β is symmetric and nondegenerate, and find all $\mathbf{v}$ with $q(\mathbf{v}) = 0$.
- (b) Show that β has a vector with $q(\mathbf{v}) = 0$ and $\mathbf{v} \neq \mathbf{0}$ that lies in no hyperbolic plane, and identify which line of the proof of [Lemma 14.7.4](#) fails.
- (c) Explain in one sentence why the third description of “totally isotropic” given after [Definition 14.7.2](#) is unavailable here.

Solution.

- (a) The Gram matrix $\mathbf{I}_2$ is symmetric and invertible, so β is symmetric and nondegenerate (Proposition 14.1.2). Here $q(\mathbf{x}) = x_1^2 + x_2^2 = (x_1 + x_2)^2$ over $\mathbb{F}_2$, which vanishes exactly when $x_1 = x_2$. So $q(\mathbf{v}) = 0$ for $\mathbf{v} = \mathbf{0}$ and for $\mathbf{v} = (1, 1)$.
- (b) Take $\mathbf{v} = (1, 1)$. A hyperbolic plane containing it would need $\mathbf{f}$ with $q(\mathbf{f}) = 0$ and $\beta(\mathbf{v}, \mathbf{f}) = 1$. The only nonzero vector with $q = 0$ is $\mathbf{v}$ itself, and $\beta(\mathbf{v}, \mathbf{v}) = q(\mathbf{v}) = 0 \neq 1$. The step that fails is the correction $\mathbf{f} = \mathbf{u} - \frac{1}{2}q(\mathbf{u})\mathbf{e}$: the element $2 = 1 + 1$ is 0 in $\mathbb{F}_2$ and is not invertible, so $\frac{1}{2}$ does not exist.
- (c) Because it is Theorem 14.3.5, which needs $\text{char } F \neq 2$ to divide by 2 ; over $\mathbb{F}_2$ the line $\text{span}((1, 1))$ has $q \equiv 0$ on it and is indeed totally isotropic here, but the implication “ q vanishes on W implies β vanishes on $W \times W$ ” is not available in general.