

13.8 Products Entry by Entry

Matrix multiplication is not the only way to multiply two matrices of the same size. There is a much simpler product — multiply the entries in matching positions — which is useless for composing linear maps and extremely useful everywhere else. This section introduces it and proves the one theorem that makes it interesting: it preserves positivity. A determinant inequality follows, and it contains Hadamard’s inequality of Section 7 as a special case.

Throughout, $F = \mathbb{R}$ or $F = \mathbb{C}$.

13.8.1 The entrywise product

Two matrices of the same shape can be multiplied the way two lists of numbers are: position by position. The operation has no interpretation in terms of linear maps, which is one reason it looks like a curiosity at first sight, and it is written with a small circle to keep it away from ordinary multiplication.

The Hadamard product multiplies matrices the way a spreadsheet would.

Definition 13.8.1. Hadamard Product

Let $\mathbf{A}, \mathbf{B} \in M_{m \times n}(F)$. Their **Hadamard product**, or **entrywise product**, is the matrix $\mathbf{A} \circ \mathbf{B} \in M_{m \times n}(F)$ with

$$(\mathbf{A} \circ \mathbf{B})_{ij} = a_{ij}b_{ij} \quad \text{for all } i, j.$$

Both factors must have **the same shape**, and the result has that shape too; there is no rule like “columns of the first match rows of the second”. The operation is commutative and associative, and it is bilinear — $(\mathbf{A} + \mathbf{A}') \circ \mathbf{B} = \mathbf{A} \circ \mathbf{B} + \mathbf{A}' \circ \mathbf{B}$ and $(c\mathbf{A}) \circ \mathbf{B} = c(\mathbf{A} \circ \mathbf{B})$ — because each of these is a statement about numbers, checked one entry at a time. Its identity element is the all-ones matrix $\mathbf{J}$, not $\mathbf{I}$.

Examples.

- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ -3 & 8 \end{pmatrix}.$
- $\mathbf{A} \circ \mathbf{I} = \text{diag}(a_{11}, \dots, a_{nn})$: multiplying entrywise by the identity **erases** everything off the diagonal, where ordinary multiplication by $\mathbf{I}$ changes nothing.
- $\mathbf{A} \circ \mathbf{J} = \mathbf{A}$, so $\mathbf{J}$ is the identity for $\circ$.
- For two diagonal matrices, $\mathbf{D} \circ \mathbf{D}' = \mathbf{DD}'$: both products are $\text{diag}(d_1d'_1, \dots, d_nd'_n)$. For matrices in general the two have nothing to do with each other, though they can coincide by accident: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ have equal Hadamard and matrix products without either being diagonal.
- For column vectors $\mathbf{u}, \mathbf{w} \in F^n$, regarded as $n \times 1$ matrices, $\mathbf{u} \circ \mathbf{w} = (u_1w_1, \dots, u_nw_n).$

Nonexample by minimal change. Take $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \mathbf{A}$. Then $\mathbf{AB} = \mathbf{I}$, but $\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The two products differ, and they differ in the most basic way: $\mathbf{A}$ is its own inverse and its own Hadamard square.

Warning. $\mathbf{A} \circ \mathbf{B}$ is not $\mathbf{AB}$, and almost nothing carries over. Two invertible matrices can have a singular entrywise product:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

where the two factors have determinants -3 and 3 and the product has determinant 0 . There is no formula for $\det(\mathbf{A} \circ \mathbf{B})$ in terms of $\det \mathbf{A}$ and $\det \mathbf{B}$, no relation between the eigenvalues, and no rule $(\mathbf{A} \circ \mathbf{B})^{-1} = \mathbf{A}^{-1} \circ \mathbf{B}^{-1}$. The one structural property that does survive is the subject of this section, and it is a genuine surprise.

The name is the same one that Section 7 attached to a determinant inequality. That is a coincidence of a busy mathematician, not of the mathematics: Hadamard's inequality and the Hadamard product are unrelated statements, and this section will nonetheless prove one from the other.

13.8.2 The Schur product theorem

Positivity is a statement about the quadratic form $\mathbf{x}^* \mathbf{A} \mathbf{x}$, and the quadratic form of $\mathbf{A} \circ \mathbf{B}$ has no visible relation to those of $\mathbf{A}$ and $\mathbf{B}$. So the following theorem cannot be proved by staring at the form. What works is to break both matrices into the simplest positive pieces there are — rank-one matrices $\mathbf{u} \mathbf{u}^*$ — and to notice that the entrywise product of two such pieces is again one.

Lemma 13.8.1. Entrywise product of two rank-one positive matrices

Let $\mathbf{u}, \mathbf{w} \in F^n$. Then

$$(\mathbf{u} \mathbf{u}^*) \circ (\mathbf{w} \mathbf{w}^*) = (\mathbf{u} \circ \mathbf{w})(\mathbf{u} \circ \mathbf{w})^*.$$

Proof. Compare the (i, j) entries. The (i, j) entry of $\mathbf{u} \mathbf{u}^*$ is $u_i \overline{u_j}$ and that of $\mathbf{w} \mathbf{w}^*$ is $w_i \overline{w_j}$, so the left-hand side has (i, j) entry

$$u_i \overline{u_j} w_i \overline{w_j} = (u_i w_i) \overline{(u_j w_j)},$$

the rearrangement using commutativity of F and $\overline{u_j} \overline{w_j} = \overline{u_j w_j}$. The vector $\mathbf{u} \circ \mathbf{w}$ has i -th entry $u_i w_i$, so the right-hand side has (i, j) entry $(u_i w_i) \overline{(u_j w_j)}$ as well. The two matrices agree entry by entry. ■

That one line is the whole theorem; everything else is bookkeeping over the spectral decompositions.

Theorem 13.8.2. Schur Product Theorem

Let $\mathbf{A}, \mathbf{B} \in M_n(F)$.

- (a) If $\mathbf{A} \succeq \mathbf{O}$ and $\mathbf{B} \succeq \mathbf{O}$, then $\mathbf{A} \circ \mathbf{B} \succeq \mathbf{O}$.
 (b) If $\mathbf{A} \succ \mathbf{O}$ and $\mathbf{B} \succ \mathbf{O}$, then $\mathbf{A} \circ \mathbf{B} \succ \mathbf{O}$.

Idea. For (a): the spectral theorem writes $\mathbf{A} = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^*$ and $\mathbf{B} = \sum_j \mu_j \mathbf{w}_j \mathbf{w}_j^*$ with all $\lambda_i, \mu_j \geq 0$. Expand $\mathbf{A} \circ \mathbf{B}$ by bilinearity into n^2 terms; by Lemma 13.8.1 each term is a nonnegative multiple of a matrix $\mathbf{z}\mathbf{z}^*$, and those are positive semidefinite. A sum of positive semidefinite matrices is positive semidefinite, so we are done.

For (b), do not redo the argument with strict inequalities: the vectors $\mathbf{u}_i \circ \mathbf{w}_j$ need not span, and the bookkeeping becomes unpleasant. Instead peel a multiple of the identity off $\mathbf{B}$. If $\varepsilon > 0$ is the smallest eigenvalue of $\mathbf{B}$, then $\mathbf{B} = (\mathbf{B} - \varepsilon \mathbf{I}) + \varepsilon \mathbf{I}$ with $\mathbf{B} - \varepsilon \mathbf{I} \succeq \mathbf{O}$, so

$$\mathbf{A} \circ \mathbf{B} = \mathbf{A} \circ (\mathbf{B} - \varepsilon \mathbf{I}) + \varepsilon (\mathbf{A} \circ \mathbf{I}),$$

and $\mathbf{A} \circ \mathbf{I}$ is the diagonal matrix of the a_{ii} , which are strictly positive. Part (a) handles the first term, and a positive definite matrix plus a positive semidefinite one is positive definite.

Proof.

- (a) Both $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, so each has an orthonormal basis of eigenvectors. Let $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ be one for $\mathbf{A}$, with $\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i$, and let $\mathbf{U}$ be the unitary matrix with these columns, so that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$ with $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$ by Corollary 12.4.3 over $\mathbb{C}$ and Corollary 12.5.2 over $\mathbb{R}$. The columns of $\mathbf{U}\mathbf{D}$ are $\lambda_i \mathbf{u}_i$ and the rows of $\mathbf{U}^*$ are $\mathbf{u}_i^*$, so Corollary 8.1.3 gives

$$\mathbf{A} = \sum_{i=1}^n \lambda_i \mathbf{u}_i \mathbf{u}_i^*,$$

which over $\mathbb{R}$ is Corollary 12.5.4. In the same way $\mathbf{B} = \sum_j \mu_j \mathbf{w}_j \mathbf{w}_j^*$. Every $\lambda_i \geq 0$ and every $\mu_j \geq 0$, by Theorem 13.1.4 (b).

The Hadamard product is bilinear, so expanding both sums,

$$\mathbf{A} \circ \mathbf{B} = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j [(\mathbf{u}_i \mathbf{u}_i^*) \circ (\mathbf{w}_j \mathbf{w}_j^*)] = \sum_{i,j} \lambda_i \mu_j \mathbf{z}_{ij} \mathbf{z}_{ij}^*,$$

where $\mathbf{z}_{ij} = \mathbf{u}_i \circ \mathbf{w}_j$ and the second equality is Lemma 13.8.1. Each $\mathbf{z}_{ij} \mathbf{z}_{ij}^*$ is positive semidefinite by Example (Five Positive Matrices) (b), and each coefficient $\lambda_i \mu_j$ is a nonnegative real number. A nonnegative combination of positive semidefinite matrices is positive semidefinite (Exercise (C2: The cone of positive matrices) (a)): the combination is Hermitian, being a real combination of Hermitian matrices, and for every $\mathbf{x}$,

$$\left\langle \left(\sum_{i,j} \lambda_i \mu_j \mathbf{z}_{ij} \mathbf{z}_{ij}^* \right) \mathbf{x}, \mathbf{x} \right\rangle = \sum_{i,j} \lambda_i \mu_j \langle \mathbf{z}_{ij} \mathbf{z}_{ij}^* \mathbf{x}, \mathbf{x} \rangle \geq 0,$$

every term being a product of nonnegative reals. So $\mathbf{A} \circ \mathbf{B} \succeq \mathbf{O}$.

- (b) Let $\varepsilon > 0$ be the smallest eigenvalue of $\mathbf{B}$, which is positive by Theorem 13.1.5 (b). The matrix $\mathbf{B} - \varepsilon \mathbf{I}$ is Hermitian with eigenvalues $\mu_j - \varepsilon \geq 0$, so $\mathbf{B} - \varepsilon \mathbf{I} \succeq \mathbf{O}$ by Theorem 13.1.4 (b). Since $\circ$ is bilinear and $\mathbf{A} \circ \mathbf{I} = \text{diag}(a_{11}, \dots, a_{nn})$,

$$\mathbf{A} \circ \mathbf{B} = \mathbf{A} \circ (\mathbf{B} - \varepsilon \mathbf{I}) + \varepsilon \text{diag}(a_{11}, \dots, a_{nn}).$$

The first term is positive semidefinite by (a). In the second, $a_{ii} = \langle \mathbf{A}\mathbf{e}_i, \mathbf{e}_i \rangle > 0$ because $\mathbf{A} \succ \mathbf{O}$ and $\mathbf{e}_i \neq \mathbf{0}$, so $\varepsilon \operatorname{diag}(a_{11}, \dots, a_{nn}) \succ \mathbf{O}$ by Example (Five Positive Matrices) (a). For $\mathbf{x} \neq \mathbf{0}$, the quadratic form of the sum is ≥ 0 plus > 0 , hence > 0 . So $\mathbf{A} \circ \mathbf{B} \succ \mathbf{O}$, and the theorem is proved. ■

The theorem is a genuine addition to the list of operations that preserve positivity. Sums and nonnegative multiples were easy (Exercise (C2: The cone of positive matrices)), congruence was easy (Proposition 13.1.2), and the ordinary product is not on the list at all, since $\mathbf{AB}$ need not even be Hermitian. The entrywise product is on the list, and nothing in the definition of positivity suggests it should be.

Example. $\mathbf{A} \ 3 \times 3$ Entrywise Product

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Check that $\mathbf{A} \succ \mathbf{O}$ and $\mathbf{B} \succ \mathbf{O}$, compute $\mathbf{A} \circ \mathbf{B}$, and verify directly that it is positive definite.

Solution. The leading principal minors of $\mathbf{A}$ are 2, $4 - 1 = 3$ and, expanding along the first row, $2(4 - 1) - 1(2 - 0) = 4$. All are positive, so $\mathbf{A} \succ \mathbf{O}$ by Sylvester's criterion (Theorem 13.1.5 (d)).

Those of $\mathbf{B}$ are 1, $2 - 1 = 1$, and $1(6 - 1) + 1(-3 + 1) + 1(1 - 2) = 5 - 2 - 1 = 2$. All positive, so $\mathbf{B} \succ \mathbf{O}$.

Entry by entry,

$$\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 6 \end{pmatrix},$$

whose leading principal minors are 2, $8 - 1 = 7$, and $2(24 - 1) + 1(-6 - 0) = 46 - 6 = 40$. All positive, so $\mathbf{A} \circ \mathbf{B} \succ \mathbf{O}$, as Theorem 13.8.2 (b) promised. Note that the eigenvalues of $\mathbf{A} \circ \mathbf{B}$ have nothing to do with those of $\mathbf{A}$ and $\mathbf{B}$, and that $\det(\mathbf{A} \circ \mathbf{B}) = 40$ is much larger than $\det \mathbf{A} \det \mathbf{B} = 4 \cdot 2 = 8$. The next theorem explains part of that: it forces $\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{A} \prod_i b_{ii} = 24$. Getting down to the product $\det \mathbf{A} \det \mathbf{B}$ needs Hadamard's inequality (Theorem 13.7.3) as well, which is Exercise (C2: The product of the determinants).

Quick check. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Compute $\mathbf{A} \circ \mathbf{B}$ and decide whether it is positive definite. Does this contradict Theorem 13.8.2 (b)?

Solution. $\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, with leading principal minors 2 and 3, so it is positive definite. There is no contradiction: Theorem 13.8.2 (b) is an implication, not an equivalence. Here $\mathbf{A}$ is only positive semidefinite ($\mathbf{A} = \mathbf{1}\mathbf{1}^\top$, so $\det \mathbf{A} = 0$), and the theorem's hypothesis fails — but the conclusion happens to hold anyway. Part (a) is what applies, and it correctly predicts $\mathbf{A} \circ \mathbf{B} \succeq \mathbf{O}$.

13.8.3 Oppenheim's inequality

Hadamard's inequality of Section 7 compared $\det \mathbf{A}$ with the product of the diagonal entries of $\mathbf{A}$. Oppenheim's inequality does the same for a Hadamard product, and it contains Hadamard's as the special case $\mathbf{B} = \mathbf{I}$.

Theorem 13.8.3. Oppenheim's Inequality

Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ with $\mathbf{A} \succeq \mathbf{O}$ and $\mathbf{B} \succeq \mathbf{O}$. Then

$$\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{A} \cdot b_{11}b_{22} \cdots b_{nn}.$$

Idea. Induct on n , taking both matrices positive definite first. Split off the $(1,1)$ entry of each: the Schur complements $\mathbf{A}/\alpha$ and $\mathbf{B}/\beta$ are positive definite of size $n-1$, and a short computation, whose only real step is Lemma 13.8.1 again, shows that the Schur complement of the corner in $\mathbf{A} \circ \mathbf{B}$ is

$$(\mathbf{A} \circ \mathbf{B})/(\alpha\beta) = (\mathbf{A}/\alpha) \circ \mathbf{B}' + \frac{1}{\alpha} (\mathbf{a}\mathbf{a}^*) \circ (\mathbf{B}/\beta),$$

a positive definite matrix plus a positive semidefinite one. So its determinant is at least that of the first term, by Lemma 13.7.1, and the inductive hypothesis handles that. The semidefinite case follows by replacing $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} + t\mathbf{I}$ and $\mathbf{B} + t\mathbf{I}$ and letting t shrink to 0.

Proof. Case 1: $\mathbf{A} \succ \mathbf{O}$ and $\mathbf{B} \succ \mathbf{O}$. Induct on n . For $n=1$ both sides are $a_{11}b_{11}$.

Let $n \geq 2$, and split both matrices with a $1 \times (n-1)$ partition:

$$\mathbf{A} = \begin{pmatrix} \alpha & \mathbf{a}^* \\ \mathbf{a} & \mathbf{A}' \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \beta & \mathbf{b}^* \\ \mathbf{b} & \mathbf{B}' \end{pmatrix},$$

with $\alpha = a_{11} > 0$, $\beta = b_{11} > 0$, and $\mathbf{a}, \mathbf{b} \in F^{n-1}$. The Schur complements are

$$\mathbf{A}/\alpha = \mathbf{A}' - \frac{1}{\alpha} \mathbf{a}\mathbf{a}^*, \quad \mathbf{B}/\beta = \mathbf{B}' - \frac{1}{\beta} \mathbf{b}\mathbf{b}^*,$$

and both are positive definite by Theorem 13.6.1 (b). Since conjugation acts entrywise, $\mathbf{a}^* \circ \mathbf{b}^* = (\mathbf{a} \circ \mathbf{b})^*$, so

$$\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} \alpha\beta & (\mathbf{a} \circ \mathbf{b})^* \\ \mathbf{a} \circ \mathbf{b} & \mathbf{A}' \circ \mathbf{B}' \end{pmatrix},$$

and $\alpha\beta > 0$, so this matrix too has a Schur complement at its corner:

$$(\mathbf{A} \circ \mathbf{B})/(\alpha\beta) = \mathbf{A}' \circ \mathbf{B}' - \frac{1}{\alpha\beta} (\mathbf{a} \circ \mathbf{b})(\mathbf{a} \circ \mathbf{b})^*.$$

By [Lemma 13.8.1](#) the subtracted term is $\frac{1}{\alpha\beta}(\mathbf{a}\mathbf{a}^*) \circ (\mathbf{b}\mathbf{b}^*)$. Substituting $\mathbf{A}' = \mathbf{A}/\alpha + \frac{1}{\alpha}\mathbf{a}\mathbf{a}^*$ and expanding by bilinearity,

$$\begin{aligned} \mathbf{A}' \circ \mathbf{B}' &= (\mathbf{A}/\alpha) \circ \mathbf{B}' + \frac{1}{\alpha} (\mathbf{a}\mathbf{a}^*) \circ \mathbf{B}' \\ &= (\mathbf{A}/\alpha) \circ \mathbf{B}' + \frac{1}{\alpha} (\mathbf{a}\mathbf{a}^*) \circ \left(\mathbf{B}/\beta + \frac{1}{\beta} \mathbf{b}\mathbf{b}^* \right), \end{aligned}$$

and the last term of this is exactly $\frac{1}{\alpha\beta}(\mathbf{a}\mathbf{a}^*) \circ (\mathbf{b}\mathbf{b}^*)$. It therefore cancels, leaving

$$(\mathbf{A} \circ \mathbf{B})/(\alpha\beta) = \underbrace{(\mathbf{A}/\alpha) \circ \mathbf{B}'}_{\succ \mathbf{O}} + \underbrace{\frac{1}{\alpha} (\mathbf{a}\mathbf{a}^*) \circ (\mathbf{B}/\beta)}_{\succeq \mathbf{O}}. \quad (*)$$

The first term is positive definite by [Theorem 13.8.2](#) (b), since $\mathbf{A}/\alpha \succ \mathbf{O}$ and $\mathbf{B}' \succ \mathbf{O}$ (the latter by [Theorem 13.6.1](#) (a) applied to $\mathbf{B}$). The second is positive semidefinite by [Theorem 13.8.2](#) (a), since $\mathbf{a}\mathbf{a}^* \succeq \mathbf{O}$ by Example (Five Positive Matrices) (b) and $\mathbf{B}/\beta \succ \mathbf{O}$, and $1/\alpha > 0$.

Now compute. By [Theorem 8.3.2](#) applied to $\mathbf{A} \circ \mathbf{B}$, then (*) with [Lemma 13.7.1](#), then the inductive hypothesis applied to the pair $\mathbf{A}/\alpha$ and $\mathbf{B}'$:

$$\begin{aligned} \det(\mathbf{A} \circ \mathbf{B}) &= \alpha\beta \cdot \det[(\mathbf{A} \circ \mathbf{B})/(\alpha\beta)] \\ &\geq \alpha\beta \cdot \det[(\mathbf{A}/\alpha) \circ \mathbf{B}'] \\ &\geq \alpha\beta \cdot \det(\mathbf{A}/\alpha) \cdot b_{22} \cdots b_{nn}. \end{aligned}$$

By [Theorem 8.3.2](#) again, now for $\mathbf{A}$, we have $\det(\mathbf{A}/\alpha) = \det \mathbf{A}/\alpha$. Substituting and canceling α ,

$$\det(\mathbf{A} \circ \mathbf{B}) \geq \beta \det \mathbf{A} \cdot b_{22} \cdots b_{nn} = \det \mathbf{A} \cdot b_{11} b_{22} \cdots b_{nn},$$

which completes the induction.

Case 2: $\mathbf{A} \succeq \mathbf{O}$ and $\mathbf{B} \succeq \mathbf{O}$. For real $t > 0$, the matrices $\mathbf{A} + t\mathbf{I}$ and $\mathbf{B} + t\mathbf{I}$ are positive definite, as in the proof of [Corollary 13.7.2](#). Case 1 gives

$$\det[(\mathbf{A} + t\mathbf{I}) \circ (\mathbf{B} + t\mathbf{I})] \geq \det(\mathbf{A} + t\mathbf{I}) \cdot \prod_{i=1}^n (b_{ii} + t).$$

Every entry of $(\mathbf{A} + t\mathbf{I}) \circ (\mathbf{B} + t\mathbf{I})$ is a polynomial in t , so both sides are polynomial, hence continuous, functions of t (Theorem 7.4.3). Letting t decrease to 0 preserves the inequality and gives the statement. This proves the theorem. ■

Remark. Taking $\mathbf{B} = \mathbf{I}$ recovers Hadamard's inequality. Indeed $\mathbf{A} \circ \mathbf{I} = \text{diag}(a_{11}, \dots, a_{nn})$ and $\prod_i b_{ii} = 1$, so the conclusion reads $a_{11} \cdots a_{nn} \geq \det \mathbf{A}$, which is Theorem 13.7.3 (a). Swapping the roles of $\mathbf{A}$ and $\mathbf{B}$ gives the companion bound $\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{B} \cdot \prod_i a_{ii}$, and combining either one with Hadamard's inequality gives $\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{A} \det \mathbf{B}$; that last consequence is [Exercise \(C2: The product of the determinants\)](#).

13.8.4 Where this gets used

The natural home of the Hadamard product is data. A covariance matrix is a Gram matrix (Theorem 11.2.5), hence positive semidefinite, and rescaling each variable to unit variance turns it into a **correlation matrix**: positive semidefinite with every diagonal entry equal to 1. The Schur product theorem says that the entrywise product of two correlation matrices is another one — the diagonal entries stay $1 \cdot 1 = 1$ — so one can damp the correlations of one model by those of another and still have a legitimate covariance structure. Nothing in the definition of “positive semidefinite” makes that obvious, and it is the reason the theorem is quoted far outside linear algebra.

Warning. **Entrywise products preserve positivity; entrywise inverses do not.** Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which is positive definite, with leading principal minors 2 and 3. Replacing each entry by its reciprocal gives $\begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$, whose determinant is $\frac{1}{4} - 1 = -\frac{3}{4} < 0$: not positive semidefinite. The entrywise reciprocal is not the Hadamard inverse of anything useful, and the matrix inverse $\mathbf{A}^{-1}$, which **is** positive definite (Exercise (B3: The inverse of a positive definite matrix)), has nothing to do with it.

13.8.5 Exercises

13.8.5.1 A. Check your understanding

Exercise. A1

- Define $\mathbf{A} \circ \mathbf{B}$, and say what shapes the two factors must have.
- What is $\mathbf{A} \circ \mathbf{I}$? What is $\mathbf{A} \circ \mathbf{J}$?
- State the Schur product theorem.
- True or false: if $\mathbf{A} \succ \mathbf{O}$, then the matrix with entries $1/a_{ij}$ (assuming no entry is 0) is positive definite. Justify your answer.
- What does Oppenheim's inequality say when $\mathbf{B} = \mathbf{I}$?

Solution.

- (a) For $\mathbf{A}, \mathbf{B} \in M_{m \times n}(F)$ of the **same** shape, $(\mathbf{A} \circ \mathbf{B})_{ij} = a_{ij}b_{ij}$, and $\mathbf{A} \circ \mathbf{B}$ has that same shape (Definition 13.8.1).
- (b) $\mathbf{A} \circ \mathbf{I} = \text{diag}(a_{11}, \dots, a_{nn})$, and $\mathbf{A} \circ \mathbf{J} = \mathbf{A}$, since $\mathbf{J}$ has every entry 1.
- (c) If $\mathbf{A} \succeq \mathbf{O}$ and $\mathbf{B} \succeq \mathbf{O}$, then $\mathbf{A} \circ \mathbf{B} \succeq \mathbf{O}$; if both are positive definite, so is $\mathbf{A} \circ \mathbf{B}$ (Theorem 13.8.2).
- (d) False. For $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \succ \mathbf{O}$, the entrywise reciprocal is $\begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$, with determinant $-3/4 < 0$.
- (e) By Theorem 13.8.3 it says $\det(\mathbf{A} \circ \mathbf{I}) \geq \det \mathbf{A} \cdot 1$, that is $a_{11} \cdots a_{nn} \geq \det \mathbf{A}$: Hadamard's inequality, Theorem 13.7.3 (a).

13.8.5.2 B. Practice

Exercise. B1: Compute and decide

For each pair below, compute $\mathbf{A} \circ \mathbf{B}$ and determine whether it is positive definite, positive semidefinite but not definite, or neither. Say in each case whether Theorem 13.8.2 applies.

- (a) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
- (b) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.
- (c) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$.

Solution.

- (a) $\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Both factors are $\mathbf{1}\mathbf{1}^\top \succeq \mathbf{O}$ (Example (Five Positive Matrices) (b)), so Theorem 13.8.2 (a) applies and predicts $\succeq \mathbf{O}$. The product has determinant 0, so it is positive semidefinite but not definite.
- (b) $\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, whose determinant is $1 - 4 = -3 < 0$: neither. The theorem does not apply, since $\mathbf{B}$ is not positive semidefinite (its determinant is -3). This is what the hypothesis is for.
- (c) $\mathbf{A} \succ \mathbf{O}$ (minors 2, 3) and $\mathbf{B} \succ \mathbf{O}$ (minors 3, 2), so Theorem 13.8.2 (b) applies. Indeed $\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix}$, with minors 6 and 11, is positive definite. Theorem 13.8.3 also checks out: $11 \geq \det \mathbf{A} \cdot b_{11}b_{22} = 3 \cdot 3 = 9$.

Exercise. B2: Entrywise squares

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

- (a) Show that $\mathbf{A} \succ \mathbf{O}$.
- (b) Compute $\mathbf{A} \circ \mathbf{A}$, the matrix of squared entries, and verify that it is positive definite.
- (c) Explain why (b) needed no computation.

Solution.

- (a) The leading principal minors are 1, $2 - 1 = 1$ and, expanding along the first row, $1(6 - 4) - 1(3 - 2) + 1(2 - 2) = 1$. All positive, so $\mathbf{A} \succ \mathbf{O}$ by Theorem 13.1.5 (d).
- (b) $\mathbf{A} \circ \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 9 \end{pmatrix}$, with leading principal minors 1, $4 - 1 = 3$, and $1(36 - 16) - 1(9 - 4) + 1(4 - 4) = 20 - 5 = 15$. All positive, so it is positive definite.
- (c) Theorem 13.8.2 (b) with $\mathbf{B} = \mathbf{A}$ gives $\mathbf{A} \circ \mathbf{A} \succ \mathbf{O}$ immediately.

13.8.5.3 C. Going deeper

Exercise. C1: Polynomials applied entrywise

Let $\mathbf{A} \in M_n(F)$ with $\mathbf{A} \succeq \mathbf{O}$.

- (a) Prove that the entrywise power $\mathbf{A}^{\circ m}$, with (i, j) entry a_{ij}^m , is positive semidefinite for every integer $m \geq 1$.
- (b) Let $p(x) = c_0 + c_1x + \cdots + c_dx^d$ have **nonnegative real** coefficients. Prove that the matrix $p^\circ(\mathbf{A})$ with (i, j) entry $p(a_{ij})$ is positive semidefinite.
- (c) Give an example of $\mathbf{A} \succeq \mathbf{O}$ and a polynomial p with a negative coefficient for which $p^\circ(\mathbf{A})$ is not positive semidefinite.

Solution.

- (a) Induct on m . For $m = 1$ the claim is the hypothesis. If $\mathbf{A}^{\circ m} \succeq \mathbf{O}$, then $\mathbf{A}^{\circ(m+1)} = \mathbf{A}^{\circ m} \circ \mathbf{A} \succeq \mathbf{O}$ by Theorem 13.8.2 (a).
- (b) By definition of the entries, $p^\circ(\mathbf{A}) = c_0\mathbf{J} + c_1\mathbf{A} + c_2\mathbf{A}^{\circ 2} + \cdots + c_d\mathbf{A}^{\circ d}$, since each side has (i, j) entry $\sum_k c_k a_{ij}^k$ (with $a_{ij}^0 = 1$, which is the (i, j) entry of $\mathbf{J}$). Now $\mathbf{J} = \mathbf{1}\mathbf{1}^\top \succeq \mathbf{O}$ by Example (Five Positive Matrices) (b), each $\mathbf{A}^{\circ k} \succeq \mathbf{O}$ by (a), and each $c_k \geq 0$. A nonnegative combination of positive semidefinite matrices is positive semidefinite (Exercise (C2: The cone of positive matrices) (a)).
- (c) Take $\mathbf{A} = \mathbf{J} \in M_2(\mathbb{R})$, which is positive semidefinite, and $p(x) = x - 2$. Then $p^\circ(\mathbf{A})$ has every entry -1 , that is $p^\circ(\mathbf{A}) = -\mathbf{J}$, and $\langle -\mathbf{J}\mathbf{e}_1, \mathbf{e}_1 \rangle = -1 < 0$. So the hypothesis on the coefficients cannot be dropped.

Exercise. C2: The product of the determinants

Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ with $\mathbf{A} \succeq \mathbf{O}$ and $\mathbf{B} \succeq \mathbf{O}$. Prove that

$$\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{A} \cdot \det \mathbf{B}.$$

Is the inequality ever an equality with both matrices positive definite?

Solution. By Theorem 13.8.3, $\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{A} \cdot \prod_i b_{ii}$. By Theorem 13.7.3 (a) applied to $\mathbf{B} \succeq \mathbf{O}$, $\prod_i b_{ii} \geq \det \mathbf{B}$. Both $\det \mathbf{A} \geq 0$ and $\prod_i b_{ii} \geq 0$ (the diagonal entries of a positive semidefinite matrix are ≥ 0 by Exercise (C1: Zero trace forces zero) (a)), so multiplying the

second inequality by $\det \mathbf{A} \geq 0$ and chaining,

$$\det(\mathbf{A} \circ \mathbf{B}) \geq \det \mathbf{A} \prod_i b_{ii} \geq \det \mathbf{A} \det \mathbf{B}.$$

Yes, equality can occur: take $\mathbf{A} = \mathbf{B} = \mathbf{I}_n$, so that $\mathbf{A} \circ \mathbf{B} = \mathbf{I}_n$ and all three quantities are 1. More generally, take $\mathbf{A} \succ \mathbf{O}$ diagonal and $\mathbf{B} = \mathbf{I}_n$.

Exercise. C3: The rank of an entrywise product

Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ with $\mathbf{A} \succeq \mathbf{O}$ and $\mathbf{B} \succeq \mathbf{O}$, of ranks r and s . Prove that

$$\text{rank}(\mathbf{A} \circ \mathbf{B}) \leq rs.$$

Hint: how many terms in the proof of Theorem 13.8.2 (a) are actually nonzero?

Solution. Write $\mathbf{A} = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^*$ and $\mathbf{B} = \sum_j \mu_j \mathbf{w}_j \mathbf{w}_j^*$ as in the proof of Theorem 13.8.2 (a). The number of nonzero eigenvalues of $\mathbf{A}$ is $\text{rank } \mathbf{A} = r$: diagonalizing, $\mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{U}^*$ with $\mathbf{U}$ invertible, so $\text{rank } \mathbf{A} = \text{rank } \mathbf{D}$ by Theorem 3.5.9, and $\text{rank } \mathbf{D}$ counts the nonzero λ_i . Likewise exactly s of the μ_j are nonzero. Dropping the vanishing terms,

$$\mathbf{A} \circ \mathbf{B} = \sum_{\lambda_i \neq 0} \sum_{\mu_j \neq 0} \lambda_i \mu_j \mathbf{z}_{ij} \mathbf{z}_{ij}^*, \quad \mathbf{z}_{ij} = \mathbf{u}_i \circ \mathbf{w}_j,$$

a sum of rs matrices, each of rank at most 1, since every column of $\mathbf{z}_{ij} \mathbf{z}_{ij}^*$ lies in $\text{span}(\mathbf{z}_{ij})$. The rank of a sum of two matrices is at most the sum of their ranks (Exercise (C1: Rank of a sum)); applying that repeatedly to the rs summands gives $\text{rank}(\mathbf{A} \circ \mathbf{B}) \leq rs$.