

11.5 The Riesz Representation Theorem

Chapter 5 built the dual space V^* of all linear functionals on V , proved that $\dim V^* = \dim V$ in finite dimension, and then delivered a warning: the resulting isomorphism $V \cong V^*$ is not canonical. Every basis of V gives one, different bases give different ones, and no argument singles out a preferred choice (Corollary 5.1.4, and the contrast with the natural map $\text{ev}_V: V \rightarrow V^{**}$ of Theorem 5.4.6). The chapter closed by promising that an inner product would supply the missing ingredient. This section keeps the promise.

The mechanism is simple to state. Fix $w \in V$ and let $\varphi_w(v) = \langle v, w \rangle$. Because the inner product is linear in the **first** slot, φ_w is a linear functional. The Riesz Representation Theorem says that in finite dimension there are no others.

Throughout, V is a finite-dimensional inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$, unless the text says otherwise.

11.5.1 Every functional is an inner product

In F^n , Theorem 5.1.1 already told us that every functional has the form $\mathbf{x} \mapsto a_1x_1 + \cdots + a_nx_n$ for a unique row of coefficients. With the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i \overline{y_i}$, that reads $\varphi(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w} \rangle$ with $\mathbf{w} = (\overline{a_1}, \dots, \overline{a_n})$. So in coordinates the theorem is a restatement of a fact we already have; the content is that it survives the removal of coordinates.

Theorem 11.5.1. Riesz Representation Theorem

Let V be a finite-dimensional inner product space over F and let $\varphi \in V^*$. Then there is a **unique** $w \in V$ such that

$$\varphi(v) = \langle v, w \rangle \quad \text{for every } v \in V.$$

If $(e_1, \dots, e_n)$ is any orthonormal basis of V , then $w = \sum_{i=1}^n \overline{\varphi(e_i)} e_i$.

Idea. Fix an orthonormal basis. Then a linear functional is determined by the n numbers $\varphi(e_i)$, and a vector is determined by its n coordinates, which Theorem 11.2.1 reads off as inner products. Matching the two lists tells us what w must be, and the only care needed is the conjugate: w sits in the second slot, so its coordinates come out conjugated. Uniqueness is the standard move — if two vectors pair identically with everything, subtract and pair the difference with itself.

Proof. Existence. Let $(e_1, \dots, e_n)$ be an orthonormal basis of V , which exists by Theorem 11.2.2, and put

$$w = \sum_{i=1}^n \overline{\varphi(e_i)} e_i.$$

For each j , conjugate-linearity in the second slot and $\langle e_j, e_i \rangle = \delta_{ij}$ give

$$\langle e_j, w \rangle = \sum_{i=1}^n \overline{\overline{\varphi(e_i)}} \langle e_j, e_i \rangle = \varphi(e_j).$$

So the two linear functionals $v \mapsto \langle v, w \rangle$ and φ agree on a basis, hence are equal by Theo-

rem 2.1.5. (The first is linear because the inner product is linear in the first slot.)

Uniqueness. Suppose $\langle v, w \rangle = \langle v, w' \rangle$ for every v . Then $\langle v, w - w' \rangle = 0$ for every v , by additivity in the second slot. Taking $v = w - w'$ gives $\|w - w'\|^2 = 0$, so $w = w'$ by positive definiteness. ■

Call w the **Riesz vector** of φ . The formula is worth reading twice: the coordinates of w in an orthonormal basis are the conjugates of the values of φ on that basis. Over $\mathbb{R}$ the conjugation is invisible, which is why real textbooks can omit it and complex ones cannot.

Example. Two Riesz vectors

- (a) On $V = \mathbb{R}[x]_{\leq 1}$ with $\langle p, q \rangle = \int_0^1 p(t)q(t) dt$, find the Riesz vector of $\varphi(p) = p(0)$.
- (b) On $V = M_n(\mathbb{C})$ with the Frobenius inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \text{tr}(\mathbf{B}^* \mathbf{A})$, find the Riesz vector of $\varphi(\mathbf{A}) = \text{tr } \mathbf{A}$.

Solution.

- (a) We look for $w = a + bx$ with $\langle p, w \rangle = p(0)$ for all p . By linearity it is enough to impose this on the basis $(1, x)$:

$$\int_0^1 (a + bt) dt = a + \frac{b}{2} = 1, \quad \int_0^1 t(a + bt) dt = \frac{a}{2} + \frac{b}{3} = 0.$$

The second equation gives $a = -\frac{2}{3}b$; substituting into the first gives $-\frac{2}{3}b + \frac{1}{2}b = -\frac{1}{6}b = 1$, so $b = -6$ and $a = 4$. Hence $w = 4 - 6x$.

Check. $\int_0^1 (4 - 6t) dt = 4 - 3 = 1$ and $\int_0^1 t(4 - 6t) dt = 2 - 2 = 0$, as required; and for a general $p = c + dx$ the pairing gives $c \cdot 1 + d \cdot 0 = c = p(0)$. Note how little w resembles “evaluation at 0”: the Riesz vector is manufactured by the inner product, not by the point 0.

- (b) We look for $\mathbf{W}$ with $\text{tr}(\mathbf{W}^* \mathbf{A}) = \text{tr } \mathbf{A}$ for every $\mathbf{A}$. Taking $\mathbf{W} = \mathbf{I}_n$ works, since $\mathbf{I}_n^* = \mathbf{I}_n$ and $\text{tr}(\mathbf{I}_n \mathbf{A}) = \text{tr } \mathbf{A}$. By uniqueness, $\mathbf{W} = \mathbf{I}_n$ is the answer. (This example was not solved by the basis formula, but by guessing and then invoking uniqueness — a legitimate and often faster route.)

11.5.2 The Riesz map, and the data Chapter 5 was missing

Uniqueness in [Theorem 11.5.1](#) lets us read the theorem as a statement about a map.

Corollary 11.5.2. The Riesz map

Let V be a finite-dimensional inner product space over F . Define

$$\Phi: V \rightarrow V^*, \quad \Phi(w) = \langle \cdot, w \rangle.$$

Then Φ is a bijection, and for all $w, w' \in V$ and $c \in F$,

$$\Phi(w + w') = \Phi(w) + \Phi(w'), \quad \Phi(cw) = \bar{c} \Phi(w).$$

In particular Φ is an isomorphism of vector spaces when $F = \mathbb{R}$, and is **conjugate-linear** when $F = \mathbb{C}$.

Proof. Each $\Phi(w)$ lies in V^* because the inner product is linear in the first slot. Surjectivity is the existence half of [Theorem 11.5.1](#) and injectivity is the uniqueness half (if $\Phi(w) = \Phi(w')$, then $w = w'$). For the two rules, evaluate at an arbitrary v : additivity in the second slot gives $\langle v, w + w' \rangle = \langle v, w \rangle + \langle v, w' \rangle$, and conjugate-homogeneity in the second slot gives $\langle v, cw \rangle = \bar{c} \langle v, w \rangle$. Over $\mathbb{R}$ we have $\bar{c} = c$, so Φ is linear, and a linear bijection is an isomorphism (Theorem 0.4.7). ■

This is the payoff Chapter 5 promised. There, $V \cong V^*$ was proved by choosing a basis $\mathcal{B}$ and sending $v_i \mapsto \varphi_i$, and the warning was that the isomorphism changes when $\mathcal{B}$ does: a vector does not come with “its” functional. Here Φ is written down with no basis at all. It is not free of choices — it depends on $\langle \cdot, \cdot \rangle$ — but the choice has been made once, in advance, and it is a choice of *geometry* rather than of coordinates. **The inner product is exactly the extra data that turns $V \cong V^*$ from a counting fact into a formula.**

The two descriptions fit together neatly. Let $\mathcal{B} = (v_1, \dots, v_n)$ be an **orthonormal** basis of V , with dual basis $(\varphi_1, \dots, \varphi_n)$ (Theorem 5.1.3). Then $\Phi(v_i)$ is the functional $v \mapsto \langle v, v_i \rangle$, which sends v_j to δ_{ij} ; that is,

$$\Phi(v_i) = \varphi_i \quad (i = 1, \dots, n),$$

over either field. Over $\mathbb{R}$, where Φ is linear, this says that Φ is Chapter 5’s basis-dependent isomorphism $v_i \mapsto \varphi_i$, for the inner product that declares $\mathcal{B}$ orthonormal. Choosing a basis and choosing an inner product are two ways of supplying the same missing data.

Quick check. Let $\mathcal{B} = (v_1, \dots, v_n)$ be an orthonormal basis of a **complex** inner product space and let $(\varphi_1, \dots, \varphi_n)$ be its dual basis. What is the Riesz vector of $i\varphi_1$?

Solution. By the identity $\Phi(v_i) = \varphi_i$ above, which holds over $\mathbb{C}$ as well, φ_1 has Riesz vector v_1 . For $i\varphi_1$, use conjugate-homogeneity in reverse: $\Phi(-iv_1) = \overline{(-i)}\Phi(v_1) = i\varphi_1$. So the Riesz vector of $i\varphi_1$ is $-iv_1$, not iv_1 . Alternatively, apply the formula of [Theorem 11.5.1](#): the coordinates of the Riesz vector are $\overline{(i\varphi_1)(v_j)} = i\overline{\delta_{1j}} = -i\delta_{1j}$.

Warning. Over $\mathbb{C}$, the Riesz map is not linear, so it is not an isomorphism of complex vector spaces. It is a bijection, and it is linear over $\mathbb{R}$, but $\Phi(iw) = -i\Phi(w)$. Complex spaces V and V^* are still isomorphic — they have the same dimension — but the isomorphism produced by an inner product is the conjugate-linear Φ . Most of what we do with Φ (Section 6 builds the adjoint from it) is unaffected, because the conjugation cancels when Φ is used twice.

Warning. The Riesz vector depends on the inner product, not on φ alone. On $\mathbb{R}^2$ let $\varphi(\mathbf{x}) = x_1 + x_2$. With the dot product, $\varphi(\mathbf{x}) = \langle \mathbf{x}, (1, 1) \rangle$, so $\mathbf{w} = (1, 1)$. With the weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + x_2y_2$, we need $2w_1 = 1$ and $w_2 = 1$, so $\mathbf{w} = (\frac{1}{2}, 1)$. Same functional, different representing vector. Whenever two inner products are in play, the notation must say which Φ is meant.

11.5.3 Annihilators become orthogonal complements

Section 3 warned that the annihilator $U^0 \subseteq V^*$ and the orthogonal complement $U^\perp \subseteq V$ are different objects that happen to have the same dimension. The Riesz map explains the coincidence: it carries one onto the other.

Theorem 11.5.3. Perp corresponds to the annihilator

Let V be a finite-dimensional inner product space over F , let U be a subspace, and let Φ be the Riesz map. Then

$$\Phi(U^\perp) = U^0.$$

Equivalently, a functional $\varphi \in V^*$ annihilates U if and only if its Riesz vector lies in $U^\perp$.

Proof. Let $w \in V$. Then

$$\Phi(w) \in U^0 \iff \langle u, w \rangle = 0 \text{ for every } u \in U \iff w \in U^\perp,$$

the first equivalence being the definition of the annihilator (Definition 5.2.1) applied to the functional $\Phi(w) = \langle \cdot, w \rangle$, and the second the definition of the orthogonal complement (Definition 11.3.1). Since Φ is a bijection (Corollary 11.5.2), the set $\Phi(U^\perp)$ is therefore exactly U^0 . ■

The dimension count now has a reason rather than a coincidence: $\dim U^0 = \dim V - \dim U$ (Theorem 5.2.2) and $\dim U^\perp = \dim V - \dim U$ (Theorem 11.3.2 (c)) agree because the two spaces correspond under a bijection. (Over $\mathbb{C}$ the bijection is only conjugate-linear, but a conjugate-linear bijection still carries subspaces to subspaces of the same dimension, since $c\Phi(w) = \Phi(\bar{c}w)$.)

Here is the correspondence in $\mathbb{R}^3$. Let $U = \text{span}((1, 2, 0))$ with the dot product. Then $U^\perp = \{(a, b, c) : a + 2b = 0\} = \text{span}((2, -1, 0), (0, 0, 1))$, and U^0 consists of the functionals $\mathbf{x} \mapsto \alpha x_1 + \beta x_2 + \gamma x_3$ with $\alpha + 2\beta = 0$, spanned by $2x_1 - x_2$ and x_3 . These are precisely $\Phi((2, -1, 0))$ and $\Phi((0, 0, 1))$.

Section 6 uses Theorem 11.5.3 in the same way for maps: the dual map $T' : W^* \rightarrow V^*$ of Chapter 5 and the adjoint $T^* : W \rightarrow V$ are the same map, read through the two Riesz maps.

11.5.4 What happens in infinite dimension

The proof used an orthonormal basis of the whole of V , so finite dimension was essential, and the theorem genuinely fails without it.

Example. A functional with no Riesz vector

Let $V = \mathbb{R}[x]$ with the coefficient inner product $\langle p, q \rangle = \sum_{k \geq 0} a_k b_k$ for $p = \sum_k a_k x^k$ and $q = \sum_k b_k x^k$ (a finite sum, and an inner product by Exercise (C2: A subspace with no orthogonal complement) (a)). Show that $\varphi(p) = p(1)$ has no Riesz vector.

Solution. Suppose $w = \sum_k b_k x^k$ satisfied $\langle p, w \rangle = p(1)$ for every $p \in V$. Taking $p = x^i$ gives

$$b_i = \langle x^i, w \rangle = \varphi(x^i) = 1^i = 1 \quad \text{for every } i \geq 0.$$

So every coefficient of w equals 1. But a polynomial has only finitely many nonzero coefficients, so no such w exists.

What fails is not “infinite dimension” as a slogan. Two things are missing at once, and the same example shows both. First, V has gaps: Exercise (C2: A subspace with no orthogonal complement) exhibits polynomials $p_m \in \ker \varphi$ with $\|p_m - 1\| = 1/\sqrt{m}$, so the vectors p_m crowd around the polynomial 1 without any limit being forced to lie in $\ker \varphi$. Second, φ is wildly discontinuous with respect to $\|\cdot\|$: along the same sequence, $\varphi(p_m) = 0$ for every m , while $\varphi(1) = 1$.

Repairing both defects — completing the space, and restricting attention to functionals φ with $|\varphi(v)| \leq C\|v\|$ — restores the theorem. That statement is the Riesz representation theorem for Hilbert spaces, and it belongs to analysis, not to linear algebra; we record its existence and go no further. What survives here with no hypotheses at all is the easy direction: for every $w \in V$, the map $\langle \cdot, w \rangle$ is a linear functional. Only the converse needs finite dimension.

11.5.5 Exercises

11.5.5.1 A. Check your understanding

Exercise. A1

- State the Riesz Representation Theorem, including every hypothesis.
- Write down the formula for the Riesz vector of φ in terms of an orthonormal basis, and say where the conjugate comes from.
- True or false: over $\mathbb{C}$, the Riesz map $\Phi: V \rightarrow V^*$ is a linear isomorphism. Justify your answer.
- In what sense does an inner product answer the question Chapter 5 left open about $V \cong V^*$?
- Under Φ , which subspace of V^* corresponds to $U^\perp$?
- Give an example showing that the theorem fails in infinite dimension.

Solution.

- If V is a **finite-dimensional** inner product space over $\mathbb{R}$ or $\mathbb{C}$ and $\varphi \in V^*$, then there is a unique $w \in V$ with $\varphi(v) = \langle v, w \rangle$ for all $v \in V$ ([Theorem 11.5.1](#)).
- $w = \sum_i \overline{\varphi(e_i)} e_i$ for any orthonormal basis $(e_1, \dots, e_n)$. The conjugate appears because

w occupies the second slot, where scalars come out conjugated: to make $\langle e_j, w \rangle$ equal $\varphi(e_j)$, the coefficient of e_j in w must be $\overline{\varphi(e_j)}$.

- (c) False. Φ is a bijection, but $\Phi(cw) = \bar{c}\Phi(w)$, so it is conjugate-linear and not $\mathbb{C}$ -linear (Corollary 11.5.2). It is linear over $\mathbb{R}$, and over $\mathbb{R}$ it is an isomorphism.
- (d) Chapter 5 showed $V \cong V^*$ by counting dimensions, with an isomorphism depending on a chosen basis and no preferred choice available (Corollary 5.1.4). An inner product supplies a specific map $\Phi(w) = \langle \cdot, w \rangle$, written without reference to any basis. The inner product is the extra data; picking an orthonormal basis and picking an inner product that makes a given basis orthonormal are the same act.
- (e) U^0 , the annihilator of U (Theorem 11.5.3).
- (f) On $\mathbb{R}[x]$ with the coefficient inner product, $\varphi(p) = p(1)$ has no Riesz vector: such a vector would need every coefficient equal to 1 (Example (A functional with no Riesz vector)).

11.5.5.2 B. Practice

Exercise. B1: Riesz vectors for evaluation

On $V = \mathbb{R}[x]_{\leq 2}$ use the node inner product $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Find the Riesz vectors of $\varphi(p) = p(0)$ and of $\psi(p) = p(1)$.

Solution. Since $-1, 0, 1$ are three distinct nodes and $\dim V = 3$, a polynomial of V is determined by its values there (Theorem 3.7.3). Writing w for a candidate Riesz vector of φ , the requirement $\langle p, w \rangle = p(0)$ reads

$$p(-1)w(-1) + p(0)w(0) + p(1)w(1) = p(0) \quad \text{for all } p \in V.$$

Taking for p the three Lagrange polynomials at the nodes, which have value 1 at one node and 0 at the others (Definition 6.6.1), the three equations become $w(-1) = 0$, $w(0) = 1$, $w(1) = 0$. The unique quadratic with those values is $w = 1 - x^2$.

For ψ the same argument gives $w(-1) = 0$, $w(0) = 0$, $w(1) = 1$, whose interpolant is $w = \frac{1}{2}x(x+1)$.

Check. For $p = 3 + 2x - x^2$ we get $p(-1) = 0$, $p(0) = 3$, $p(1) = 4$. Pairing with $1 - x^2$, whose values are 0, 1, 0, gives $3 = p(0)$; pairing with $\frac{1}{2}x(x+1)$, whose values are 0, 0, 1, gives $4 = p(1)$.

Exercise. B2: Riesz vectors on matrices

On $M_n(\mathbb{C})$ with the Frobenius inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \text{tr}(\mathbf{B}^* \mathbf{A})$, find the Riesz vector of:

- (a) $\varphi(\mathbf{A}) = a_{12}$ (assume $n \geq 2$);
- (b) $\varphi_{\mathbf{C}}(\mathbf{A}) = \text{tr}(\mathbf{C}\mathbf{A})$, for a fixed $\mathbf{C} \in M_n(\mathbb{C})$.

Solution.

- (a) Let $\mathbf{E}_{12}$ be the matrix unit with a 1 in position $(1, 2)$ and zeros elsewhere. Then

$\mathbf{E}_{12}^* = \mathbf{E}_{21}$, and $(\mathbf{E}_{21}\mathbf{A})_{kk} = \sum_j (\mathbf{E}_{21})_{kj} a_{jk}$, which is 0 unless $k = 2$, where it equals a_{12} . Hence $\text{tr}(\mathbf{E}_{12}^*\mathbf{A}) = a_{12} = \varphi(\mathbf{A})$, and by uniqueness in [Theorem 11.5.1](#) the Riesz vector is $\mathbf{E}_{12}$.

- (b) We need $\mathbf{W}$ with $\text{tr}(\mathbf{W}^*\mathbf{A}) = \text{tr}(\mathbf{C}\mathbf{A})$ for all $\mathbf{A}$. Taking $\mathbf{W}^* = \mathbf{C}$, that is $\mathbf{W} = \mathbf{C}^*$, works. By uniqueness it is the only answer. Part (a) is the case $\mathbf{C} = \mathbf{E}_{21}$, and the example in the text is the case $\mathbf{C} = \mathbf{I}_n$.

Exercise. B3: The correspondence in $\mathbb{R}^4$

In $\mathbb{R}^4$ with the dot product, let $U = \text{span}((1, 0, 1, 0), (0, 1, 0, 1))$. Find a basis of $U^\perp$ and a basis of U^0 , and verify that Φ carries the first to the second.

Solution. A vector (a, b, c, d) lies in $U^\perp$ exactly when $a + c = 0$ and $b + d = 0$ (Proposition 11.3.1, testing against the two spanning vectors). So

$$U^\perp = \text{span}((1, 0, -1, 0), (0, 1, 0, -1)),$$

of dimension $2 = 4 - 2$.

A functional $\varphi(\mathbf{x}) = \alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4$ lies in U^0 exactly when it kills both spanning vectors, that is, when $\alpha + \gamma = 0$ and $\beta + \delta = 0$. So $U^0 = \text{span}(\varphi_1, \varphi_2)$ with

$$\varphi_1(\mathbf{x}) = x_1 - x_3, \quad \varphi_2(\mathbf{x}) = x_2 - x_4.$$

Over $\mathbb{R}$, $\Phi(\mathbf{w})(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w} \rangle = \sum_i x_i w_i$, so $\Phi((1, 0, -1, 0)) = \varphi_1$ and $\Phi((0, 1, 0, -1)) = \varphi_2$. Since Φ is linear over $\mathbb{R}$ and bijective, it carries the span of the two vectors onto the span of the two functionals, confirming [Theorem 11.5.3](#).

11.5.5.3 C. Going deeper

Exercise. C1: Riesz without an orthonormal basis

Let V be a finite-dimensional inner product space and let $\varphi \in V^*$ be nonzero. Write $K = \ker \varphi$.

- Prove that $\dim K^\perp = 1$.
- Let u span $K^\perp$ with $\|u\| = 1$. Prove that $w = \overline{\varphi(u)}u$ satisfies $\varphi(v) = \langle v, w \rangle$ for all $v \in V$.
- Deduce [Theorem 11.5.1](#), including the case $\varphi = 0$.

Hint for (b): split v using $V = K \oplus K^\perp$.

Solution.

- Since $\varphi \neq 0$, its image is a nonzero subspace of the 1-dimensional space F , so $\text{im } \varphi = F$ and $\text{rank } \varphi = 1$. By Theorem 2.3.1, $\dim K = \dim V - 1$, and by Theorem 11.3.2 (c), $\dim K^\perp = \dim V - \dim K = 1$.
- By Theorem 11.3.2 (a), $V = K \oplus K^\perp$, so each $v \in V$ is $v = k + cu$ with $k \in K$ and

$c \in F$. Then $\varphi(v) = \varphi(k) + c\varphi(u) = c\varphi(u)$. On the other side,

$$\langle v, w \rangle = \langle k + cu, \overline{\varphi(u)}u \rangle = \varphi(u)(\langle k, u \rangle + c\langle u, u \rangle) = \varphi(u)c,$$

using conjugate-homogeneity in the second slot (which conjugates $\overline{\varphi(u)}$ back to $\varphi(u)$), then $\langle k, u \rangle = 0$ because $u \in K^\perp$ and $k \in K$, and $\langle u, u \rangle = \|u\|^2 = 1$. The two sides agree.

- (c) If $\varphi = 0$, then $w = 0$ represents it. If $\varphi \neq 0$, part (a) supplies a unit vector u spanning the line $K^\perp$ — any nonzero vector of that line divided by its norm — and part (b) supplies w . Uniqueness is unchanged: if $\langle v, w \rangle = \langle v, w' \rangle$ for all v , take $v = w - w'$ to get $\|w - w'\|^2 = 0$. Note that this proof never mentions a basis of V ; it uses only the orthogonal decomposition along the hyperplane $\ker \varphi$.

Exercise. C2: Evaluation at 0 is not represented

Let $V = \mathbb{R}[x]$ with $\langle p, q \rangle = \int_0^1 p(t)q(t) dt$, and let $\varphi(p) = p(0)$. Suppose, for a contradiction, that $w = \sum_{j=0}^d c_j x^j$ satisfied $\langle p, w \rangle = p(0)$ for every $p \in V$.

- (a) Show that $\sum_{j=0}^d \frac{c_j}{k+j+1} = 0$ for every integer $k \geq 1$, and that the same sum equals 1 for $k = 0$.
- (b) Let $N(s) = \sum_{j=0}^d c_j \prod_{i \neq j} (s+i+1)$, the product running over $i \in \{0, \dots, d\} \setminus \{j\}$. Show that $N(k) = 0$ for every integer $k \geq 1$, and deduce that N is the zero polynomial.
- (c) Deduce that every $c_j = 0$, and hence a contradiction.

Solution.

- (a) Take $p = x^k$. Then $p(0) = 0$ for $k \geq 1$ and $p(0) = 1$ for $k = 0$, while

$$\langle x^k, w \rangle = \int_0^1 t^k \sum_{j=0}^d c_j t^j dt = \sum_{j=0}^d c_j \int_0^1 t^{k+j} dt = \sum_{j=0}^d \frac{c_j}{k+j+1}.$$

- (b) For an integer $k \geq 1$, multiply the identity of (a) by $\prod_{i=0}^d (k+i+1)$, a nonzero number. Each term $c_j/(k+j+1)$ becomes $c_j \prod_{i \neq j} (k+i+1)$, so the result is exactly $N(k) = 0$. Thus the polynomial N , of degree at most d , has the infinitely many distinct roots $1, 2, 3, \dots$. A nonzero polynomial of degree d has at most d distinct roots (Corollary 6.1.4), so $N = 0$.
- (c) Evaluate N at $s = -(j_0 + 1)$ for a fixed $j_0 \in \{0, \dots, d\}$. Every product $\prod_{i \neq j} (s+i+1)$ with $j \neq j_0$ contains the factor $s + j_0 + 1 = 0$ and so vanishes; the term $j = j_0$ contributes $c_{j_0} \prod_{i \neq j_0} (i - j_0)$, and that product is nonzero because the factors $i - j_0$ are nonzero integers. Since $N = 0$, we get $c_{j_0} = 0$. This holds for every j_0 , so $w = 0$. But then the case $k = 0$ of (a) reads $0 = 1$, a contradiction. Hence no Riesz vector exists.

The moral matches [Example \(A functional with no Riesz vector\)](#): point evaluation is a perfectly good linear functional, but on an infinite-dimensional space it is not represented by any vector, whichever of these two inner products we use.