

11.10 Orthogonal Polynomials

Section 2 ran Gram–Schmidt on $(1, x, x^2)$ with $\langle p, q \rangle = \int_{-1}^1 pq$, and stopped at degree 2 only because the space did (Example (Gram–Schmidt on $1, x, x^2$)). Nothing in the process cared. Feed it $(1, x, x^2, x^3, \dots)$ and it returns an infinite orthogonal sequence, one polynomial of each degree. Three things make such sequences worth a section. Each member is computable from the two before it. Each member of degree k has k distinct real roots inside the interval. And those roots are the best possible points at which to sample a function to estimate its integral. The first fact rests on one small identity; the other two rest on the first.

Throughout this section the field is $\mathbb{R}$.

11.10.1 Two ways to make an inner product on polynomials

Both constructions we need appeared in Section 1.

From a weight. Fix an interval with endpoints $-\infty \leq a < b \leq \infty$ and a **weight** w : a function continuous and **strictly positive** on the open interval (a, b) , with $\int_a^b |t|^k w(t) dt$ finite for every $k \in \mathbb{N}$. Define

$$\langle p, q \rangle = \int_a^b p(t) q(t) w(t) dt.$$

Finiteness of the moments makes this a real number for all $p, q \in \mathbb{R}[x]$, and bilinearity and symmetry come from linearity of the integral. For positivity, let $p \neq 0$. Then $p^2 w$ is continuous and ≥ 0 on (a, b) , and > 0 off the finitely many roots of p (Corollary 6.1.4), so $\langle p, p \rangle > 0$. That last step is the one calculus fact this section leans on: a continuous nonnegative function on an interval whose integral is 0 vanishes identically. As in Definition 11.1.1 it follows from the local fact that a continuous function nonzero at a point stays bounded away from 0 on a small interval around it, which then contributes a positive amount to the integral.

From nodes. Fix distinct $c_0 < \dots < c_n$ in $\mathbb{R}$ and put $\langle p, q \rangle = \sum_{i=0}^n p(c_i) q(c_i)$. Section 1 checked that this is an inner product on $\mathbb{R}[x]_{\leq n}$, positivity coming from the root bound. It is not one on $\mathbb{R}[x]$: the polynomial $(x - c_0) \cdots (x - c_n)$ is nonzero and pairs with itself to give 0. So in the node case everything below stops at degree n .

The two constructions share one property, and it is the engine of the section.

Proposition 11.10.1. Multiplication by x is symmetric

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}[x]$ coming from a weight, or the node inner product on $\mathbb{R}[x]_{\leq n}$. Then

$$\langle xp, q \rangle = \langle p, xq \rangle$$

for all polynomials p, q in the space (in the node case, for all p, q of degree at most $n - 1$, so that xp and xq still lie in it).

Proof. In the weight case both sides equal $\int_a^b t p(t) q(t) w(t) dt$, and in the node case both sides equal $\sum_{i=0}^n c_i p(c_i) q(c_i)$. ■

The proof is one line, but the statement is not a triviality: multiplication by x moves from one slot to the other for free. That is exactly the identity $\langle Tu, v \rangle = \langle u, Tv \rangle$ defining a **self-adjoint** operator (Definition 11.6.2), and it is the reason the recurrence below has three terms instead of $k + 2$.

Warning. **Multiplication by x is not an operator on $\mathbb{R}[x]_{\leq n}$.** It raises degree, so it maps that space out of itself; on $\mathbb{R}[x]$ it is an operator, but there the space is infinite-dimensional, and Definition 11.6.2 was stated for a finite-dimensional one. “Self-adjoint” is here a name for [Proposition 11.10.1](#), not a literal application of the definition. It is the identity, not the name, that the proofs use.

11.10.2 Orthogonal polynomial sequences

Applied to the independent list $(1, x, x^2, \dots)$, Gram–Schmidt never stalls. What comes out is one polynomial of each degree, pairwise orthogonal, and that combination deserves a name.

An orthogonal polynomial sequence is one polynomial of every degree, no two of them pairing to anything but zero.

Definition 11.10.1. Orthogonal Polynomial Sequence

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}[x]$, or on $\mathbb{R}[x]_{\leq n}$. An **orthogonal polynomial sequence** for it is a sequence $p_0, p_1, p_2, \dots$ of polynomials (a finite list $p_0, \dots, p_n$ in the second case) such that

(OP1) $\deg p_k = k$ for **every** index k ;

(OP2) $\langle p_j, p_k \rangle = 0$ whenever $j \neq k$.

The sequence is **monic** if every p_k has leading coefficient 1.

In words. (OP1) says one polynomial of each degree: no gaps, no repeats, no accidental drop in degree. (OP2) says the members are pairwise orthogonal. Together they say more than either alone, because by (OP1) the list $(p_0, \dots, p_{k-1})$ is a basis of $\mathbb{R}[x]_{\leq k-1}$, so (OP2) upgrades to

$$\langle p_k, q \rangle = 0 \quad \text{for every } q \in \mathbb{R}[x]_{\leq k-1},$$

and that is the form in which the condition always gets used.

Two things need checking: that such a sequence exists, and that fixing the scale pins it down.

Proposition 11.10.2. Existence and uniqueness of the monic sequence

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}[x]$. For each $k \in \mathbb{N}$ there is **exactly one** monic p_k of degree k with $\langle p_k, q \rangle = 0$ for every $q \in \mathbb{R}[x]_{\leq k-1}$. These p_k form the unique monic orthogonal polynomial sequence, and $(p_0, \dots, p_k)$ is an orthogonal basis of $\mathbb{R}[x]_{\leq k}$ for every k . The same holds on $\mathbb{R}[x]_{\leq n}$ for $k \leq n$.

Idea. Existence is Gram–Schmidt, once we know the k -th output really has degree k ; it does, because it lies in $\mathbb{R}[x]_{\leq k}$ but not in the span of its predecessors, which is $\mathbb{R}[x]_{\leq k-1}$. Uniqueness is the standard move: two candidates differ by a polynomial of lower degree that

is orthogonal to everything of lower degree, hence to itself.

Proof. Existence. The list $(1, x, \dots, x^k)$ is independent, so Theorem 11.2.2 applies and returns an orthonormal list $(e_0, \dots, e_k)$ with $\text{span}(e_0, \dots, e_j) = \text{span}(1, x, \dots, x^j) = \mathbb{R}[x]_{\leq j}$ for every $j \leq k$. Fix j . Then $e_j \in \mathbb{R}[x]_{\leq j}$, and $e_j \notin \mathbb{R}[x]_{\leq j-1} = \text{span}(e_0, \dots, e_{j-1})$, since an orthonormal list is independent (Theorem 11.1.4). Hence $\deg e_j = j$ exactly. Let $a_j \neq 0$ be its leading coefficient and put $p_j = e_j/a_j$, monic of degree j . For $q \in \mathbb{R}[x]_{\leq j-1} = \text{span}(e_0, \dots, e_{j-1})$ we get $\langle p_j, q \rangle = 0$ by orthogonality of the list.

Uniqueness. Suppose p and p' are both monic of degree k and orthogonal to every element of $\mathbb{R}[x]_{\leq k-1}$. Their leading terms cancel, so $p - p' \in \mathbb{R}[x]_{\leq k-1}$. Therefore $\langle p - p', p - p' \rangle = \langle p, p - p' \rangle - \langle p', p - p' \rangle = 0$, and positive definiteness gives $p = p'$.

Finally, $(p_0, \dots, p_k)$ is an orthogonal list of nonzero vectors, hence independent (Theorem 11.1.4), with $k + 1 = \dim \mathbb{R}[x]_{\leq k}$ members, hence a basis (Theorem 1.6.4). ■

Examples. For $\langle p, q \rangle = \int_{-1}^1 pq$ the first three monic members are $1, x, x^2 - \frac{1}{3}$ (Example (Gram–Schmidt on $1, x, x^2$)); these are the monic **Legendre polynomials**. For the node inner product with nodes $-1, 0, 1$ on $\mathbb{R}[x]_{\leq 2}$ we have $\langle 1, 1 \rangle = 3$, $\langle x, 1 \rangle = 0$, $\langle x, x \rangle = 2$, $\langle x^2, 1 \rangle = 2$, $\langle x^2, x \rangle = 0$, so $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - \frac{2}{3}$ — and there the sequence ends, since $\mathbb{R}[x]_{\leq 2}$ holds nothing of degree 3. The degenerate member is always $p_0 = 1$, the only monic polynomial of degree 0, with an empty condition on it.

A nonexample by minimal change. Replace the Legendre $p_2 = x^2 - \frac{1}{3}$ by $\tilde{p}_2 = x^2 + x - \frac{1}{3}$. Still monic of degree 2, so (OP1) survives; but $\langle \tilde{p}_2, x \rangle = \langle x, x \rangle = \frac{2}{3} \neq 0$, so (OP2) fails. Adding a multiple of a lower member is exactly the freedom orthogonality removes.

Why monic? Rescaling each p_k by a nonzero λ_k gives another orthogonal polynomial sequence, so a normalization has to be chosen, and traditions differ: the classical Legendre polynomials P_k are fixed by $P_k(1) = 1$, and Gram–Schmidt as run in Section 2 fixes $\|p_k\| = 1$. Monic is the convenient choice here, because it is the one for which the recurrence below needs no extra factor.

Warning. **The sequence belongs to the inner product, not to the polynomials.** “The orthogonal polynomials” is meaningless without a weight and an interval. Exercise B2 of Section 2 (Exercise (B2: Gram–Schmidt on polynomials)) ran the same process with $\int_0^1$ instead of $\int_{-1}^1$ and got $1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}$: same degrees, different polynomials.

Quick check. For $\langle p, q \rangle = \int_{-1}^1 pq$, how many polynomials of degree 2 with $\|p\| = 1$ are orthogonal to $\mathbb{R}[x]_{\leq 1}$? Write them down.

Solution. Exactly two. If p has degree 2 and $p \perp \mathbb{R}[x]_{\leq 1}$, let $\lambda \neq 0$ be its leading coefficient; then p/λ is monic of degree 2 and still orthogonal to $\mathbb{R}[x]_{\leq 1}$, so $p/\lambda = x^2 - \frac{1}{3}$ by [Proposition 11.10.2](#). Thus $p = \lambda(x^2 - \frac{1}{3})$, and $\|p\| = 1$ leaves $\lambda = \pm 1/\|x^2 - \frac{1}{3}\|$. Since Example

(Gram–Schmidt on $1, x, x^2$) computed $\|x^2 - \frac{1}{3}\|^2 = \frac{8}{45}$, the two answers are $\pm \frac{\sqrt{10}}{4}(3x^2 - 1)$. Fixing the scale by the norm leaves a sign; fixing it by “monic” leaves nothing.

11.10.3 The three-term recurrence

Computing p_n by Gram–Schmidt costs one inner product for every earlier member, so reaching degree n costs about $n^2/2$ integrals. Almost all of them return 0. The next theorem says which ones do not, and reduces the cost to two integrals per step.

Theorem 11.10.3. Three-Term Recurrence

Let (p_k) be the monic orthogonal polynomial sequence of [Proposition 11.10.2](#), for an inner product satisfying [Proposition 11.10.1](#). For a weight on an interval this covers every k ; for the node inner product on n nodes, where [Proposition 11.10.1](#) holds only up to degree $n-1$, read the statement for $k+1 \leq n$. Put $p_{-1} = 0$ and, for $k \geq 0$,

$$a_k = \frac{\langle xp_k, p_k \rangle}{\langle p_k, p_k \rangle}, \quad b_k = \frac{\langle p_k, p_k \rangle}{\langle p_{k-1}, p_{k-1} \rangle} \quad (k \geq 1).$$

Then $p_0 = 1$, $p_1 = x - a_0$, and

$$p_{k+1} = (x - a_k)p_k - b_k p_{k-1} \quad \text{for every } k \geq 1.$$

Moreover $b_k > 0$ for every $k \geq 1$.

Idea. The polynomial xp_k is monic of degree $k+1$, so it is p_{k+1} plus something of degree at most k , and that something expands in the orthogonal basis $(p_0, \dots, p_k)$ with coefficients $\langle xp_k, p_j \rangle / \langle p_j, p_j \rangle$ ([Corollary 11.3.4](#)). The point is that almost every coefficient is 0: move the x across with [Proposition 11.10.1](#), and the coefficient becomes $\langle p_k, xp_j \rangle$, which vanishes as soon as $\deg(xp_j) = j+1$ is still below k . Only $j = k$ and $j = k-1$ can survive.

Proof. $p_0 = 1$ is the unique monic polynomial of degree 0, and $p_1 = x - a_0$ is the unique monic polynomial of degree 1 with $\langle p_1, p_0 \rangle = 0$, since $\langle x - a_0, 1 \rangle = \langle x, 1 \rangle - a_0 \langle 1, 1 \rangle = 0$ exactly for $a_0 = \langle x \cdot 1, 1 \rangle / \langle 1, 1 \rangle$.

Now fix $k \geq 1$. By [Proposition 11.10.2](#), $(p_0, \dots, p_{k+1})$ is an orthogonal basis of $\mathbb{R}[x]_{\leq k+1}$, and xp_k lies in that space. By [Corollary 11.3.4](#) applied inside $\mathbb{R}[x]_{\leq k+1}$,

$$xp_k = \sum_{j=0}^{k+1} c_j p_j, \quad c_j = \frac{\langle xp_k, p_j \rangle}{\langle p_j, p_j \rangle}.$$

Both xp_k and p_{k+1} are monic of degree $k+1$, and every other term of the sum has degree at most k ; comparing coefficients of x^{k+1} gives $c_{k+1} = 1$.

Let $j \leq k-2$. By [Proposition 11.10.1](#), $\langle xp_k, p_j \rangle = \langle p_k, xp_j \rangle$, and $\deg(xp_j) = j+1 \leq k-1$, so $xp_j \in \mathbb{R}[x]_{\leq k-1}$ and the pairing is 0 by [Proposition 11.10.2](#). Hence $c_j = 0$ for every $j \leq k-2$, and

$$xp_k = p_{k+1} + c_k p_k + c_{k-1} p_{k-1},$$

which rearranges to $p_{k+1} = (x - c_k)p_k - c_{k-1}p_{k-1}$.

It remains to identify the two coefficients. By definition $c_k = \langle xp_k, p_k \rangle / \langle p_k, p_k \rangle = a_k$. For the other, xp_{k-1} is monic of degree k , so $xp_{k-1} - p_k$ has degree at most $k-1$ and therefore pairs to 0 with p_k . Hence, using [Proposition 11.10.1](#) again,

$$\langle xp_k, p_{k-1} \rangle = \langle p_k, xp_{k-1} \rangle = \langle p_k, p_k \rangle,$$

so $c_{k-1} = \langle p_k, p_k \rangle / \langle p_{k-1}, p_{k-1} \rangle = b_k$. Finally b_k is a quotient of $\|p_k\|^2$ by $\|p_{k-1}\|^2$, both strictly positive because p_k and p_{k-1} are nonzero, so $b_k > 0$. ■

Two inner products per step generate the whole sequence. The positivity of b_k is not decoration either: it is the sign that drives the interlacing argument of [Exercise \(C1: Interlacing roots\)](#).

Remark. If the interval is symmetric, $(a, b) = (-c, c)$, and the weight is even, $w(-t) = w(t)$, then $a_k = 0$ for every k and $p_k(-x) = (-1)^k p_k(x)$. Induct on k : $p_0 = 1$ is even, and $a_0 = \langle x, 1 \rangle / \langle 1, 1 \rangle = 0$ because $tw(t)$ is odd, so $p_1 = x$ is odd. If p_{k-1} and p_k have parities $(-1)^{k-1}$ and $(-1)^k$, then $tp_k(t)^2w(t)$ is odd, so $a_k = 0$, and $p_{k+1} = xp_k - b_k p_{k-1}$ is a combination of two polynomials of parity $(-1)^{k+1}$.

Example. The monic Legendre polynomials, continued

For $\langle p, q \rangle = \int_{-1}^1 pq$, Example (Gram–Schmidt on $1, x, x^2$) gave $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - \frac{1}{3}$, with $\langle p_0, p_0 \rangle = 2$, $\langle p_1, p_1 \rangle = \frac{2}{3}$, and $\langle p_2, p_2 \rangle = \frac{8}{45}$. Use [Theorem 11.10.3](#) to obtain p_3 , and check the recurrence reproduces p_2 .

Solution. The weight $w = 1$ is even on $(-1, 1)$, so every $a_k = 0$ by the remark above. Then

$$b_1 = \frac{\langle p_1, p_1 \rangle}{\langle p_0, p_0 \rangle} = \frac{2/3}{2} = \frac{1}{3}, \quad b_2 = \frac{\langle p_2, p_2 \rangle}{\langle p_1, p_1 \rangle} = \frac{8/45}{2/3} = \frac{4}{15}.$$

The recurrence at $k = 1$ gives $p_2 = x \cdot x - \frac{1}{3} \cdot 1 = x^2 - \frac{1}{3}$, which is what Gram–Schmidt produced. At $k = 2$,

$$p_3 = x \left(x^2 - \frac{1}{3} \right) - \frac{4}{15} x = x^3 - \frac{3}{5} x.$$

Check. $\langle p_3, 1 \rangle = \langle p_3, x^2 \rangle = 0$ because the integrands are odd, and

$$\langle p_3, x \rangle = \int_{-1}^1 \left(t^4 - \frac{3}{5} t^2 \right) dt = \frac{2}{5} - \frac{3}{5} \cdot \frac{2}{3} = 0.$$

So $p_3 \perp \mathbb{R}[x]_{\leq 2}$, as it should be. Two integrals produced a polynomial that Gram–Schmidt would have charged three for, and the saving grows with the degree.

11.10.4 Where the roots are

The roots of $p_2 = x^2 - \frac{1}{3}$ are $\pm 1/\sqrt{3} \approx \pm 0.577$, and those of $p_3 = x^3 - \frac{3}{5}x$ are 0 and $\pm\sqrt{3/5} \approx \pm 0.775$. All five lie inside $(-1, 1)$, where the weight lives. No coincidence. The argument first needs one fact about real polynomials, in which $\text{mult}_r(f)$ is the multiplicity of r as a root of f

(Definition 6.4.1).

Lemma 11.10.4. Even multiplicities mean no sign change

Let $I \subseteq \mathbb{R}$ be an open interval and let $f \in \mathbb{R}[x]$ be nonzero. If $\text{mult}_r(f)$ is even for every $r \in I$, then $f \geq 0$ throughout I , or $f \leq 0$ throughout I .

Proof. By Theorem 6.4.3 over $\mathbb{R}$, write $f = (x - d_1)^{m_1} \cdots (x - d_l)^{m_l} g$, where $d_1, \dots, d_l$ are the distinct real roots of f , $m_i = \text{mult}_{d_i}(f)$, and g has no real root. Separate the factors according to whether $d_i \in I$:

$$f = h \cdot \left(\prod_{d_i \in I} (x - d_i)^{m_i} \right), \quad h = g \cdot \prod_{d_i \notin I} (x - d_i)^{m_i}.$$

Each m_i with $d_i \in I$ is even by hypothesis, so the second factor is the square of a real polynomial and is ≥ 0 everywhere. The polynomial h has no root in I : its roots are the d_i outside I , and g has none. Being continuous and nowhere zero on the interval I , h has constant sign there by the intermediate value theorem. Hence f has that same sign throughout I , weakly, with equality only at the d_i inside I . ■

Theorem 11.10.5. Roots of an Orthogonal Polynomial

Let w be a weight on (a, b) , let $\langle p, q \rangle = \int_a^b p q w$, and let (p_k) be the monic orthogonal polynomial sequence. For every $k \geq 1$, the polynomial p_k has k **distinct real** roots, and every one of them lies in the **open** interval (a, b) .

Idea. Suppose p_k changed sign at only a few points $c_1, \dots, c_m$ of (a, b) , with $m < k$. Build the polynomial $q = (x - c_1) \cdots (x - c_m)$. It is cheap — degree at most $k - 1$ — so p_k is orthogonal to it. But q was designed to change sign at exactly the points where p_k does, so the product $p_k q$ changes sign nowhere, and a function of one sign that is not identically zero cannot integrate to zero against a positive weight. The contradiction forces $m \geq k$, and k distinct roots is all a polynomial of degree k can hold.

Proof. Fix $k \geq 1$. Let $c_1 < \cdots < c_m$ be the roots of p_k in (a, b) whose multiplicity is **odd** (possibly $m = 0$), and set $q = (x - c_1) \cdots (x - c_m)$, with $q = 1$ when $m = 0$.

Suppose, for contradiction, that $m \leq k - 1$. Then $q \in \mathbb{R}[x]_{\leq k-1}$, so $\langle p_k, q \rangle = 0$ by [Proposition 11.10.2](#).

Put $f = p_k q$, a nonzero polynomial (Theorem 0.8.3). Let $r \in (a, b)$ be a root of f . By Theorem 6.4.2, $\text{mult}_r(f) = \text{mult}_r(p_k) + \text{mult}_r(q)$. If $r = c_i$ for some i , then $\text{mult}_r(q) = 1$ and $\text{mult}_r(p_k)$ is odd by the choice of the c_i , so the sum is even. If r is none of the c_i , then $q(r) \neq 0$, so $\text{mult}_r(q) = 0$ and $\text{mult}_r(p_k)$ is even, again by the choice of the c_i . Either way $\text{mult}_r(f)$ is even.

By [Lemma 11.10.4](#), $f \geq 0$ throughout (a, b) or $f \leq 0$ throughout (a, b) ; replacing q by $-q$ if necessary, which changes neither $\langle p_k, q \rangle = 0$ nor $\deg q$, we may assume $f \geq 0$ there. Then fw

is continuous and ≥ 0 on (a, b) , and it is > 0 off the finitely many roots of f , because $w > 0$. By the calculus fact recalled when the inner product was set up — a continuous nonnegative function with integral 0 vanishes identically — an integrand of that shape cannot integrate to 0. Therefore

$$\langle p_k, q \rangle = \int_a^b f(t) w(t) dt > 0,$$

contradicting $\langle p_k, q \rangle = 0$.

Hence $m \geq k$. Each c_i is a root of p_k , and p_k is nonzero of degree k , so $m \leq k$ by Corollary 6.1.4; thus $m = k$ and p_k has k distinct roots $c_1, \dots, c_k$, all in (a, b) . By Corollary 6.1.4 again, $(x - c_1) \cdots (x - c_k)$ divides p_k , and both are monic of degree k , so $p_k = (x - c_1) \cdots (x - c_k)$. In particular each root is simple and p_k has no roots outside (a, b) , real or complex. This proves the theorem. ■

So the weight hands us k points inside its interval, chosen by the inner product rather than by us. The next subsection says what they are good for.

Quick check. The monic Laguerre polynomial of degree 2 for the weight $w(t) = e^{-t}$ on $(0, \infty)$ is $p_2 = x^2 - 4x + 2$. Check that [Theorem 11.10.5](#) is not lying.

Solution. The roots are $2 \pm \sqrt{2}$, that is ≈ 0.586 and ≈ 3.414 : real, distinct, and both in $(0, \infty)$. An unbounded interval is no obstacle, since the theorem asks only that the weight be positive with finite moments, and $\int_0^\infty t^k e^{-t} dt = k!$.

11.10.5 Gaussian quadrature

Chapter 6 wrote $\int_0^1 p$ as a fixed combination of $p(0)$, $p(\frac{1}{2})$ and $p(1)$, valid for every $p \in \mathbb{R}[x]_{\leq 2}$ (Example (Integrals and Slopes from Values)). The construction works for any prescribed nodes: integrate the Lagrange interpolation formula (Theorem 6.6.1) and k distinct nodes give a rule exact on $\mathbb{R}[x]_{\leq k-1}$.

But a k -point rule has $2k$ numbers at its disposal, k nodes and k coefficients, and only k of them were spent. Spending the nodes too should buy degree $2k - 1$. Where to put them is the question, and the answer is the one place we have not chosen: the roots of p_k .

Theorem 11.10.6. Gaussian Quadrature

Let w be a weight on (a, b) with $\langle p, q \rangle = \int_a^b pqw$, let $k \geq 1$, and let $t_1 < \cdots < t_k$ be the roots of the monic orthogonal polynomial p_k , which lie in (a, b) by [Theorem 11.10.5](#). Let $\ell_1, \dots, \ell_k \in \mathbb{R}[x]_{\leq k-1}$ be the Lagrange polynomials for these nodes (Definition 6.6.1) and set

$$\alpha_i = \int_a^b \ell_i(t) w(t) dt \quad (i = 1, \dots, k).$$

Then every α_i is **strictly positive**, and

$$\int_a^b f(t) w(t) dt = \sum_{i=1}^k \alpha_i f(t_i)$$

for **every** $f \in \mathbb{R}[x]_{\leq 2k-1}$.

Idea. Two moves, one for each side of the identity. On the left, divide f by p_k : $f = qp_k + r$. The degree bookkeeping — $\deg f \leq 2k-1$ against $\deg p_k = k$ — forces $\deg q \leq k-1$, which is exactly the condition under which $\int qp_k w = \langle p_k, q \rangle$ vanishes. So the integral of f is the integral of its remainder. On the right, p_k vanishes at every node, so the sum also only sees the remainder. And the remainder has degree at most $k-1$, which is precisely the range on which k nodes reproduce a polynomial exactly. The two sides meet there.

Proof. Let $f \in \mathbb{R}[x]_{\leq 2k-1}$. Since $p_k \neq 0$, Theorem 6.1.1 gives $q, r \in \mathbb{R}[x]$ with

$$f = qp_k + r, \quad \deg r < k.$$

If $q = 0$ then $q \in \mathbb{R}[x]_{\leq k-1}$ trivially. Otherwise $qp_k = f - r$ has degree at most $2k-1$, while $\deg(qp_k) = \deg q + k$ by Theorem 0.8.3; hence $\deg q \leq k-1$. Either way $q \in \mathbb{R}[x]_{\leq k-1}$, so Proposition 11.10.2 gives $\langle p_k, q \rangle = 0$, that is, $\int_a^b qp_k w = 0$. Therefore

$$\int_a^b fw = \int_a^b qp_k w + \int_a^b rw = \int_a^b rw. \quad (*)$$

At each node, $p_k(t_i) = 0$, so

$$f(t_i) = q(t_i)p_k(t_i) + r(t_i) = r(t_i) \quad (i = 1, \dots, k). \quad (**)$$

Since $\deg r \leq k-1$ and $t_1, \dots, t_k$ are k distinct points, Theorem 6.6.1 (b) gives $r = \sum_i r(t_i)\ell_i$. Integrating against w and using the definition of α_i ,

$$\int_a^b rw = \sum_{i=1}^k r(t_i) \alpha_i = \sum_{i=1}^k \alpha_i f(t_i),$$

the last step by (**). Combining with (*) proves the quadrature identity.

For positivity, fix j and apply the identity to $f = \ell_j^2$, which is legitimate because $\deg \ell_j^2 = 2k-2 \leq 2k-1$. By Theorem 6.6.1 (a), $\ell_j(t_i) = \delta_{ij}$, so the right-hand side collapses to α_j , while the left-hand side is $\langle \ell_j, \ell_j \rangle = \|\ell_j\|^2$. Since $\ell_j \neq 0$, positive definiteness gives $\alpha_j = \|\ell_j\|^2 > 0$. ■

Each hypothesis did one job. Orthogonality to lower degrees killed the quotient on the left; the nodes being roots of p_k killed it again on the right. That is also why the nodes cannot be moved: shift one and the second cancellation fails. Exercise (C2: The ceiling is $2k-1$) shows $2k-1$ is a ceiling, not just what this proof happens to reach.

Example. Two-point Gauss–Legendre

Take $w = 1$ on $(-1, 1)$ and $k = 2$. Find the nodes and weights, and test the resulting rule on $f = 1 + 2x + 3x^2 + 4x^3$ and on $f = x^4$.

Solution. The monic p_2 is $x^2 - \frac{1}{3}$ (Example (Gram–Schmidt on $1, x, x^2$)), so the nodes are $t_1 = -1/\sqrt{3}$ and $t_2 = 1/\sqrt{3}$. The Lagrange polynomials are $\ell_1 = (x - t_2)/(t_1 - t_2)$ and $\ell_2 = (x - t_1)/(t_2 - t_1)$, with $t_1 - t_2 = -2/\sqrt{3}$. Since $\int_{-1}^1 t \, dt = 0$ and $\int_{-1}^1 1 \, dt = 2$,

$$\alpha_1 = \frac{0 - 2t_2}{t_1 - t_2} = \frac{-2/\sqrt{3}}{-2/\sqrt{3}} = 1,$$

and $\alpha_2 = 1$ by the same computation with the roles swapped. The rule is

$$\int_{-1}^1 f(t) \, dt = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \text{for } f \in \mathbb{R}[x]_{\leq 3}.$$

Two evaluations, no coefficients to remember, every cubic exactly.

The cubic test. Exactly, $\int_{-1}^1 (1 + 2t + 3t^2 + 4t^3) \, dt = 2 + 0 + 3 \cdot \frac{2}{3} + 0 = 4$. By the rule, $f(-1/\sqrt{3}) = 2 - \frac{10\sqrt{3}}{9}$ and $f(1/\sqrt{3}) = 2 + \frac{10\sqrt{3}}{9}$, whose sum is 4. The odd parts cancel and the constant and quadratic parts contribute $1 + 1$ and $1 + 1$.

The quartic test. Exactly, $\int_{-1}^1 t^4 \, dt = \frac{2}{5}$. By the rule, $(1/\sqrt{3})^4 + (-1/\sqrt{3})^4 = \frac{2}{9}$. These differ, so degree $4 = 2k$ is already out of reach. Compare Example (Integrals and Slopes from Values), where the nodes were fixed in advance and **three** of them guaranteed exactness only up to degree 2 — guaranteed by the construction, that is; that particular symmetric rule happens to integrate cubics exactly as well, which is luck of the symmetry and not something the construction promises.

Warning. **“Exact” means exact on polynomials.** Theorem 11.10.6 says nothing about $\int_a^b f w$ for a general f , and nothing about what happens as k grows; on a nonpolynomial the rule returns an approximation whose error is a question for analysis. What linear algebra delivers is exactness on a subspace, and the reason that subspace is the largest available.

11.10.6 The classical families

Four weights account for most of the orthogonal polynomials that occur in practice. The table records them; the entries are statements, and Exercise (B2: Two rows of the table) asks you to verify two of the rows by integration.

Family	Interval	Weight $w(t)$	p_1	p_2
Legendre	$(-1, 1)$	1	x	$x^2 - \frac{1}{3}$
Chebyshev	$(-1, 1)$	$(1 - t^2)^{-1/2}$	x	$x^2 - \frac{1}{2}$
Hermite	$(-\infty, \infty)$	e^{-t^2}	x	$x^2 - \frac{1}{2}$
Laguerre	$(0, \infty)$	e^{-t}	$x - 1$	$x^2 - 4x + 2$

The last two columns give the monic members of degrees 1 and 2; $p_0 = 1$ in every row. The Chebyshev weight is unbounded at ± 1 and the Hermite and Laguerre intervals are unbounded, so in those three rows the integrals defining the inner product are improper; they converge, which is the finiteness condition our definition of a weight demanded. The first three weights are even on symmetric intervals, so their a_k all vanish and their members alternate in parity; the Laguerre weight is not, and $a_0 = 1$ there. Section 9 of this chapter meets the Chebyshev family again from the Fourier side, where the substitution $t = \cos \theta$ turns its awkward weight into $d\theta$.

11.10.7 Exercises

11.10.7.1 A. Check your understanding

Exercise. A1

- Define what it means for $p_0, p_1, p_2, \dots$ to be a **monic orthogonal polynomial sequence** for an inner product on $\mathbb{R}[x]$.
- State the three-term recurrence, including the formulas for a_k and b_k and the starting values.
- True or false: $b_k > 0$ for every $k \geq 1$. Justify your answer.
- For $\langle p, q \rangle = \int_{-1}^1 pq$, how many roots does p_7 have, of what kind, and where?
- True or false: for **every** inner product on $\mathbb{R}[x]$ one has $\langle xp, q \rangle = \langle p, xq \rangle$. Justify your answer.
- How many nodes does a Gaussian rule need in order to integrate every polynomial of degree at most 11 exactly?

Solution.

- A sequence with $\deg p_k = k$ and leading coefficient 1 for every k , and $\langle p_j, p_k \rangle = 0$ whenever $j \neq k$ ([Definition 11.10.1](#)).
- With $p_{-1} = 0$, $p_0 = 1$, and $p_1 = x - a_0$, one has $p_{k+1} = (x - a_k)p_k - b_k p_{k-1}$ for $k \geq 1$, where $a_k = \langle xp_k, p_k \rangle / \langle p_k, p_k \rangle$ and $b_k = \langle p_k, p_k \rangle / \langle p_{k-1}, p_{k-1} \rangle$ ([Theorem 11.10.3](#)).
- True. Both $\langle p_k, p_k \rangle = \|p_k\|^2$ and $\langle p_{k-1}, p_{k-1} \rangle = \|p_{k-1}\|^2$ are strictly positive, since p_k and p_{k-1} are nonzero and the inner product is positive definite.
- Seven roots, all real, all distinct, all in the open interval $(-1, 1)$ ([Theorem 11.10.5](#)). In particular p_7 has no complex roots and no repeated ones.
- False. Take the coefficient inner product $\langle p, q \rangle = \sum_k u_k v_k$ on $\mathbb{R}[x]$, where $p = \sum_k u_k x^k$ and $q = \sum_k v_k x^k$ (Example (A functional with no Riesz vector)). Then $\langle x \cdot 1, x \rangle = \langle x, x \rangle = 1$, while $\langle 1, x \cdot x \rangle = \langle 1, x^2 \rangle = 0$. The identity is a property of the weight and node constructions ([Proposition 11.10.1](#)), not of inner products in general.
- Six. A k -point rule is exact up to degree $2k - 1$, and $2 \cdot 6 - 1 = 11$.

11.10.7.2 B. Practice

Exercise. B1: A weighted interval

On $\mathbb{R}[x]$ put $\langle p, q \rangle = \int_0^1 p(t)q(t)t \, dt$. Compute the monic p_0 , p_1 , and p_2 , and verify directly

that $p_2 \perp \mathbb{R}[x]_{\leq 1}$.

Solution. The moments are $\langle x^k, 1 \rangle = \int_0^1 t^{k+1} dt = 1/(k+2)$, so $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ for $k = 0, 1, 2, 3$.

$p_0 = 1$. Then $a_0 = \langle x, 1 \rangle / \langle 1, 1 \rangle = (1/3)/(1/2) = \frac{2}{3}$, so $p_1 = x - \frac{2}{3}$, and

$$\langle p_1, p_1 \rangle = \frac{1}{4} - \frac{4}{3} \cdot \frac{1}{3} + \frac{4}{9} \cdot \frac{1}{2} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}.$$

Next $\langle xp_1, p_1 \rangle = \int_0^1 (t^4 - \frac{4}{3}t^3 + \frac{4}{9}t^2) dt = \frac{1}{5} - \frac{1}{3} + \frac{4}{27} = \frac{2}{135}$, so

$$a_1 = \frac{2/135}{1/36} = \frac{8}{15}, \quad b_1 = \frac{1/36}{1/2} = \frac{1}{18},$$

and [Theorem 11.10.3](#) gives

$$p_2 = \left(x - \frac{8}{15}\right)\left(x - \frac{2}{3}\right) - \frac{1}{18} = x^2 - \frac{6}{5}x + \frac{3}{10}.$$

Verification. $\langle p_2, 1 \rangle = \int_0^1 (t^3 - \frac{6}{5}t^2 + \frac{3}{10}t) dt = \frac{1}{4} - \frac{2}{5} + \frac{3}{20} = 0$, and $\langle p_2, x \rangle = \int_0^1 (t^4 - \frac{6}{5}t^3 + \frac{3}{10}t^2) dt = \frac{1}{5} - \frac{3}{10} + \frac{1}{10} = 0$. Since $(1, x)$ spans $\mathbb{R}[x]_{\leq 1}$, this is the whole check.

Exercise. B2: Two rows of the table

Verify the degree-1 and degree-2 entries of the table for two of the classical families.

- (a) Chebyshev: $w(t) = (1 - t^2)^{-1/2}$ on $(-1, 1)$. You may use, without proof, that $\int_{-1}^1 (1 - t^2)^{-1/2} dt = \pi$ and $\int_{-1}^1 t^2 (1 - t^2)^{-1/2} dt = \pi/2$.
- (b) Laguerre: $w(t) = e^{-t}$ on $(0, \infty)$. You may use, without proof, that $\int_0^\infty t^k e^{-t} dt = k!$.

Solution.

- (a) The weight is even, so $\langle x^j, x^i \rangle = 0$ whenever $i + j$ is odd. Hence $\langle x, 1 \rangle = 0$, and x is monic of degree 1 and orthogonal to $\mathbb{R}[x]_{\leq 0}$, so $p_1 = x$ by [Proposition 11.10.2](#). Write $p_2 = x^2 + \beta x + \gamma$. Then $\langle p_2, x \rangle = \langle x^2, x \rangle + \beta \langle x, x \rangle + \gamma \langle 1, x \rangle = 0 + \beta \cdot \frac{\pi}{2} + 0$, forcing $\beta = 0$; and $\langle p_2, 1 \rangle = \frac{\pi}{2} + 0 + \gamma\pi = 0$, forcing $\gamma = -\frac{1}{2}$. So $p_2 = x^2 - \frac{1}{2}$.
- (b) The moments are $\langle x^k, 1 \rangle = k!$, so $1, 1, 2, 6$ for $k = 0, 1, 2, 3$. For $p_1 = x + \beta$: $\langle p_1, 1 \rangle = 1 + \beta = 0$, so $p_1 = x - 1$. For $p_2 = x^2 + \beta x + \gamma$, the two conditions $\langle p_2, 1 \rangle = 0$ and $\langle p_2, x \rangle = 0$ read

$$2 + \beta + \gamma = 0, \quad 6 + 2\beta + \gamma = 0.$$

Subtracting gives $\beta = -4$, then $\gamma = 2$. So $p_2 = x^2 - 4x + 2$, whose roots $2 \pm \sqrt{2}$ we already located in the Quick check above.

Exercise. B3: One more Legendre polynomial

Continue [Example \(The monic Legendre polynomials, continued\)](#): compute $\langle p_3, p_3 \rangle$ for $p_3 = x^3 - \frac{3}{5}x$, then use [Theorem 11.10.3](#) to find p_4 . Verify that $\langle p_4, 1 \rangle = \langle p_4, x^2 \rangle = 0$.

Solution.

$$\langle p_3, p_3 \rangle = \int_{-1}^1 \left(t^6 - \frac{6}{5}t^4 + \frac{9}{25}t^2 \right) dt = \frac{2}{7} - \frac{6}{5} \cdot \frac{2}{5} + \frac{9}{25} \cdot \frac{2}{3} = \frac{2}{7} - \frac{6}{25} = \frac{8}{175}.$$

Since the weight is even, $a_3 = 0$, and $b_3 = \langle p_3, p_3 \rangle / \langle p_2, p_2 \rangle = (8/175) / (8/45) = \frac{9}{35}$. Hence

$$p_4 = x \left(x^3 - \frac{3}{5}x \right) - \frac{9}{35} \left(x^2 - \frac{1}{3} \right) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

Verification. $\langle p_4, 1 \rangle = \frac{2}{5} - \frac{6}{7} \cdot \frac{2}{3} + \frac{3}{35} \cdot 2 = \frac{14-20+6}{35} = 0$, and

$$\langle p_4, x^2 \rangle = \frac{2}{7} - \frac{6}{7} \cdot \frac{2}{5} + \frac{3}{35} \cdot \frac{2}{3} = \frac{10-12+2}{35} = 0.$$

Orthogonality to x and x^3 is automatic by parity.

Exercise. B4: A two-point rule on the unit interval

Work on $(0, 1)$ with $w = 1$. Exercise B2 of Section 2 (Exercise (B2: Gram–Schmidt on polynomials)) found the monic orthogonal polynomials $1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}$.

- (a) Write down the two-point Gaussian rule for $\int_0^1 f$.
- (b) Check it on $f = x^3$, and show that it fails on $f = x^4$.

Solution.

- (a) The nodes are the roots of $x^2 - x + \frac{1}{6}$, namely

$$t_{1,2} = \frac{1}{2} \mp \frac{\sqrt{3}}{6}, \quad t_2 - t_1 = \frac{\sqrt{3}}{3},$$

both in $(0, 1)$, as [Theorem 11.10.5](#) promises. With $\ell_1 = (x - t_2)/(t_1 - t_2)$ and $\int_0^1 t \, dt = \frac{1}{2}$,

$$\alpha_1 = \frac{\frac{1}{2} - t_2}{t_1 - t_2} = \frac{-\sqrt{3}/6}{-\sqrt{3}/3} = \frac{1}{2},$$

and $\alpha_2 = \frac{1}{2}$ likewise. So $\int_0^1 f = \frac{1}{2}(f(t_1) + f(t_2))$ for $f \in \mathbb{R}[x]_{\leq 3}$.

- (b) Write $t_{1,2} = \frac{1}{2} \mp u$ with $u = \sqrt{3}/6$, so $u^2 = \frac{1}{12}$. The odd powers of u cancel in a sum over t_1 and t_2 , so

$$t_1^3 + t_2^3 = 2 \left(\frac{1}{8} + 3 \cdot \frac{1}{2} \cdot \frac{1}{12} \right) = 2 \cdot \frac{1}{4} = \frac{1}{2},$$

and half of that is $\frac{1}{4} = \int_0^1 t^3 \, dt$. For the quartic,

$$t_1^4 + t_2^4 = 2 \left(\frac{1}{16} + 6 \cdot \frac{1}{4} \cdot \frac{1}{12} + \frac{1}{144} \right) = 2 \cdot \frac{28}{144} = \frac{7}{18},$$

so the rule returns $\frac{7}{36}$, whereas $\int_0^1 t^4 \, dt = \frac{1}{5}$. The rule is exact to degree $3 = 2k - 1$ and no further.

11.10.7.3 C. Going deeper

Exercise. C1: Interlacing roots

Let w be a weight on (a, b) and let (p_k) be its monic orthogonal polynomial sequence. The roots of p_k are said to **interlace** those of p_{k+1} when exactly one root of p_k lies strictly between each pair of consecutive roots of p_{k+1} . This is true for every k ; prove the first two cases by hand.

- (a) Let r be the root of p_1 and $s_1 < s_2$ the roots of p_2 . Prove that $s_1 < r < s_2$.
- (b) Let $u_1 < u_2 < u_3$ be the roots of p_3 . Prove that $u_1 < s_1 < u_2 < s_2 < u_3$.

Hint: evaluate the three-term recurrence at a root.

Solution.

- (a) By [Theorem 11.10.3](#) with $k = 1$, $p_2 = (x - a_1)p_1 - b_1p_0 = (x - a_1)(x - r) - b_1$. Evaluating at r gives $p_2(r) = -b_1 < 0$, since $b_1 > 0$. By [Theorem 11.10.5](#), $p_2 = (x - s_1)(x - s_2)$ with $s_1 < s_2$ real, and this product is negative exactly on (s_1, s_2) . Hence $s_1 < r < s_2$.
- (b) By [Theorem 11.10.3](#) with $k = 2$, $p_3 = (x - a_2)p_2 - b_2p_1$. At a root of p_2 the first term dies, so

$$p_3(s_i) = -b_2 p_1(s_i) = -b_2(s_i - r) \quad (i = 1, 2).$$

By (a), $s_1 - r < 0$ and $s_2 - r > 0$, and $b_2 > 0$, so $p_3(s_1) > 0$ and $p_3(s_2) < 0$.

By [Theorem 11.10.5](#), $p_3 = (x - u_1)(x - u_2)(x - u_3)$ with $u_1 < u_2 < u_3$. Such a product is negative on $(-\infty, u_1)$, positive on (u_1, u_2) , negative on (u_2, u_3) , and positive on (u_3, ∞) . So $p_3(s_1) > 0$ puts s_1 in (u_1, u_2) or in (u_3, ∞) , and $p_3(s_2) < 0$ puts s_2 in $(-\infty, u_1)$ or in (u_2, u_3) . The option $s_1 > u_3$ is impossible: it would give $s_2 > s_1 > u_3$, where p_3 is positive, contradicting $p_3(s_2) < 0$. The option $s_2 < u_1$ is impossible too: it would give $s_1 < s_2 < u_1$, where p_3 is negative, contradicting $p_3(s_1) > 0$. Hence $u_1 < s_1 < u_2$ and $u_2 < s_2 < u_3$, which is the assertion.

The general case is not proved this way. The coefficients a_k and b_k assemble into a symmetric tridiagonal matrix whose characteristic polynomial is p_k , and the interlacing then follows from the interlacing theorem for the eigenvalues of a symmetric matrix and its leading submatrices, which is Chapter 17's business.

Exercise. C2: The ceiling is $2k - 1$

Let w be a weight on (a, b) and let $k \geq 1$.

- (a) Let $t_1, \dots, t_k$ be **any** k distinct real numbers and $\alpha_1, \dots, \alpha_k$ **any** real numbers. Prove that there is an $f \in \mathbb{R}[x]_{\leq 2k}$ with $\int_a^b f w \neq \sum_i \alpha_i f(t_i)$.
- (b) Hence deduce that the Gaussian rule of [Theorem 11.10.6](#) is exact on $\mathbb{R}[x]_{\leq 2k-1}$ and on no larger space $\mathbb{R}[x]_{\leq d}$.

Solution.

- (a) Put $g = (x - t_1) \cdots (x - t_k)$ and $f = g^2$, a polynomial of degree $2k$. Every node is a root of g , so $f(t_i) = 0$ for every i and the right-hand side is 0, whatever the α_i are. The left-hand side is $\int_a^b g^2 w = \langle g, g \rangle$, and $g \neq 0$, so $\langle g, g \rangle > 0$ by positive definiteness.

The two sides differ.

- (b) Exactness on $\mathbb{R}[x]_{\leq 2k-1}$ is [Theorem 11.10.6](#). Suppose the rule were exact on $\mathbb{R}[x]_{\leq d}$ for some $d \geq 2k$. Since $\mathbb{R}[x]_{\leq 2k} \subseteq \mathbb{R}[x]_{\leq d}$, it would then be exact on every polynomial of degree at most $2k$. But its nodes are k distinct real numbers ([Theorem 11.10.5](#)), so (a) supplies a polynomial of degree $2k$ on which it is not. Hence no such d exists, and $2k - 1$ is the largest degree for which the rule is exact. The concrete instance $k = 2$, $f = x^4$ is the failure computed in [Example \(Two-point Gauss–Legendre\)](#).