

8.5 Commutators and Trace-Zero Matrices

The expression $\mathbf{AB} - \mathbf{BA}$ has come up several times already. In Chapter 0 its trace was 0, which showed that $\mathbf{AB} - \mathbf{BA} = \mathbf{I}_n$ is impossible over $\mathbb{R}$ (Exercise (C1: $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ is impossible, or is it?)). In Chapter 2 the same trace argument showed that the relation $DS - SD = \text{id}$ between differentiation and multiplication by x on $F[x]$ has no finite-dimensional analogue in characteristic 0 (Exercise (C1: The Heisenberg relation)). Both arguments use one direction only: every matrix of the form $\mathbf{AB} - \mathbf{BA}$ has trace zero. This section asks whether the converse holds. Is every trace-zero matrix of the form $\mathbf{AB} - \mathbf{BA}$? The answer is yes over a field of characteristic 0, and the proof is a good example of the block techniques of this chapter: move the matrix into a convenient shape by a similarity, one row and column at a time, and then solve an equation that has become entrywise.

Which fields. Commutators, their basic rules, the first similarity lemma, and the final theorem on spans work over **every** field F . The lemma on zero diagonals and Shoda's theorem assume that F has **characteristic** 0. We point out exactly where that hypothesis is used, and what happens in characteristic p .

8.5.1 Commutators

Two matrices $\mathbf{A}$ and $\mathbf{B}$ commute when $\mathbf{AB} = \mathbf{BA}$, and most pairs do not. The difference $\mathbf{AB} - \mathbf{BA}$ measures the failure, and it keeps recurring: in the trace arguments above, in the relation between differentiation and multiplication by x , and in the description of trace-like functionals (Exercise (C2: The trace is the only trace-like functional)). So we give it a name.

The commutator of two matrices is the amount by which they fail to commute.

Definition 8.5.1. Commutator

Let $n \geq 1$ and $\mathbf{A}, \mathbf{B} \in M_n(F)$. The **commutator** of $\mathbf{A}$ and $\mathbf{B}$ is

$$[\mathbf{A}, \mathbf{B}] := \mathbf{AB} - \mathbf{BA} \in M_n(F).$$

A matrix $\mathbf{C} \in M_n(F)$ is a **commutator** if $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ for **some** $\mathbf{X}, \mathbf{Y} \in M_n(F)$. For operators $S, T \in \mathcal{L}(V)$ on a vector space V , we write likewise $[S, T] := ST - TS \in \mathcal{L}(V)$.

In words: $[\mathbf{A}, \mathbf{B}]$ is “ $\mathbf{A}$ then $\mathbf{B}$ ” minus “ $\mathbf{B}$ then $\mathbf{A}$ ”, in that order, and it is the zero matrix exactly when $\mathbf{A}$ and $\mathbf{B}$ commute. The second sentence of the definition is about a single matrix: being a commutator means that **there exist** two matrices whose commutator it is, and it says nothing about which ones.

Examples.

1. **Degenerate cases.** $[\mathbf{A}, \mathbf{A}] = \mathbf{O}$ and $[\mathbf{A}, c\mathbf{I}_n] = c\mathbf{A} - c\mathbf{A} = \mathbf{O}$ for every $c \in F$. So the zero matrix is a commutator, in many ways. In $M_1(F) = F$ every commutator is $ab - ba = 0$, so for $n = 1$ the only commutator is 0; this is the smallest case, and it already matches the trace condition, since the only 1×1 matrix of trace 0 is $\mathbf{O}_1$.
2. **Matrix units.** Recall that $\mathbf{E}_{ij}\mathbf{E}_{kl}$ is $\mathbf{E}_{il}$ if $j = k$ and $\mathbf{O}$ otherwise (Definition 0.9.9). So for $i \neq k$,

$$[\mathbf{E}_{ij}, \mathbf{E}_{jk}] = \mathbf{E}_{ik} - \mathbf{O} = \mathbf{E}_{ik}, \quad \text{and} \quad [\mathbf{E}_{ij}, \mathbf{E}_{ji}] = \mathbf{E}_{ii} - \mathbf{E}_{jj} \quad (i \neq j).$$

For instance $[\mathbf{E}_{12}, \mathbf{E}_{21}] = \text{diag}(1, -1)$ in $M_2(F)$, and $[\mathbf{E}_{11}, \mathbf{E}_{12}] = \mathbf{E}_{12}$. Both have trace 0.

3. **Operators on an infinite-dimensional space.** On $F[x]$, with D differentiation and S multiplication by x , Exercise (C1: The Heisenberg relation) (a) shows $[D, S] = \text{id}$. So the identity is a commutator of operators on $F[x]$. This does not contradict anything, because there is no trace on $\mathcal{L}(F[x])$: the trace was defined only on finite-dimensional spaces (Definition 4.5.2).

Nonexample by minimal change. Change the minus sign to a plus: $\mathbf{AB} + \mathbf{BA}$. It is still linear in $\mathbf{A}$ and in $\mathbf{B}$, and it is still built from the two products $\mathbf{AB}$ and $\mathbf{BA}$. What is lost is the trace: $\text{tr}(\mathbf{AB} + \mathbf{BA}) = 2 \text{tr}(\mathbf{AB})$, which has no reason to vanish. For example $\mathbf{E}_{12}\mathbf{E}_{21} + \mathbf{E}_{21}\mathbf{E}_{12} = \mathbf{E}_{11} + \mathbf{E}_{22} = \mathbf{I}_2$, with trace 2. It also no longer detects commuting: $\mathbf{AA} + \mathbf{AA} = 2\mathbf{A}^2$ is nonzero for $\mathbf{A} = \mathbf{I}_n$ over $\mathbb{Q}$, although $\mathbf{A}$ commutes with itself.

Why this definition. The order in $\mathbf{AB} - \mathbf{BA}$ is a convention; the other order gives $[\mathbf{B}, \mathbf{A}] = -[\mathbf{A}, \mathbf{B}]$, which is a commutator too. The notion is invariant under a change of basis: a similarity applied to $\mathbf{X}$ and $\mathbf{Y}$ applies to $[\mathbf{X}, \mathbf{Y}]$. So “being a commutator” is a property of the operator $\mathbf{x} \mapsto \mathbf{C}\mathbf{x}$, not of the basis used to write $\mathbf{C}$, and we are free to choose a convenient basis. This is what the proof of the main theorem exploits.

The basic rules are collected here.

Proposition 8.5.1. Rules for Commutators

Let $\mathbf{A}, \mathbf{A}', \mathbf{B} \in M_n(F)$, $c \in F$, and let $\mathbf{P} \in M_n(F)$ be invertible. Then:

- (a) $[\mathbf{A} + \mathbf{A}', \mathbf{B}] = [\mathbf{A}, \mathbf{B}] + [\mathbf{A}', \mathbf{B}]$, $[c\mathbf{A}, \mathbf{B}] = c[\mathbf{A}, \mathbf{B}]$, and $[\mathbf{B}, \mathbf{A}] = -[\mathbf{A}, \mathbf{B}]$; in particular the commutator is linear in each slot.
- (b) $\mathbf{P}^{-1}[\mathbf{A}, \mathbf{B}]\mathbf{P} = [\mathbf{P}^{-1}\mathbf{A}\mathbf{P}, \mathbf{P}^{-1}\mathbf{B}\mathbf{P}]$.
- (c) $\text{tr}[\mathbf{A}, \mathbf{B}] = 0$.

Proof.

- (a) By Theorem 0.9.4, $(\mathbf{A} + \mathbf{A}')\mathbf{B} - \mathbf{B}(\mathbf{A} + \mathbf{A}') = (\mathbf{AB} - \mathbf{BA}) + (\mathbf{A}'\mathbf{B} - \mathbf{BA}')$ and $(c\mathbf{A})\mathbf{B} - \mathbf{B}(c\mathbf{A}) = c(\mathbf{AB} - \mathbf{BA})$. The last identity is $\mathbf{BA} - \mathbf{AB} = -(\mathbf{AB} - \mathbf{BA})$. Linearity in the second slot follows from the first slot and antisymmetry.
- (b) Inserting $\mathbf{PP}^{-1} = \mathbf{I}_n$, $\mathbf{P}^{-1}\mathbf{A}\mathbf{B}\mathbf{P} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{B}\mathbf{P})$, and likewise for $\mathbf{BA}$. Subtract, using distributivity.
- (c) By Theorem 0.9.6 (1) and (3), $\text{tr}(\mathbf{AB} - \mathbf{BA}) = \text{tr}(\mathbf{AB}) - \text{tr}(\mathbf{BA}) = 0$. ■

So every commutator lies in the subspace of trace-zero matrices of Example (Trace-Free Matrices). The question of this section is whether it fills that subspace.

8.5.2 The identity in characteristic p

Rule (c) is the whole content of the Heisenberg exercises: if $[\mathbf{X}, \mathbf{Y}] = \mathbf{I}_n$, then $n \cdot 1 = \text{tr} \mathbf{I}_n = 0$, which is impossible when F has characteristic 0 (Definition 0.7.2). The obstruction is the number $n \cdot 1$, not the identity matrix itself. In characteristic p , the $p \times p$ identity has trace $p \cdot 1 = 0$, and the obstruction disappears. In fact the pair D, S of Example 3 survives in a finite form.

Example. The Identity as a Commutator in Characteristic p

Let F have prime characteristic p (for instance $F = \mathbb{F}_p$). Let $\mathbf{D}_p \in M_p(F)$ have $(k, k+1)$ -entry $k \cdot 1$ for $1 \leq k \leq p-1$, and let $\mathbf{S}_p \in M_p(F)$ have $(k+1, k)$ -entry 1 for $1 \leq k \leq p-1$, all other entries 0. Show that $[\mathbf{D}_p, \mathbf{S}_p] = \mathbf{I}_p$.

Solution. The columns of these matrices describe differentiation and multiplication by x on the list $1, x, \dots, x^{p-1}$, where x^p is replaced by 0. By Theorem 0.9.2, $\mathbf{S}_p \mathbf{e}_k = \mathbf{e}_{k+1}$ for $k < p$ and $\mathbf{S}_p \mathbf{e}_p = \mathbf{0}$, while $\mathbf{D}_p \mathbf{e}_1 = \mathbf{0}$ and $\mathbf{D}_p \mathbf{e}_k = (k-1)\mathbf{e}_{k-1}$ for $k \geq 2$. Hence

$$\mathbf{D}_p \mathbf{S}_p \mathbf{e}_k = k \mathbf{e}_k \quad (k < p), \quad \mathbf{D}_p \mathbf{S}_p \mathbf{e}_p = \mathbf{0}, \quad \mathbf{S}_p \mathbf{D}_p \mathbf{e}_k = (k-1) \mathbf{e}_k \quad (1 \leq k \leq p),$$

where for $k=1$ we used $\mathbf{S}_p \mathbf{D}_p \mathbf{e}_1 = \mathbf{0} = 0 \mathbf{e}_1$. Subtracting, $[\mathbf{D}_p, \mathbf{S}_p] \mathbf{e}_k = \mathbf{e}_k$ for $k < p$, and $[\mathbf{D}_p, \mathbf{S}_p] \mathbf{e}_p = -(p-1)\mathbf{e}_p = (1-p)\mathbf{e}_p = \mathbf{e}_p$, because $p \cdot 1 = 0$ in F (Definition 0.7.2, Lemma 0.7.4). So every column of $[\mathbf{D}_p, \mathbf{S}_p]$ is the corresponding column of $\mathbf{I}_p$. For $p=2$ this is the pair of Exercise (C1: The Heisenberg relation) (d), transposed.

Warning. “The identity is never a commutator” is a statement about characteristic 0. The trace test says that a commutator has trace 0, and $\text{tr } \mathbf{I}_n = n \cdot 1$. Over $\mathbb{R}$ this is $n \neq 0$; over $\mathbb{F}_p$ it is 0 whenever p divides n , and then the test is silent. The example above shows that $\mathbf{I}_p$ really is a commutator over $\mathbb{F}_p$. Whenever an argument ends with “ $n \cdot 1 = 0$, contradiction”, check which fields it covers.

8.5.3 Trace-zero matrices with zero diagonal

We want to write a given trace-zero $\mathbf{A}$ as $[\mathbf{X}, \mathbf{Y}]$. Look first at the easiest choice of $\mathbf{X}$, a diagonal matrix $\mathbf{X} = \text{diag}(x_1, \dots, x_n)$. For any $\mathbf{Y} = (y_{ij})$, Definition 0.9.9 gives $(\mathbf{X}\mathbf{Y})_{ij} = x_i y_{ij}$ and $(\mathbf{Y}\mathbf{X})_{ij} = y_{ij} x_j$, so

$$[\mathbf{X}, \mathbf{Y}]_{ij} = (x_i - x_j) y_{ij}. \quad (*)$$

The diagonal entries of $[\mathbf{X}, \mathbf{Y}]$ are all 0, and each off-diagonal entry can be anything, provided $x_i \neq x_j$. So a diagonal $\mathbf{X}$ with distinct entries produces **exactly** the matrices with zero diagonal. A trace-zero matrix need not have zero diagonal, but by Proposition 8.5.1 (b) we may replace $\mathbf{A}$ by any matrix similar to it. The plan is therefore:

- ① find a similarity that makes the diagonal of $\mathbf{A}$ zero; ② solve $(*)$ entrywise.

Step ① is the heart of the matter. We do it one diagonal entry at a time, starting with the $(1, 1)$ -entry.

A matrix is **scalar** if it equals $\lambda \mathbf{I}_n$ for some $\lambda \in F$. A scalar matrix is similar only to itself, since $\mathbf{P}^{-1}(\lambda \mathbf{I}_n)\mathbf{P} = \lambda \mathbf{I}_n$, so its diagonal cannot be changed. Every other matrix can at least be given a zero in the corner. This first lemma holds over every field.

Lemma 8.5.2. A Non-Scalar Matrix Has a Similar Matrix with Zero Corner

Let F be any field, let $n \geq 2$, and let $\mathbf{A} \in M_n(F)$ be **not** scalar. Then there is an invertible

$\mathbf{P} \in M_n(F)$ such that the first column of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is $\mathbf{e}_2$. In particular

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & \mathbf{r}^\top \\ \mathbf{e}_1 & \mathbf{A}_1 \end{pmatrix}$$

for some $\mathbf{r} \in F^{n-1}$ and $\mathbf{A}_1 \in M_{n-1}(F)$ with $\text{tr } \mathbf{A}_1 = \text{tr } \mathbf{A}$, where $\mathbf{e}_1 \in F^{n-1}$.

Idea. Work backwards from the goal. The first column of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is the coordinate vector of $\mathbf{A}\mathbf{p}_1$ in the basis formed by the columns $\mathbf{p}_1, \dots, \mathbf{p}_n$ of $\mathbf{P}$. For it to be $\mathbf{e}_2$, we need $\mathbf{A}\mathbf{p}_1 = \mathbf{p}_2$. So we want a vector $\mathbf{v}$ such that $\mathbf{v}$ and $\mathbf{A}\mathbf{v}$ are independent, and then we build a basis starting $\mathbf{v}, \mathbf{A}\mathbf{v}$. Such a $\mathbf{v}$ exists unless $\mathbf{A}$ sends every vector to a multiple of itself, and a short argument shows that this forces $\mathbf{A}$ to be scalar.

Proof. We first show that there is $\mathbf{v} \in F^n$ with $(\mathbf{v}, \mathbf{A}\mathbf{v})$ linearly independent. Suppose not. Then for every $\mathbf{v} \neq \mathbf{0}$ there are $a, b \in F$, not both zero, with $a\mathbf{v} + b\mathbf{A}\mathbf{v} = \mathbf{0}$. If $b = 0$, then $a \neq 0$ and $a\mathbf{v} = \mathbf{0}$, which forces $\mathbf{v} = \mathbf{0}$ (Theorem 1.1.9), a contradiction; so $b \neq 0$ and $\mathbf{A}\mathbf{v} = \lambda_{\mathbf{v}}\mathbf{v}$ with $\lambda_{\mathbf{v}} = -a/b$. Apply this to $\mathbf{e}_i, \mathbf{e}_j$, and $\mathbf{e}_i + \mathbf{e}_j$ for $i \neq j$:

$$\lambda_{\mathbf{e}_i+\mathbf{e}_j}\mathbf{e}_i + \lambda_{\mathbf{e}_i+\mathbf{e}_j}\mathbf{e}_j = \mathbf{A}(\mathbf{e}_i + \mathbf{e}_j) = \mathbf{A}\mathbf{e}_i + \mathbf{A}\mathbf{e}_j = \lambda_{\mathbf{e}_i}\mathbf{e}_i + \lambda_{\mathbf{e}_j}\mathbf{e}_j.$$

Since $\mathbf{e}_i, \mathbf{e}_j$ are linearly independent, comparing coefficients (Theorem 1.4.3) gives $\lambda_{\mathbf{e}_i} = \lambda_{\mathbf{e}_i+\mathbf{e}_j} = \lambda_{\mathbf{e}_j}$. Hence all $\lambda_{\mathbf{e}_i}$ equal a common λ , so $\mathbf{A}\mathbf{e}_i = \lambda\mathbf{e}_i$ for every i , and by Theorem 0.9.2 the columns of $\mathbf{A}$ are those of $\lambda\mathbf{I}_n$. So $\mathbf{A} = \lambda\mathbf{I}_n$ is scalar, contrary to the hypothesis.

Take such a $\mathbf{v}$. By Theorem 1.6.5, the independent list $(\mathbf{v}, \mathbf{A}\mathbf{v})$ extends to a basis $(\mathbf{v}, \mathbf{A}\mathbf{v}, \mathbf{w}_3, \dots, \mathbf{w}_n)$ of F^n . Let $\mathbf{P}$ be the matrix with these columns; it is invertible by Theorem 3.4.4. Then $\mathbf{P}\mathbf{e}_1 = \mathbf{v}$ and $\mathbf{P}\mathbf{e}_2 = \mathbf{A}\mathbf{v}$ (Theorem 0.9.3), so the first column of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{e}_1 = \mathbf{P}^{-1}\mathbf{A}\mathbf{v} = \mathbf{P}^{-1}\mathbf{P}\mathbf{e}_2 = \mathbf{e}_2.$$

Partitioning $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ after the first row and column gives the displayed form. Finally, $\text{tr}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \text{tr } \mathbf{A}$ by Theorem 4.2.5, and $\text{tr}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = 0 + \text{tr } \mathbf{A}_1$ by Definition 0.9.12, so $\text{tr } \mathbf{A}_1 = \text{tr } \mathbf{A}$. ■

Quick check. Over $\mathbb{Q}$, let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$. Using $\mathbf{v} = \mathbf{e}_2$ in the proof of Lemma 8.5.2, find $\mathbf{P}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, and check that its diagonal is zero.

Solution. $\mathbf{A}\mathbf{e}_2 = (1, -2)$, which is not a multiple of $\mathbf{e}_2$, so $(\mathbf{e}_2, \mathbf{A}\mathbf{e}_2)$ is independent and already a basis of $\mathbb{Q}^2$. Then $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$, with $\det \mathbf{P} = -1$ and $\mathbf{P}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$. We get $\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$. The first column is $\mathbf{e}_2$, as the lemma promises; the $(2, 2)$ -entry is 0 as well, because the trace is 0 and trace is a similarity invariant.

The lemma controls one diagonal entry and hands us a smaller matrix $\mathbf{A}_1$ with the same trace. That is exactly the setting for induction on the size. The only thing that can stop the induction is a scalar $\mathbf{A}_1$, and here the trace saves us, provided the field has characteristic 0.

Lemma 8.5.3. Trace Zero Means Similar to Zero Diagonal

Let F be a field of **characteristic** 0, let $n \geq 1$, and let $\mathbf{A} \in M_n(F)$ with $\text{tr } \mathbf{A} = 0$. Then $\mathbf{A}$ is similar to a matrix all of whose diagonal entries are 0.

Idea. Induction on n . If $\mathbf{A}$ is scalar, $\mathbf{A} = \lambda \mathbf{I}_n$ with $n\lambda = 0$, and characteristic 0 forces $\lambda = 0$: then $\mathbf{A} = \mathbf{O}$ already has zero diagonal. Otherwise, Lemma 8.5.2 puts a 0 in the corner and leaves a trace-zero block $\mathbf{A}_1$ of size $n - 1$. The induction hypothesis conjugates $\mathbf{A}_1$ by some $\mathbf{Q}$ into zero diagonal. Conjugating the whole matrix by the block diagonal matrix $(1) \oplus \mathbf{Q}$ does this to the lower block without disturbing the corner.

Proof. We use induction on n (Theorem 0.2.7). For $n = 1$, $\mathbf{A} = (\text{tr } \mathbf{A}) = \mathbf{O}_1$, which has zero diagonal.

Let $n \geq 2$, and assume the statement for trace-zero matrices of size $n - 1$.

Case 1: $\mathbf{A}$ is scalar, say $\mathbf{A} = \lambda \mathbf{I}_n$. Then $0 = \text{tr } \mathbf{A} = n \cdot \lambda = (n \cdot 1)\lambda$ by Definition 0.9.12 and Lemma 0.7.4. Since F has characteristic 0 and $n \geq 1$, $n \cdot 1 \neq 0$ (Definition 0.7.2), so $\lambda = 0$. Hence $\mathbf{A} = \mathbf{O}$, which has zero diagonal and is similar to itself.

Case 2: $\mathbf{A}$ is not scalar. By Lemma 8.5.2 there is an invertible $\mathbf{P}$ with

$$\mathbf{M} := \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & \mathbf{r}^\top \\ \mathbf{e}_1 & \mathbf{A}_1 \end{pmatrix}, \quad \text{tr } \mathbf{A}_1 = \text{tr } \mathbf{A} = 0.$$

By the induction hypothesis there is an invertible $\mathbf{Q} \in M_{n-1}(F)$ such that $\mathbf{Z}_1 := \mathbf{Q}^{-1}\mathbf{A}_1\mathbf{Q}$ has zero diagonal. Let $\mathbf{R} = (1) \oplus \mathbf{Q} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{Q} \end{pmatrix}$. By Theorem 8.1.4, $\mathbf{R}$ is invertible with $\mathbf{R}^{-1} = (1) \oplus \mathbf{Q}^{-1}$. By Theorem 8.1.1,

$$\begin{aligned} \mathbf{R}^{-1}\mathbf{M}\mathbf{R} &= \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{Q}^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{r}^\top \\ \mathbf{e}_1 & \mathbf{A}_1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{Q} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{r}^\top \mathbf{Q} \\ \mathbf{Q}^{-1}\mathbf{e}_1 & \mathbf{Q}^{-1}\mathbf{A}_1\mathbf{Q} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{r}^\top \mathbf{Q} \\ \mathbf{Q}^{-1}\mathbf{e}_1 & \mathbf{Z}_1 \end{pmatrix}. \end{aligned}$$

Its $(1,1)$ -entry is 0, and its other diagonal entries are those of $\mathbf{Z}_1$, which are 0. Since $\mathbf{R}^{-1}\mathbf{M}\mathbf{R} = (\mathbf{P}\mathbf{R})^{-1}\mathbf{A}(\mathbf{P}\mathbf{R})$ by Theorem 0.9.7 (3), and $\mathbf{P}\mathbf{R}$ is invertible, $\mathbf{A}$ is similar to a matrix with zero diagonal. This completes the induction. ■

Remark. The characteristic was used only in Case 1, and only through $k \cdot 1 \neq 0$ for the sizes $k = 2, \dots, n$ that the induction passes through. So the lemma holds, with the same proof, whenever F has characteristic 0 or characteristic $p > n$. Some hypothesis on the

characteristic is needed: for $n = p$, over $\mathbb{F}_p$, the matrix $\mathbf{I}_p$ has trace $p \cdot 1 = 0$, but it is scalar, so it is similar only to itself, and its diagonal entries are 1. The same happens for $\lambda \mathbf{I}_n$ with $\lambda \neq 0$ whenever p divides n .

8.5.4 Shoda's theorem

With a zero diagonal available, the diagonal choice of $\mathbf{X}$ in (*) finishes the job. All we need are n distinct scalars for the diagonal of $\mathbf{X}$, and characteristic 0 provides them.

Theorem 8.5.4. Shoda's Theorem

Let F be a field of **characteristic** 0, let $n \geq 1$, and let $\mathbf{A} \in M_n(F)$. Then $\mathbf{A}$ is a commutator, that is $\mathbf{A} = [\mathbf{X}, \mathbf{Y}]$ for some $\mathbf{X}, \mathbf{Y} \in M_n(F)$, if and only if $\text{tr } \mathbf{A} = 0$.

Idea. One direction is [Proposition 8.5.1](#) (c). For the other: ① by [Lemma 8.5.3](#), $\mathbf{A} = \mathbf{PZP}^{-1}$ with $\mathbf{Z}$ of zero diagonal; ② take $\mathbf{X}_0 = \text{diag}(0, 1, \dots, n-1)$, whose entries are distinct in characteristic 0; ③ solve $[\mathbf{X}_0, \mathbf{Y}_0] = \mathbf{Z}$ entrywise with (*), dividing by $x_i - x_j \neq 0$; ④ conjugate back with [Proposition 8.5.1](#) (b).

Proof. ($\Rightarrow$) If $\mathbf{A} = [\mathbf{X}, \mathbf{Y}]$, then $\text{tr } \mathbf{A} = 0$ by [Proposition 8.5.1](#) (c).

($\Leftarrow$) Suppose $\text{tr } \mathbf{A} = 0$. By [Lemma 8.5.3](#) there is an invertible $\mathbf{P}$ such that $\mathbf{Z} = (z_{ij}) := \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ has $z_{ii} = 0$ for all i .

For $1 \leq i \leq n$ let $x_i := (i-1) \cdot 1 \in F$, and $\mathbf{X}_0 := \text{diag}(x_1, \dots, x_n)$. For $i < j$, [Lemma 0.7.4](#) gives $x_j - x_i = (j-i) \cdot 1$, which is nonzero because $j-i \geq 1$ and F has characteristic 0 ([Definition 0.7.2](#)). So the x_i are distinct, and we may define $\mathbf{Y}_0 = (y_{ij}) \in M_n(F)$ by

$$y_{ij} := \frac{z_{ij}}{x_i - x_j} \quad (i \neq j), \quad y_{ii} := 0.$$

By (*), $[\mathbf{X}_0, \mathbf{Y}_0]_{ij} = (x_i - x_j)y_{ij} = z_{ij}$ for $i \neq j$, and $[\mathbf{X}_0, \mathbf{Y}_0]_{ii} = 0 = z_{ii}$. Hence $[\mathbf{X}_0, \mathbf{Y}_0] = \mathbf{Z}$.

Let $\mathbf{X} = \mathbf{P}\mathbf{X}_0\mathbf{P}^{-1}$ and $\mathbf{Y} = \mathbf{P}\mathbf{Y}_0\mathbf{P}^{-1}$. Applying [Proposition 8.5.1](#) (b) with the invertible matrix $\mathbf{P}^{-1}$,

$$[\mathbf{X}, \mathbf{Y}] = [\mathbf{P}\mathbf{X}_0\mathbf{P}^{-1}, \mathbf{P}\mathbf{Y}_0\mathbf{P}^{-1}] = \mathbf{P}[\mathbf{X}_0, \mathbf{Y}_0]\mathbf{P}^{-1} = \mathbf{PZP}^{-1} = \mathbf{A}.$$

This proves that $\mathbf{A}$ is a commutator. ■

The proof used characteristic 0 twice: in the lemma, to rule out nonzero scalar blocks of trace zero, and here, to find n distinct diagonal entries. The second use needs only $(j-i) \cdot 1 \neq 0$ for $1 \leq j-i \leq n-1$, and with the remark after [Lemma 8.5.3](#), the same proof works whenever the characteristic of F is 0 or larger than n . Neither use is essential for the **theorem**. In fact the statement “a square matrix over a field is a commutator if and only if its trace is 0” is true over **every** field. The proof in positive characteristic needs a different normal form and more work, and we do not give it. [Example \(The Identity as a Commutator in Characteristic \$p\$ \)](#) is a small

instance: $\mathbf{I}_p$ over $\mathbb{F}_p$ is a commutator, although [Lemma 8.5.3](#) fails for it.

The proof is constructive, and it is worth running once in full.

Example. Writing a Trace-Zero 3×3 Matrix as a Commutator

Over $\mathbb{Q}$, write $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$ as a commutator $[\mathbf{X}, \mathbf{Y}]$.

Solution. The trace is $1 + 1 - 2 = 0$. We follow the proofs step by step.

First corner. $\mathbf{A}$ is not scalar. Take $\mathbf{v} = \mathbf{e}_1$; then $\mathbf{A}\mathbf{v} = (1, 0, 1)$ is not a multiple of $\mathbf{e}_1$. Extend $(\mathbf{e}_1, \mathbf{A}\mathbf{e}_1)$ by $\mathbf{e}_2$, which is not in their span, to a basis, and let

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{M} = \mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(To check $\mathbf{M}$ without inverting, verify $\mathbf{P}_1\mathbf{M} = \mathbf{A}\mathbf{P}_1$: both equal $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.) The first

column is $\mathbf{e}_2$, and the lower block is $\mathbf{A}_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, with trace 0.

Second corner. $\mathbf{A}_1$ is not scalar. With $\mathbf{w} = \mathbf{e}_1 \in \mathbb{Q}^2$, $\mathbf{A}_1\mathbf{w} = (-1, 1)$ is independent of $\mathbf{e}_1$, so

$$\mathbf{Q} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Z}_1 = \mathbf{Q}^{-1}\mathbf{A}_1\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assemble. With $\mathbf{r}^\top = (2, 2)$, the proof of [Lemma 8.5.3](#) gives $\mathbf{r}^\top\mathbf{Q} = (2, 0)$ and $\mathbf{Q}^{-1}\mathbf{e}_1 = \mathbf{e}_1$, so for $\mathbf{P} = \mathbf{P}_1((1) \oplus \mathbf{Q})$,

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{Z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solve entrywise. With $\mathbf{X}_0 = \text{diag}(0, 1, 2)$, the rule $y_{ij} = z_{ij}/(x_i - x_j)$ gives $y_{12} = \frac{2}{-1} = -2$, $y_{21} = \frac{1}{1} = 1$, $y_{23} = \frac{1}{-1} = -1$, $y_{32} = \frac{1}{1} = 1$, and $y_{13} = y_{31} = 0$:

$$\mathbf{Y}_0 = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Conjugate back. $\mathbf{X} = \mathbf{P}\mathbf{X}_0\mathbf{P}^{-1}$ and $\mathbf{Y} = \mathbf{P}\mathbf{Y}_0\mathbf{P}^{-1}$ are

$$\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix}.$$

Check: $\mathbf{XY} = \begin{pmatrix} 1 & -3 & -3 \\ 0 & 2 & 2 \\ 1 & -3 & -3 \end{pmatrix}$ and $\mathbf{YX} = \begin{pmatrix} 0 & -5 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{pmatrix}$, and $\mathbf{XY} - \mathbf{YX} = \mathbf{A}$.

Nothing about these $\mathbf{X}$ and $\mathbf{Y}$ is unique. Adding to $\mathbf{Y}$ any matrix that commutes with $\mathbf{X}$, for instance a polynomial in $\mathbf{X}$, gives another solution, by [Proposition 8.5.1](#) (a).

8.5.5 The span of the commutators

Shoda's theorem is about single commutators. A weaker question, with an easier answer that holds over **every** field, is which matrices are **sums** of commutators. Since the trace-zero matrices form a subspace and contain all commutators, the span of the commutators lies inside it. Matrix units show that it is all of it.

Theorem 8.5.5. Commutators Span the Trace-Zero Matrices

Let F be any field and $n \geq 1$. The span of all commutators $[\mathbf{X}, \mathbf{Y}]$ with $\mathbf{X}, \mathbf{Y} \in M_n(F)$ is the subspace $\{\mathbf{A} \in M_n(F) : \text{tr } \mathbf{A} = 0\}$. More precisely, every trace-zero matrix is a linear combination of the commutators

$$\mathbf{E}_{ij} = [\mathbf{E}_{ii}, \mathbf{E}_{ij}] \quad (i \neq j) \quad \text{and} \quad \mathbf{E}_{ii} - \mathbf{E}_{nn} = [\mathbf{E}_{in}, \mathbf{E}_{ni}] \quad (1 \leq i \leq n-1).$$

Proof. Let $W = \{\mathbf{A} \in M_n(F) : \text{tr } \mathbf{A} = 0\}$, a subspace by Example (Trace-Free Matrices). By [Proposition 8.5.1](#) (c), every commutator lies in W , so their span is contained in W (Theorem 1.3.1, the span being the smallest subspace containing them).

For the reverse inclusion, the displayed identities hold by Example 2 after [Definition 8.5.1](#): for $i \neq j$, $[\mathbf{E}_{ii}, \mathbf{E}_{ij}] = \mathbf{E}_{ij}$, and for $i < n$, $[\mathbf{E}_{in}, \mathbf{E}_{ni}] = \mathbf{E}_{ii} - \mathbf{E}_{nn}$. Now let $\mathbf{A} = (a_{ij}) \in W$. Writing $\mathbf{A}$ in the basis of matrix units (Example (Standard Bases)) and using $a_{nn} = -\sum_{i < n} a_{ii}$,

$$\mathbf{A} = \sum_{i \neq j} a_{ij} \mathbf{E}_{ij} + \sum_{i=1}^{n-1} a_{ii} \mathbf{E}_{ii} + a_{nn} \mathbf{E}_{nn} = \sum_{i \neq j} a_{ij} \mathbf{E}_{ij} + \sum_{i=1}^{n-1} a_{ii} (\mathbf{E}_{ii} - \mathbf{E}_{nn}).$$

So $\mathbf{A}$ is a linear combination of commutators, hence lies in their span. (For $n = 1$, $W = \{\mathbf{O}\}$ and both sides are $\{\mathbf{O}\}$.)

■

The theorem lists $n(n-1) + (n-1) = n^2 - 1$ commutators spanning the trace-zero matrices. Since $\text{tr}: M_n(F) \rightarrow F$ is surjective ($\text{tr}(c\mathbf{E}_{11}) = c$), the Rank–Nullity Theorem (Theorem 2.3.1) gives that its kernel has dimension $n^2 - 1$, so by Theorem 1.6.4 (b) these commutators form a basis of it. Over a field of characteristic 0, Shoda's theorem says much more: the sum of commutators produced here can always be collapsed into a single commutator. In general the

set of commutators is not obviously closed under addition, and that is exactly what makes Shoda's theorem a theorem.

Warning. **The span of a set is not the set.** [Theorem 8.5.5](#) shows that every trace-zero matrix is a **sum** of commutators, over any field, by a two-line argument. It does not show that every trace-zero matrix **is** a commutator. For comparison, in $M_2(F)$ the matrices of rank at most 1 span everything (the four matrix units have rank 1), yet $\mathbf{I}_2$ has rank 2: a set can span a subspace without filling it. For commutators it happens that the set does fill the subspace, but the proof needs the similarity argument of [Lemma 8.5.3](#), not just matrix units.

The span theorem gives a clean proof of a fact from Chapter 4: a linear functional φ on $M_n(F)$ with $\varphi(\mathbf{AB}) = \varphi(\mathbf{BA})$ is a multiple of the trace. Such a φ kills every commutator, hence their span, which is the kernel of the trace; [Exercise \(C1: Trace-like functionals, again\)](#) below completes the argument.

8.5.6 Exercises

8.5.6.1 A. Check your understanding

Exercise. A1

- Define the commutator $[\mathbf{A}, \mathbf{B}]$, and state what it means for a matrix to be a commutator.
- State Shoda's theorem, including the hypothesis on the field.
- True or false: $[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{A}]$ for all $\mathbf{A}, \mathbf{B} \in M_n(F)$. Justify your answer.
- True or false: over every field, every trace-zero matrix is similar to a matrix with zero diagonal. Justify your answer.
- Name the two places where the proof of Shoda's theorem uses characteristic 0.
- Explain why [Theorem 8.5.5](#) does not by itself prove Shoda's theorem.

Solution.

- $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ ([Definition 8.5.1](#)). A matrix $\mathbf{C} \in M_n(F)$ is a commutator if $\mathbf{C} = [\mathbf{X}, \mathbf{Y}]$ for some $\mathbf{X}, \mathbf{Y} \in M_n(F)$.
- If F has characteristic 0 and $\mathbf{A} \in M_n(F)$, then $\mathbf{A} = [\mathbf{X}, \mathbf{Y}]$ for some $\mathbf{X}, \mathbf{Y} \in M_n(F)$ if and only if $\text{tr } \mathbf{A} = 0$ ([Theorem 8.5.4](#)).
- False. $[\mathbf{B}, \mathbf{A}] = -[\mathbf{A}, \mathbf{B}]$ ([Proposition 8.5.1 \(a\)](#)), so equality holds only when $[\mathbf{A}, \mathbf{B}] = \mathbf{O}$. For example $[\mathbf{E}_{12}, \mathbf{E}_{21}] = \text{diag}(1, -1)$ and $[\mathbf{E}_{21}, \mathbf{E}_{12}] = \text{diag}(-1, 1)$ over $\mathbb{Q}$.
- False. Over $\mathbb{F}_2$, $\mathbf{I}_2$ has trace $1 + 1 = 0$, but it is scalar, so every matrix similar to it equals $\mathbf{P}^{-1}\mathbf{I}_2\mathbf{P} = \mathbf{I}_2$, whose diagonal entries are 1.
- In [Lemma 8.5.3](#), to conclude from $(n \cdot 1)\lambda = 0$ that a trace-zero scalar matrix $\lambda\mathbf{I}_n$ is $\mathbf{O}$; and in [Theorem 8.5.4](#), to find n distinct scalars $x_i = (i - 1) \cdot 1$ so that we can divide by $x_i - x_j$.
- It says every trace-zero matrix is a **linear combination** of commutators. Shoda's

theorem says it is a **single** commutator. A sum of commutators need not visibly be a commutator, since $[\mathbf{X}, \mathbf{Y}] + [\mathbf{X}', \mathbf{Y}']$ has no evident expression as $[\mathbf{X}'', \mathbf{Y}'']$.

8.5.6.2 B. Practice

Exercise. B1: A 2×2 commutator

Over $\mathbb{Q}$, let $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & -3 \end{pmatrix}$.

- (a) Using $\mathbf{v} = \mathbf{e}_1$ as in Lemma 8.5.2, find an invertible $\mathbf{P}$ such that $\mathbf{Z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ has zero diagonal.
- (b) With $\mathbf{X}_0 = \text{diag}(0, 1)$, find $\mathbf{Y}_0$ with $[\mathbf{X}_0, \mathbf{Y}_0] = \mathbf{Z}$.
- (c) Hence write $\mathbf{A} = [\mathbf{X}, \mathbf{Y}]$, and check your answer by multiplying out.

Solution.

- (a) $\text{tr } \mathbf{A} = 0$, and $\mathbf{A}\mathbf{e}_1 = (3, 1)$ is not a multiple of $\mathbf{e}_1$, so $(\mathbf{e}_1, \mathbf{A}\mathbf{e}_1)$ is a basis of $\mathbb{Q}^2$. Let $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, so $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 7 \\ 1 & 0 \end{pmatrix}$ and

$$\mathbf{Z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 7 \\ 1 & 0 \end{pmatrix}.$$

The first column is $\mathbf{e}_2$ by Lemma 8.5.2, and the $(2, 2)$ -entry is 0 because the trace is 0.

- (b) By (*), $[\mathbf{X}_0, \mathbf{Y}]_{12} = (0 - 1)y_{12}$ and $[\mathbf{X}_0, \mathbf{Y}]_{21} = (1 - 0)y_{21}$, while the diagonal of $[\mathbf{X}_0, \mathbf{Y}]$ is zero. So $y_{12} = -7$, $y_{21} = 1$, and we may take $y_{11} = y_{22} = 0$: $\mathbf{Y}_0 = \begin{pmatrix} 0 & -7 \\ 1 & 0 \end{pmatrix}$.

- (c) By the proof of Theorem 8.5.4, $\mathbf{X} = \mathbf{P}\mathbf{X}_0\mathbf{P}^{-1}$ and $\mathbf{Y} = \mathbf{P}\mathbf{Y}_0\mathbf{P}^{-1}$:

$$\mathbf{X} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 3 & -7 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -16 \\ 1 & -3 \end{pmatrix},$$

using $\mathbf{P}\mathbf{X}_0 = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{P}\mathbf{Y}_0 = \begin{pmatrix} 3 & -7 \\ 1 & 0 \end{pmatrix}$. Check: $\mathbf{X}\mathbf{Y} = \begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix}$, $\mathbf{Y}\mathbf{X} = \begin{pmatrix} 0 & -7 \\ 0 & 0 \end{pmatrix}$, and $\mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & -3 \end{pmatrix} = \mathbf{A}$.

Exercise. B2: A commutator over $\mathbb{F}_2$

Work over $\mathbb{F}_2$, and let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

- (a) Show that $\text{tr } \mathbf{A} = 0$ and that $\mathbf{A}$ is not scalar.
- (b) Explain why $\mathbf{v} = \mathbf{e}_1$ cannot be used in Lemma 8.5.2, and use $\mathbf{v} = \mathbf{e}_2$ instead to find $\mathbf{P}$ with $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ of zero diagonal.

- (c) The field $\mathbb{F}_2$ has exactly two elements. Use them as the diagonal of $\mathbf{X}_0$ to write $\mathbf{A}$ as a commutator, by hand.
- (d) Explain why the same method cannot write $\mathbf{I}_2 \in M_2(\mathbb{F}_2)$ as a commutator, and recall a pair $\mathbf{X}, \mathbf{Y}$ that does.

Solution.

- (a) $\text{tr } \mathbf{A} = 1 + 1 = 0$ in $\mathbb{F}_2$. $\mathbf{A}$ has a nonzero off-diagonal entry, while every scalar matrix has zero off-diagonal entries.
- (b) $\mathbf{A}\mathbf{e}_1 = \mathbf{e}_1$, so $(\mathbf{e}_1, \mathbf{A}\mathbf{e}_1)$ is dependent. With $\mathbf{v} = \mathbf{e}_2$, $\mathbf{A}\mathbf{e}_2 = (1, 1)$, which is not a multiple of $\mathbf{e}_2$, so $(\mathbf{e}_2, \mathbf{A}\mathbf{e}_2)$ is a basis. Let $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$; since $\det \mathbf{P} = -1 = 1$, $\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ (check: $\mathbf{P}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \mathbf{I}_2$ in $\mathbb{F}_2$). Then $\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{Z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- (c) Take $\mathbf{X}_0 = \text{diag}(0, 1)$. By (*), $[\mathbf{X}_0, \mathbf{Y}]_{12} = (0-1)y_{12} = y_{12}$ (as $-1 = 1$) and $[\mathbf{X}_0, \mathbf{Y}]_{21} = y_{21}$. So $\mathbf{Y}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives $[\mathbf{X}_0, \mathbf{Y}_0] = \mathbf{Z}$. Conjugating back, $\mathbf{P}\mathbf{X}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{P}\mathbf{Y}_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so

$$\mathbf{X} = \mathbf{P}\mathbf{X}_0\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} = \mathbf{P}\mathbf{Y}_0\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Check: $\mathbf{X}\mathbf{Y} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{Y}\mathbf{X} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so $[\mathbf{X}, \mathbf{Y}] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mathbf{A}$. (Here $\mathbf{Y} = \mathbf{A}$: indeed $[\mathbf{X}, \mathbf{A}] = \mathbf{A}$.)

- (d) The method writes $\mathbf{Z} = \mathbf{P}^{-1}\mathbf{I}_2\mathbf{P}$ as $[\mathbf{X}_0, \mathbf{Y}_0]$ with $\mathbf{X}_0$ diagonal, and by (*) such a commutator has zero diagonal. But $\mathbf{I}_2$ is scalar, so $\mathbf{P}^{-1}\mathbf{I}_2\mathbf{P} = \mathbf{I}_2$ for every invertible $\mathbf{P}$, and its diagonal entries are 1. So the method cannot start. Nevertheless $\mathbf{I}_2$ is a commutator: with $\mathbf{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $[\mathbf{X}, \mathbf{Y}] = \text{diag}(-1, 1) = \mathbf{I}_2$ in $\mathbb{F}_2$ (Exercise (C1: The Heisenberg relation) (d), or [Example \(The Identity as a Commutator in Characteristic \$p\$ \)](#) with $p = 2$).

Exercise. B3: Commuting with a diagonal matrix

Let F be any field, let $d_1, \dots, d_n \in F$ be **distinct**, let $\mathbf{D} = \text{diag}(d_1, \dots, d_n)$, and define $L: M_n(F) \rightarrow M_n(F)$ by $L(\mathbf{X}) = [\mathbf{D}, \mathbf{X}]$.

- (a) Show that L is linear.
- (b) Prove that $\ker L$ is the set of diagonal matrices and $\text{im } L$ is the set of matrices with zero diagonal.
- (c) Check that your answer to (b) agrees with the Rank–Nullity Theorem.

- (d) Hence decide whether $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} \in M_3(\mathbb{Q})$ is a commutator with $\mathbf{X} = \text{diag}(1, 2, 3)$, and if so find $\mathbf{Y}$.

Solution.

- (a) By [Proposition 8.5.1](#) (a), $[\mathbf{D}, \mathbf{X} + \mathbf{X}'] = [\mathbf{D}, \mathbf{X}] + [\mathbf{D}, \mathbf{X}']$ and $[\mathbf{D}, c\mathbf{X}] = c[\mathbf{D}, \mathbf{X}]$.
 (b) By (*), $L(\mathbf{X})_{ij} = (d_i - d_j)x_{ij}$. Since $d_i - d_j \neq 0$ for $i \neq j$, $L(\mathbf{X}) = \mathbf{O}$ if and only if $x_{ij} = 0$ for all $i \neq j$, that is, $\mathbf{X}$ is diagonal. Every $L(\mathbf{X})$ has zero diagonal. Conversely, if $\mathbf{Z}$ has zero diagonal, the matrix $\mathbf{Y}$ with $y_{ij} = z_{ij}/(d_i - d_j)$ for $i \neq j$ and $y_{ii} = 0$ satisfies $L(\mathbf{Y}) = \mathbf{Z}$. So $\text{im } L$ is exactly the set of zero-diagonal matrices.
 (c) The diagonal matrices have basis $\mathbf{E}_{11}, \dots, \mathbf{E}_{nn}$, so nullity $L = n$. The zero-diagonal matrices have basis $\mathbf{E}_{ij}$ ($i \neq j$), so $\text{rank } L = n^2 - n$. Their sum is $n^2 = \dim M_n(F)$, as Theorem 2.3.1 requires.
 (d) Yes, since the matrix has zero diagonal and 1, 2, 3 are distinct. With $d = (1, 2, 3)$, $y_{ij} = z_{ij}/(d_i - d_j)$ gives

$$\mathbf{Y} = \begin{pmatrix} 0 & \frac{1}{-1} & \frac{2}{-2} \\ \frac{3}{1} & 0 & \frac{4}{-1} \\ \frac{5}{2} & \frac{6}{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 3 & 0 & -4 \\ \frac{5}{2} & 6 & 0 \end{pmatrix},$$

and by (b), $[\text{diag}(1, 2, 3), \mathbf{Y}]$ is the given matrix.

8.5.6.3 C. Going deeper

Exercise. C1: Trace-like functionals, again

Let F be any field, $n \geq 1$, and let $\varphi: M_n(F) \rightarrow F$ be linear with $\varphi(\mathbf{AB}) = \varphi(\mathbf{BA})$ for all $\mathbf{A}, \mathbf{B} \in M_n(F)$.

- (a) Prove that φ vanishes on every trace-zero matrix.
 (b) Deduce that $\varphi = c \text{tr}$, where $c = \varphi(\mathbf{E}_{11})$.
 (c) Suppose F has characteristic 0 and, in addition, $\varphi(\mathbf{I}_n) = n \cdot 1$. Deduce that $\varphi = \text{tr}$.
 (d) Show that the conclusion of (c) fails over $\mathbb{F}_p$ when $n = p$.

Hint: for (a), use [Theorem 8.5.5](#).

Solution.

- (a) For all $\mathbf{X}, \mathbf{Y}$, $\varphi([\mathbf{X}, \mathbf{Y}]) = \varphi(\mathbf{XY}) - \varphi(\mathbf{YX}) = 0$ by linearity and the hypothesis. So the kernel of φ , a subspace, contains every commutator, hence their span (Theorem 1.3.1), which is the set of trace-zero matrices by [Theorem 8.5.5](#).
 (b) Let $\mathbf{A} \in M_n(F)$. Since $\text{tr } \mathbf{E}_{11} = 1$, the matrix $\mathbf{A} - (\text{tr } \mathbf{A})\mathbf{E}_{11}$ has trace 0 (Theorem 0.9.6 (1)), so by (a) and linearity,

$$0 = \varphi(\mathbf{A} - (\text{tr } \mathbf{A})\mathbf{E}_{11}) = \varphi(\mathbf{A}) - (\text{tr } \mathbf{A})\varphi(\mathbf{E}_{11}).$$

Hence $\varphi(\mathbf{A}) = c \text{tr } \mathbf{A}$ with $c = \varphi(\mathbf{E}_{11})$. This recovers Exercise (C2: The trace is the

only trace-like functional) by a different route: the matrix-unit computations are now packaged in [Theorem 8.5.5](#).

- (c) By (b), $n \cdot 1 = \varphi(\mathbf{I}_n) = c \operatorname{tr} \mathbf{I}_n = c(n \cdot 1)$. Since F has characteristic 0, $n \cdot 1 \neq 0$, so $c = 1$ and $\varphi = \operatorname{tr}$.
- (d) Over $\mathbb{F}_p$ with $n = p$, take $\varphi = 0$. It is linear, satisfies $\varphi(\mathbf{AB}) = 0 = \varphi(\mathbf{BA})$, and $\varphi(\mathbf{I}_p) = 0 = p \cdot 1$. But $\varphi \neq \operatorname{tr}$, since $\operatorname{tr} \mathbf{E}_{11} = 1$. The step that fails is canceling $n \cdot 1 = 0$ in (c).

Exercise. C2: When $\mathbf{A}$ commutes with $[\mathbf{A}, \mathbf{B}]$

Let F be a field of characteristic 0, let $\mathbf{A}, \mathbf{B} \in M_n(F)$, put $\mathbf{C} = [\mathbf{A}, \mathbf{B}]$, and suppose that $\mathbf{AC} = \mathbf{CA}$. The goal is to prove that $\mathbf{C}$ is **nilpotent**, that is, $\mathbf{C}^s = \mathbf{O}$ for some $s \geq 1$.

- (a) Prove that $\mathbf{C}^k = [\mathbf{A}, \mathbf{BC}^{k-1}]$ for every $k \geq 1$. Deduce that $\operatorname{tr} \mathbf{C}^k = 0$ for every $k \geq 1$.
- (b) Show that there are an integer $s \geq 1$ and $g \in F[x]$ with $g(0) \neq 0$ such that $\mathbf{C}^s g(\mathbf{C}) = \mathbf{O}$.
- (c) Let $U = \operatorname{col}(\mathbf{C}^s)$ and $r = \dim U$, and suppose $r \geq 1$. Extend a basis of U to a basis of F^n , and let $\mathbf{P}$ be the matrix with these columns. Show that

$$\mathbf{P}^{-1} \mathbf{C} \mathbf{P} = \begin{pmatrix} \mathbf{D} & * \\ \mathbf{O} & \mathbf{N} \end{pmatrix}, \quad \mathbf{D} \in M_r(F),$$

with $\mathbf{N}^s = \mathbf{O}$ and $g(\mathbf{D}) = \mathbf{O}$.

- (d) Write $g = g_0 + xh$ with $g_0 = g(0) \neq 0$. Show that $\mathbf{I}_r$ is a linear combination of $\mathbf{D}^s, \mathbf{D}^{s+1}, \dots, \mathbf{D}^d$ for some $d \geq s$, and use (a) to reach a contradiction.
- (e) Conclude that $\mathbf{C}^s = \mathbf{O}$.

Hint: in (c), compare the columns of $\mathbf{P}^{-1} \mathbf{C}^k \mathbf{P}$ for $k = 1$ and $k = s$ with the coordinates of vectors of U ; in (d), take traces.

Solution.

- (a) First, $\mathbf{C}^{k-1} \mathbf{A} = \mathbf{AC}^{k-1}$ for all $k \geq 1$, by induction on k from $\mathbf{CA} = \mathbf{AC}$. Hence

$$\begin{aligned} [\mathbf{A}, \mathbf{BC}^{k-1}] &= \mathbf{ABC}^{k-1} - \mathbf{BC}^{k-1} \mathbf{A} = \mathbf{ABC}^{k-1} - \mathbf{BAC}^{k-1} \\ &= (\mathbf{AB} - \mathbf{BA}) \mathbf{C}^{k-1} = \mathbf{C}^k. \end{aligned}$$

By [Proposition 8.5.1](#) (c), $\operatorname{tr} \mathbf{C}^k = 0$.

- (b) By [Theorem 6.7.2](#) there is a nonzero $f \in F[x]$ with $f(\mathbf{C}) = \mathbf{O}$. Let $t \geq 0$ be the largest integer with x^t dividing f , and write $f = x^t g$; then $x \nmid g$, so $g(0) \neq 0$ by [Theorem 6.1.3](#). Put $s = t + 1 \geq 1$. By [Theorem 0.9.9](#), $\mathbf{C}^s g(\mathbf{C}) = \mathbf{C} f(\mathbf{C}) = \mathbf{O}$.

- (c) Let $\mathbf{P}$ have columns $\mathbf{p}_1, \dots, \mathbf{p}_n$, where $(\mathbf{p}_1, \dots, \mathbf{p}_r)$ is a basis of U ; $\mathbf{P}$ is invertible by [Theorem 3.4.4](#). For any $k \geq 1$, column j of $\mathbf{P}^{-1} \mathbf{C}^k \mathbf{P}$ is $\mathbf{P}^{-1}(\mathbf{C}^k \mathbf{p}_j)$, the vector $\mathbf{c}$ with $\mathbf{Pc} = \mathbf{C}^k \mathbf{p}_j$, that is, the coefficients of $\mathbf{C}^k \mathbf{p}_j$ in terms of $\mathbf{p}_1, \dots, \mathbf{p}_n$ ([Theorem 0.9.3](#)). If $\mathbf{y} \in U$, then $\mathbf{y} = \mathbf{C}^s \mathbf{z}$ and $\mathbf{Cy} = \mathbf{C}^s(\mathbf{Cz}) \in U$, so $\mathbf{C}$ maps U into U ; hence for $j \leq r$ the vector $\mathbf{Cp}_j$ is a combination of $\mathbf{p}_1, \dots, \mathbf{p}_r$, and column j of $\mathbf{P}^{-1} \mathbf{CP}$ has zeros below row r . This gives the block form. By [Theorem 8.1.1](#) and induction on k , $(\mathbf{P}^{-1} \mathbf{C} \mathbf{P})^k = \begin{pmatrix} \mathbf{D}^k & * \\ \mathbf{O} & \mathbf{N}^k \end{pmatrix}$, and by [Proposition 4.2.6](#) (c) this is $\mathbf{P}^{-1} \mathbf{C}^k \mathbf{P}$; more generally

$$\mathbf{P}^{-1}q(\mathbf{C})\mathbf{P} = \begin{pmatrix} q(\mathbf{D}) & * \\ \mathbf{O} & q(\mathbf{N}) \end{pmatrix} \text{ for every } q \in F[x].$$

For $k = s$, every vector $\mathbf{C}^s \mathbf{p}_j$ lies in U , so every column of $\mathbf{P}^{-1} \mathbf{C}^s \mathbf{P}$ is zero below row r , and $\mathbf{N}^s = \mathbf{O}$. For $q = g$: for $j \leq r$, $\mathbf{p}_j \in U$ is $\mathbf{C}^s \mathbf{z}$ for some $\mathbf{z}$, so $g(\mathbf{C}) \mathbf{p}_j = g(\mathbf{C}) \mathbf{C}^s \mathbf{z} = \mathbf{O}$ by (b) and Theorem 0.9.9. So the first r columns of $\mathbf{P}^{-1} g(\mathbf{C}) \mathbf{P}$ are zero, and in particular $g(\mathbf{D}) = \mathbf{O}$.

- (d) From $g(\mathbf{D}) = g_0 \mathbf{I}_r + \mathbf{D} h_1(\mathbf{D}) = \mathbf{O}$ we get $\mathbf{I}_r = \mathbf{D} h_1(\mathbf{D})$ with $h_1 = -g_0^{-1} h$. Since polynomials in $\mathbf{D}$ commute (Theorem 0.9.9),

$$\mathbf{I}_r = (\mathbf{D} h_1(\mathbf{D}))^s = \mathbf{D}^s h_1(\mathbf{D})^s = \sum_{k=s}^d c_k \mathbf{D}^k$$

for some $c_k \in F$, as every term of $x^s h_1(x)^s$ has degree at least s . For $k \geq s$, $\mathbf{N}^k = \mathbf{N}^s \mathbf{N}^{k-s} = \mathbf{O}$, so by (c), Theorem 4.2.5, and Definition 0.9.12, $\text{tr } \mathbf{C}^k = \text{tr}(\mathbf{P}^{-1} \mathbf{C}^k \mathbf{P}) = \text{tr } \mathbf{D}^k + \text{tr } \mathbf{N}^k = \text{tr } \mathbf{D}^k$. Taking traces,

$$r \cdot 1 = \text{tr } \mathbf{I}_r = \sum_{k=s}^d c_k \text{tr } \mathbf{D}^k = \sum_{k=s}^d c_k \text{tr } \mathbf{C}^k = 0$$

by (a). But $r \geq 1$ and F has characteristic 0, so $r \cdot 1 \neq 0$, a contradiction.

- (e) By (d), $r = 0$, so $\text{col}(\mathbf{C}^s) = \{\mathbf{0}\}$ and $\mathbf{C}^s = \mathbf{O}$. For example, with $\mathbf{A} = \mathbf{E}_{12} + \mathbf{E}_{23}$ and $\mathbf{B} = \text{diag}(0, 1, 2)$ in $M_3(\mathbb{Q})$, $\mathbf{C} = [\mathbf{A}, \mathbf{B}] = \mathbf{E}_{12} + \mathbf{E}_{23} = \mathbf{A}$, which commutes with $\mathbf{A}$, and indeed $\mathbf{C}^3 = \mathbf{O}$. In Chapter 9 the statement “ $\text{tr } \mathbf{C}^k = 0$ for all $k \geq 1$ forces nilpotency in characteristic 0” becomes a statement about eigenvalues.