

5.5 Duality for Subspaces and Quotients

A subspace U of V gives two new vector spaces: U itself, and the quotient V/U of Chapter 4. Each has a dual space, and so far U^* and $(V/U)^*$ have nothing to do with V^* . This section connects them. The answer is a swap: the dual of the quotient is a **subspace** of V^* , namely the annihilator U^0 , and the dual of the subspace is a **quotient** of V^* , namely V^*/U^0 . Both identifications come from the First Isomorphism Theorem, and neither needs a basis. At the end we use them to describe hyperplanes, the subspaces cut out by a single equation, in any dimension.

5.5.1 Restricting functionals to a subspace

Let U be a subspace of V . A functional $\varphi \in V^*$ measures every vector of V , so in particular it measures the vectors of U . Forgetting the rest gives the **restriction** $\varphi|_U: U \rightarrow F$, $u \mapsto \varphi(u)$, which is a functional on U . This defines the restriction map

$$\rho: V^* \rightarrow U^*, \quad \rho(\varphi) = \varphi|_U.$$

It is not a new kind of map. Let $\iota: U \rightarrow V$, $\iota(u) = u$, be the inclusion. Its dual map (Definition 5.4.1) sends φ to $\varphi \circ \iota$, and $(\varphi \circ \iota)(u) = \varphi(u)$. So $\rho = \iota'$, and in particular ρ is linear.

Two questions decide what ρ does. Which functionals restrict to zero? By Definition 5.2.1, exactly those in U^0 . And does every functional on U come from one on V ? That is an extension problem, and it has a positive answer in every dimension.

Theorem 5.5.1. Dual of a Subspace

Let V be a vector space over F , of any dimension, and let U be a subspace of V . The restriction map $\rho: V^* \rightarrow U^*$, $\rho(\varphi) = \varphi|_U$, is linear and surjective, and $\ker \rho = U^0$. Hence

$$V^*/U^0 \rightarrow U^*, \quad \varphi + U^0 \mapsto \varphi|_U$$

is a well-defined isomorphism, and $U^* \cong V^*/U^0$.

Idea. The kernel is the definition of U^0 read aloud. The real content is surjectivity: given ψ on U , we must extend it to all of V . A basis tells us how. Extend a basis of U to a basis of V ; a linear map may be prescribed freely on a basis, so declare the extension to agree with ψ on the basis of U and to be 0 on the new basis vectors. Then the First Isomorphism Theorem turns “surjective with kernel U^0 ” into the isomorphism.

Proof. We saw above that $\rho = \iota'$ is linear. For $\varphi \in V^*$, $\rho(\varphi) = 0$ means $\varphi(u) = 0$ for every $u \in U$, which is $\varphi \in U^0$ by Definition 5.2.1. So $\ker \rho = U^0$.

Surjective. Let $\psi \in U^*$. By Theorem 1.8.3, U has a basis B_U . It is a linearly independent subset of V , so by Theorem 1.8.2 there is a basis B of V with $B_U \subseteq B$. By Theorem 2.1.6 there is a linear map $\varphi: V \rightarrow F$ with

$$\varphi(b) = \psi(b) \quad \text{for } b \in B_U, \quad \varphi(b) = 0 \quad \text{for } b \in B \setminus B_U.$$

Then $\varphi|_U$ and ψ are linear maps $U \rightarrow F$ that agree on the basis B_U of U , so $\varphi|_U = \psi$ by

the uniqueness part of Theorem 2.1.6. Thus $\rho(\varphi) = \psi$, and ρ is surjective. (If V is finite-dimensional, the finite versions Corollary 1.5.5, Theorem 1.6.5 and Theorem 2.1.5 suffice, and no choice principle is needed.)

By the First Isomorphism Theorem (Theorem 4.4.7), $\varphi + \ker \rho \mapsto \rho(\varphi)$ is a well-defined isomorphism $V^*/\ker \rho \rightarrow \text{im } \rho$. Since $\ker \rho = U^0$ and $\text{im } \rho = U^*$, this is the claimed isomorphism. ■

In words: **a functional on U is a functional on V , up to adding something that vanishes on U .** Two extensions of the same ψ differ by an element of U^0 , and every element of U^0 can be added.

When V is finite-dimensional, taking dimensions recovers the count of this chapter. By Corollary 5.1.4, $\dim U^* = \dim U$ and $\dim V^* = \dim V$, and by Theorem 4.4.5, $\dim(V^*/U^0) = \dim V - \dim U^0$. So the isomorphism says $\dim U = \dim V - \dim U^0$, which is Theorem 5.2.2 again.

5.5.2 Functionals on a quotient

Now the other space. A functional ψ on V/U measures cosets. Composing with the quotient map $\pi: V \rightarrow V/U$, $\pi(v) = v + U$, turns it into a measurement of vectors, $v \mapsto \psi(v + U)$. This functional on V cannot tell apart two vectors in the same coset, so in particular it is zero on U . The theorem says that nothing is lost and nothing else appears: the functionals on V/U are **exactly** the functionals on V that vanish on U .

Theorem 5.5.2. Dual of a Quotient

Let V be a vector space over F , of any dimension, let U be a subspace, and let $\pi: V \rightarrow V/U$ be the quotient map. Then the dual map

$$\pi': (V/U)^* \rightarrow V^*, \quad \pi'(\psi) = \psi \circ \pi,$$

is injective, and its image is U^0 . Hence π' is an isomorphism from $(V/U)^*$ onto U^0 , and $(V/U)^* \cong U^0$.

Idea. Injectivity uses only that π is surjective: if $\psi \circ \pi$ is zero, then ψ is zero on every coset. For the image, one inclusion is “ π kills U ”. The other inclusion asks us to build a functional on V/U from a functional on V that vanishes on U , and that is exactly what the universal property of the quotient does.

Proof. The map π' is linear by Definition 5.4.1.

Injective. Suppose $\pi'(\psi) = \psi \circ \pi = 0$. Every element of V/U is $\pi(v)$ for some $v \in V$, because π is surjective (Theorem 4.4.4 (c)). Then $\psi(\pi(v)) = 0$ for every v , so $\psi = 0$. By Theorem 2.2.3, π' is injective.

$\text{im } \pi' \subseteq U^0$. For $u \in U$, $\pi(u) = U$ is the zero vector of V/U , so $(\psi \circ \pi)(u) = \psi(0) = 0$. Hence $\psi \circ \pi \in U^0$.

$U^0 \subseteq \text{im } \pi'$. Let $\varphi \in U^0$. Then $U \subseteq \ker \varphi$, so by Theorem 4.4.10 (a) there is a linear map $\psi: V/U \rightarrow F$ with $\psi \circ \pi = \varphi$. This ψ lies in $(V/U)^*$, and $\pi'(\psi) = \varphi$.

So π' is an injective linear map with image U^0 ; regarded as a map onto U^0 it is a linear bijection, hence an isomorphism. ■

Concretely, the inverse isomorphism $U^0 \rightarrow (V/U)^*$ is $\varphi \mapsto \bar{\varphi}$, where $\bar{\varphi}(v + U) = \varphi(v)$. The formula is well defined precisely because φ vanishes on U : if $v - v' \in U$, then $\varphi(v) - \varphi(v') = \varphi(v - v') = 0$.

The two theorems fit together as a mirror:

	subspace $U \subseteq V$	quotient V/U
its dual is	a quotient of V^* : $U^* \cong V^*/U^0$	a subspace of V^* : $(V/U)^* \cong U^0$
the map	ι' : restrict	π' : compose with π
the map is	surjective	injective

The last row is no accident: ι is injective and π is surjective, and dualizing swaps the two properties.

Alternatively, in finite dimension both theorems follow from Theorem 5.4.3, which says $\ker T' = (\text{im } T)^0$ and $\text{im } T' = (\ker T)^0$. For $T = \pi$: $\text{im } \pi = V/U$, whose annihilator is $\{0\}$, and $\ker \pi = U$, so π' is injective with image U^0 . For $T = \iota$: $\text{im } \iota = U$, so $\ker \iota' = U^0$, and $\ker \iota = \{0\}$, whose annihilator is all of U^* , so ι' is surjective. The proofs above avoid dimension counts, which is why they work in every dimension.

Quick check. Let V be a vector space of dimension 6 and U a subspace of dimension 2. Find $\dim U^0$, $\dim(V/U)^*$, $\dim U^*$, and $\dim(V^*/U^0)$, and check that the two theorems are consistent with these numbers.

Solution. By Theorem 5.2.2, $\dim U^0 = 6 - 2 = 4$. By Theorem 4.4.5 and Corollary 5.1.4, $\dim(V/U)^* = \dim(V/U) = 4$, matching $(V/U)^* \cong U^0$. Next, $\dim U^* = \dim U = 2$, and $\dim(V^*/U^0) = 6 - 4 = 2$, matching $U^* \cong V^*/U^0$.

5.5.3 A line in $\mathbb{R}^3$, in coordinates

The theorems name their isomorphisms without choosing anything. To see them act, we now choose bases and compute. Recall that a functional on $\mathbb{R}^3$ is determined by its values on $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (Theorem 5.1.1). So every $\varphi \in (\mathbb{R}^3)^*$ has the form $\varphi(x, y, z) = ax + by + cz$, with $(a, b, c) = (\varphi(\mathbf{e}_1), \varphi(\mathbf{e}_2), \varphi(\mathbf{e}_3))$. We write such a functional simply as $ax + by + cz$.

Example. Both isomorphisms for a line

Let $\mathbf{u} = (1, 2, -1)$ and $U = \text{span}(\mathbf{u}) \subseteq \mathbb{R}^3$.

- (a) Find a basis of U^0 .
- (b) Find a basis of $\mathbb{R}^3/U$, its dual basis (ψ_1, ψ_2) of $(\mathbb{R}^3/U)^*$, and the images $\pi'(\psi_1), \pi'(\psi_2)$.
- (c) Describe the restriction map $\rho: (\mathbb{R}^3)^* \rightarrow U^*$ and the isomorphism $(\mathbb{R}^3)^*/U^0 \rightarrow U^*$.

Solution.

- (a) By Theorem 5.2.3 (b), $U^0 = \{\mathbf{u}\}^0$, so $ax + by + cz \in U^0$ exactly when $a + 2b - c = 0$, that is, $c = a + 2b$. Taking $(a, b) = (1, 0)$ and $(0, 1)$ gives

$$\varphi_1 = x + z, \quad \varphi_2 = y + 2z.$$

They are independent (look at the coefficients of x and y), and $\dim U^0 = 3 - 1 = 2$ by Theorem 5.2.2, so (φ_1, φ_2) is a basis of U^0 (Theorem 1.6.4).

- (b) The list $(\mathbf{u}, \mathbf{e}_1, \mathbf{e}_2)$ is independent: in $a\mathbf{u} + b\mathbf{e}_1 + c\mathbf{e}_2 = (a + b, 2a + c, -a) = \mathbf{0}$, the third entry gives $a = 0$, and then $b = c = 0$. It has length 3, so it is a basis of $\mathbb{R}^3$, and by Theorem 4.4.5 $(\mathbf{e}_1 + U, \mathbf{e}_2 + U)$ is a basis of $\mathbb{R}^3/U$. Its dual basis consists of the ψ_i with $\psi_i(\mathbf{e}_j + U) = \delta_{ij}$ (Theorem 5.1.3).

The functional $\pi'(\psi_1) = \psi_1 \circ \pi$ on $\mathbb{R}^3$ takes the values $\psi_1(\mathbf{e}_1 + U) = 1$ at $\mathbf{e}_1$, $\psi_1(\mathbf{e}_2 + U) = 0$ at $\mathbf{e}_2$, and $\psi_1(U) = 0$ at $\mathbf{u}$. Writing it as $ax + by + cz$, this says $a = 1$, $b = 0$, and $a + 2b - c = 0$, so $c = 1$. Similarly $\pi'(\psi_2)$ has $a = 0$, $b = 1$, $c = 2$. Hence

$$\pi'(\psi_1) = x + z = \varphi_1, \quad \pi'(\psi_2) = y + 2z = \varphi_2.$$

So π' carries the dual basis of $(\mathbb{R}^3/U)^*$ onto the basis of U^0 found in (a), as Theorem 5.5.2 predicts. Read backwards: $\psi_1((x, y, z) + U) = x + z$ and $\psi_2((x, y, z) + U) = y + 2z$. These are well defined, since adding $t\mathbf{u} = (t, 2t, -t)$ changes $x + z$ by $t - t = 0$ and $y + 2z$ by $2t - 2t = 0$.

- (c) The space U^* is one-dimensional, with basis the functional ξ determined by $\xi(\mathbf{u}) = 1$ (the dual basis of $(\mathbf{u})$). A functional $\varphi = ax + by + cz$ restricts to $\varphi|_U = \varphi(\mathbf{u})\xi = (a + 2b - c)\xi$, since both sides take the value $a + 2b - c$ at $\mathbf{u}$. So ρ is “evaluate at $\mathbf{u}$ ”. Its kernel is $\{a + 2b - c = 0\} = U^0$, and it is surjective, since $\rho(x) = \xi$. The isomorphism of Theorem 5.5.1 is

$$(ax + by + cz) + U^0 \mapsto (a + 2b - c)\xi.$$

For instance $x + U^0 = -z + U^0$, because $x - (-z) = \varphi_1 \in U^0$; both restrict to ξ .

5.5.4 Why “natural” matters here

In the example, bases entered only when we wanted numbers. The maps ρ and π' were defined before any basis was chosen: restrict a functional, or compose it with π . That is what makes the two isomorphisms **natural**, in the sense of the discussion of the double dual. They are part of the structure of V and U , not of a coordinate system.

There is a tempting shortcut that is **not** natural. Since U^* is a quotient of V^* , one might try to find a copy of U^* **inside** V^* , as a subspace. This can be done, but only by making a choice. Pick a complement W of U , so $V = U \oplus W$. A functional that vanishes on W is determined by its values on U , so the restriction ρ maps W^0 isomorphically onto U^* (Exercise (C2: Complements and a choice of copy of U^*)). But different complements give different subspaces W^0 .

Warning. U^* is not a subspace of V^* . Its elements are functionals on U , which cannot be evaluated at vectors outside U . To place U^* inside V^* one must extend each functional, and extensions are not unique. In the example above, the complement $W_1 = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$ of U gives $W_1^0 = \text{span}(z)$, while $W_2 = \text{span}(\mathbf{e}_1, \mathbf{e}_3)$ gives $W_2^0 = \text{span}(y)$. These are different lines in $(\mathbb{R}^3)^*$, and both restrict isomorphically onto U^* : $z|_U = -\xi$ and $y|_U = 2\xi$. The choice-free statement is $U^* \cong V^*/U^0$, a quotient.

The same caution applies on the other side, in reverse. $(V/U)^*$ is naturally a subspace of V^* , namely U^0 , but V/U itself is not a subspace of V . Duality turns the quotient, which has no natural home inside V , into a subspace, which does.

5.5.5 Codimension and hyperplanes

The quotient measures how far a subspace is from filling the whole space. In $\mathbb{R}^3$ a plane through the origin misses “one direction” and a line misses “two directions”. For finite-dimensional V the number of missing directions is $\dim V - \dim U$. But this formula is useless in $F[x]$, where $\dim V$ and $\dim U$ may both be infinite, although $U = \{p : p(0) = 0\}$ misses only one direction: every polynomial is a constant plus an element of U . The quotient gives the right number in every case.

The codimension of U counts the directions in V that U is missing.

Definition 5.5.1. Codimension

Let U be a subspace of a vector space V over F . If V/U is finite-dimensional, the **codimension** of U in V is

$$\text{codim } U := \dim(V/U).$$

If V/U is infinite-dimensional, we say that U has **infinite codimension**.

In words: we do not compare the sizes of U and V . We collapse U to zero and measure what is left. The codimension depends on the ambient space, so when that is unclear we write “codimension of U in V ”.

Definition 5.5.2. Hyperplane

A **hyperplane** in V is a subspace H of V of codimension **exactly** 1, that is, with $\dim(V/H) = 1$.

Examples.

- **Finite dimension.** If V is finite-dimensional, then $\text{codim } U = \dim V - \dim U$ by Theorem 4.4.5. The hyperplanes of F^n are its subspaces of dimension $n - 1$: the lines through $\mathbf{0}$ in F^2 and the planes through $\mathbf{0}$ in F^3 .
- **Trace.** In $M_n(F)$ with $n \geq 1$, the trace-zero matrices are the kernel of $\text{tr}: M_n(F) \rightarrow F$. This map is linear and surjective, since $\text{tr}(c\mathbf{E}_{11}) = c$, so $M_n(F)/\ker \text{tr} \cong F$ by Theorem 4.4.7, and the trace-zero matrices form a hyperplane.
- **Polynomials.** In $F[x]$, let $U = \{p : p(0) = 0\}$. Evaluation at 0 is a linear map $F[x] \rightarrow F$ with kernel U , and it is surjective because the constant c maps to c . By Theorem 4.4.7, $F[x]/U \cong F$, so U is a hyperplane in the infinite-dimensional space $F[x]$. In the same way

$\{p : p(0) = p(1) = 0\}$ has codimension 2 ([Exercise \(B3: Dimension computations\)](#)).

- **Degenerate cases.** $\text{codim } V = 0$, since V/V is the zero space, and $\text{codim}\{0\} = \dim V$ when V is finite-dimensional. In particular V itself is **not** a hyperplane, and the only hyperplane of a one-dimensional space is $\{0\}$. The zero space $\{0\}$ has no hyperplanes at all.

Nonexample by minimal change. In $\mathbb{R}^3$, the plane $\{x + y + z = 0\}$ is a hyperplane. Replace it by the line $\{x + y + z = 0, x - y = 0\} = \text{span}((1, 1, -2))$. It is still a subspace, but its codimension is $3 - 1 = 2$, so the clause “codimension **exactly** 1” fails. Replace it instead by $\{x + y + z = 1\}$: now it is not a subspace at all, so it has no codimension. It is a coset of a hyperplane, an object we return to in the next section.

Why this definition. Codimension is the dual notion of dimension, and the duality theorems make this precise. If U has finite codimension, then by [Theorem 5.5.2](#), Corollary 5.1.4, and Theorem 2.5.4,

$$\text{codim } U = \dim(V/U) = \dim(V/U)^* = \dim U^0,$$

and this holds even when V is infinite-dimensional. So **dimension counts vectors, and codimension counts independent equations**. For finite-dimensional V this is Theorem 5.2.2 again: $\dim U^0 = \dim V - \dim U$.

Warning. In infinite dimension a hyperplane need not be “smaller” than the whole space. In $F[x]$, the hyperplane $U = \{p : p(0) = 0\} = \{xq : q \in F[x]\}$ is isomorphic to $F[x]$ itself, via $q \mapsto xq$: this map is linear, injective (if $xq = 0$ then $q = 0$), and onto U . So you cannot detect codimension by comparing U with V up to isomorphism, and the formula $\text{codim } U = \dim V - \dim U$ must never be used outside finite dimension. Always compute $\dim(V/U)$, or count equations.

Hyperplanes are exactly the subspaces cut out by **one** nontrivial linear equation. In F^n this is familiar: $a_1x_1 + \cdots + a_nx_n = 0$ with the a_i not all zero defines a subspace of dimension $n - 1$. The quotient makes the statement true in every dimension.

Theorem 5.5.3. Hyperplanes Are Kernels of Functionals

Let V be a vector space over F , of any dimension, and let H be a subspace of V . Then H is a hyperplane if and only if $H = \ker \varphi$ for some **nonzero** $\varphi \in V^*$.

Idea. A nonzero functional maps onto the one-dimensional space F , so the First Isomorphism Theorem makes $V/\ker \varphi$ one-dimensional. Conversely, if V/H is one-dimensional, it “is” F after choosing an isomorphism, and composing that isomorphism with π gives a functional whose kernel is H .

Proof. ($\Leftarrow$) Suppose $\varphi \in V^*$ is nonzero. Then $\text{im } \varphi$ is a subspace of F (Theorem 2.2.2) containing a nonzero scalar, so $\dim \text{im } \varphi = 1 = \dim F$, and $\text{im } \varphi = F$ by Theorem 1.6.8. By Theorem 4.4.7, $V/\ker \varphi \cong F$, so $\dim(V/\ker \varphi) = 1$, and $\ker \varphi$ is a hyperplane.

($\Rightarrow$) Suppose $\dim(V/H) = 1$. By Theorem 2.5.4 there is an isomorphism $\theta: V/H \rightarrow F$. Let $\varphi = \theta \circ \pi$, where $\pi: V \rightarrow V/H$ is the quotient map. It is linear as a composition of linear maps (Theorem 2.4.2), so $\varphi \in V^*$. For $v \in V$, $\varphi(v) = 0$ means $\theta(v + H) = 0$, which holds

exactly when $v + H$ is the zero coset, since θ is injective; and $v + H = H$ means $v \in H$ by Lemma 4.4.3 (b). So $\ker \varphi = H$. Finally $\varphi \neq 0$: since $V/H \neq \{0\}$, we have $H \neq V$, so some $v \notin H = \ker \varphi$. ■

The functional is not unique, but almost: two functionals with the same kernel are scalar multiples of each other (Exercise (C1: Functionals with the same kernel); [Exercise \(B2: Functionals with the same kernel\)](#) gives a second proof). So hyperplanes of V correspond to nonzero functionals **up to a nonzero scalar**, just as the plane $x + y + z = 0$ is also $2x + 2y + 2z = 0$.

A hyperplane together with any one vector outside it fills the space.

Remark. Let $H = \ker \varphi$ be a hyperplane and $v_0 \notin H$. Then $V = H \oplus \text{span}(v_0)$. Indeed $\varphi(v_0) \neq 0$, and every $v \in V$ splits as

$$v = \left(v - \frac{\varphi(v)}{\varphi(v_0)} v_0 \right) + \frac{\varphi(v)}{\varphi(v_0)} v_0,$$

where the first bracket lies in $\ker \varphi$ because φ of it is $\varphi(v) - \varphi(v) = 0$. And $H \cap \text{span}(v_0) = \{0\}$, since $\varphi(cv_0) = c\varphi(v_0) = 0$ forces $c = 0$.

Quick check. Decide which of the following are hyperplanes, and for each hyperplane give a nonzero functional with that kernel.

- (a) The symmetric matrices in $M_2(F)$.
- (b) The symmetric matrices in $M_3(F)$.
- (c) $\{p \in F[x] : p(1) = 0\}$ in $F[x]$.

Solution.

- (a) Yes. $\mathbf{A}$ is symmetric exactly when $a_{12} - a_{21} = 0$, so the symmetric matrices are the kernel of $\mathbf{A} \mapsto a_{12} - a_{21}$, which is linear and nonzero (it takes the value 1 at $\mathbf{E}_{12}$). By [Theorem 5.5.3](#) this is a hyperplane.
- (b) No. The symmetric 3×3 matrices form a subspace of dimension 6 (choose the entries on and above the diagonal freely), so the codimension is $9 - 6 = 3 \neq 1$.
- (c) Yes. It is the kernel of $p \mapsto p(1)$, which is linear (Theorem 0.8.5) and nonzero (it sends the constant 1 to 1).

5.5.6 Exercises

5.5.6.1 A. Check your understanding

Exercise. A1

- (a) State the theorem identifying $(V/U)^*$ with a subspace of V^* , including the map.

- (b) Define the codimension of a subspace, and define a hyperplane.
- (c) True or false: U^* is a subspace of V^* . Justify your answer.
- (d) Let U be a subspace of dimension 2 in F^5 . Find $\dim(F^5/U)^*$ and $\text{codim } U$.
- (e) Is $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = 0\}$ a hyperplane in $\mathbb{R}^4$? Justify your answer.
- (f) True or false: every nonzero linear functional $\varphi: V \rightarrow F$ is surjective. Justify your answer.

Solution.

- (a) For a subspace U of V with quotient map π , the dual map $\pi': (V/U)^* \rightarrow V^*$, $\psi \mapsto \psi \circ \pi$, is injective with image U^0 ; so $(V/U)^* \cong U^0$ (Theorem 5.5.2).
- (b) If V/U is finite-dimensional, $\text{codim } U = \dim(V/U)$; otherwise U has infinite codimension. A hyperplane is a subspace of codimension exactly 1.
- (c) False. Elements of U^* are defined only on U , so they are not elements of V^* . What is true is $U^* \cong V^*/U^0$, a quotient (Theorem 5.5.1).
- (d) $\dim(F^5/U)^* = \dim(F^5/U) = 5 - 2 = 3$, by Corollary 5.1.4 and Theorem 4.4.5. So $\text{codim } U = 3$.
- (e) No. The subspace is $\text{span}(\mathbf{e}_3, \mathbf{e}_4)$, of dimension 2, so its codimension is $4 - 2 = 2$.
- (f) True, in any dimension. This is Proposition 5.1.2: $\text{im } \varphi$ is a subspace of the one-dimensional space F containing a nonzero element, so it is F .

5.5.6.2 B. Practice

Exercise. B1: The dual of a quotient of $\mathbb{R}^4$

Let $U = \text{span}((1, -1, 0, 2), (0, 1, 1, -1)) \subseteq \mathbb{R}^4$, and write functionals on $\mathbb{R}^4$ as $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$.

- (a) Find a basis (φ_1, φ_2) of U^0 .
- (b) Let $\psi_1, \psi_2 \in (\mathbb{R}^4/U)^*$ be the functionals with $\pi'(\psi_i) = \varphi_i$. Compute ψ_1 and ψ_2 at the coset $(2, 0, 1, 3) + U$.
- (c) Find a basis of $\mathbb{R}^4/U$ whose dual basis is (ψ_1, ψ_2) .

Solution.

- (a) A functional lies in U^0 exactly when it vanishes at both spanning vectors (Theorem 5.2.3 (b)), that is,

$$a_1 - a_2 + 2a_4 = 0, \quad a_2 + a_3 - a_4 = 0.$$

The second gives $a_3 = a_4 - a_2$, and the first gives $a_1 = a_2 - 2a_4$. Taking $(a_2, a_4) = (1, 0)$ and $(-1, -1)$ gives $(1, 1, -1, 0)$ and $(1, -1, 0, -1)$, that is,

$$\varphi_1 = x_1 + x_2 - x_3, \quad \varphi_2 = x_1 - x_2 - x_4.$$

Check: $\varphi_1(1, -1, 0, 2) = 0$, $\varphi_1(0, 1, 1, -1) = 0$, $\varphi_2(1, -1, 0, 2) = 1 + 1 - 2 = 0$, $\varphi_2(0, 1, 1, -1) = -1 + 1 = 0$. They are independent (compare the coefficients of x_3 and x_4). The spanning vectors of U are independent (compare first entries, then second), so $\dim U^0 = 4 - 2 = 2$ by Theorem 5.2.2, and (φ_1, φ_2) is a basis by Theorem 1.6.4.

- (b) By [Theorem 5.5.2](#), $\psi_i(\mathbf{v} + U) = \varphi_i(\mathbf{v})$. So $\psi_1((2, 0, 1, 3) + U) = 2 + 0 - 1 = 1$ and $\psi_2((2, 0, 1, 3) + U) = 2 - 0 - 3 = -1$.
- (c) We need cosets c_1, c_2 with $\psi_i(c_j) = \delta_{ij}$. The vector $-\mathbf{e}_3$ has $\varphi_1(-\mathbf{e}_3) = 1$ and $\varphi_2(-\mathbf{e}_3) = 0$, and $-\mathbf{e}_4$ has $\varphi_1(-\mathbf{e}_4) = 0$, $\varphi_2(-\mathbf{e}_4) = 1$. Put $c_1 = -\mathbf{e}_3 + U$ and $c_2 = -\mathbf{e}_4 + U$. If $ac_1 + bc_2 = 0$, applying ψ_1 and ψ_2 gives $a = 0$ and $b = 0$; so (c_1, c_2) is independent, and since $\dim(\mathbb{R}^4/U) = 2$ it is a basis ([Theorem 1.6.4](#)). By construction $\psi_i(c_j) = \delta_{ij}$, so (ψ_1, ψ_2) is its dual basis ([Theorem 5.1.3](#)).

Exercise. B2: Functionals with the same kernel

Let V be a vector space over F , of any dimension, and let $\varphi, \psi \in V^*$ be nonzero. Prove that $\ker \varphi = \ker \psi$ if and only if $\psi = c\varphi$ for some nonzero $c \in F$. (Exercise (C1: Functionals with the same kernel) proved $(\Rightarrow)$ by a different route; here use the hyperplane decomposition.)

Hint: the remark after [Theorem 5.5.3](#).

Solution. $(\Leftarrow)$ If $\psi = c\varphi$ with $c \neq 0$, then $\psi(v) = 0$ if and only if $c\varphi(v) = 0$, if and only if $\varphi(v) = 0$, since c is invertible. So $\ker \psi = \ker \varphi$.

$(\Rightarrow)$ Suppose $\ker \varphi = \ker \psi = H$. Since $\varphi \neq 0$, choose v_0 with $\varphi(v_0) \neq 0$; then $v_0 \notin H$, so also $\psi(v_0) \neq 0$. Put $c = \psi(v_0)/\varphi(v_0)$, which is nonzero, and $\chi = \psi - c\varphi \in V^*$. Then χ vanishes on H , because both ψ and φ do, and $\chi(v_0) = \psi(v_0) - c\varphi(v_0) = 0$. By [Theorem 5.5.3](#) H is a hyperplane, and by the remark after it $V = H \oplus \text{span}(v_0)$. A linear map vanishing on H and on v_0 vanishes on every $h + av_0$, hence on V . So $\chi = 0$, that is, $\psi = c\varphi$.

Exercise. B3: Dimension computations

- (a) Let $\dim V = 7$ and $\dim U = 3$. Find $\dim U^*$, $\dim(V^*/U^0)$, $\dim(V/U)^*$, and $\text{codim } U$.
- (b) In $V = F[x]_{\leq 3}$, let $U = \{p : p(0) = p(1) = 0\}$. Using the map $T : V \rightarrow F^2$, $T(p) = (p(0), p(1))$, find $\text{codim } U$ and $\dim U$, and write U as an intersection of two hyperplanes.
- (c) Hence find a basis of U^0 consisting of evaluation functionals.

Solution.

- (a) $\dim U^* = 3$ by [Corollary 5.1.4](#). $\dim U^0 = 7 - 3 = 4$ by [Theorem 5.2.2](#), so $\dim(V^*/U^0) = 7 - 4 = 3$ ([Theorem 4.4.5](#)), consistent with [Theorem 5.5.1](#). $\dim(V/U)^* = \dim(V/U) = 4$, consistent with [Theorem 5.5.2](#), and $\text{codim } U = 4$.
- (b) T is linear because evaluation is ([Theorem 0.8.5](#)), and $\ker T = U$. It is surjective: $T(1) = (1, 1)$ and $T(x) = (0, 1)$ span F^2 . By [Theorem 4.4.7](#), $V/U \cong F^2$, so $\text{codim } U = 2$, and $\dim U = 4 - 2 = 2$ by [Theorem 4.4.5](#). Writing $\varepsilon_0(p) = p(0)$ and $\varepsilon_1(p) = p(1)$, we have $U = \ker \varepsilon_0 \cap \ker \varepsilon_1$, and both kernels are hyperplanes by [Theorem 5.5.3](#), since both functionals send 1 to 1.
- (c) Both ε_0 and ε_1 lie in U^0 . They are independent: if $a\varepsilon_0 + b\varepsilon_1 = 0$, evaluating at x gives $b = 0$, and then evaluating at 1 gives $a = 0$. Since $\dim U^0 = 4 - 2 = 2$, $(\varepsilon_0, \varepsilon_1)$ is a basis of U^0 ([Theorem 1.6.4](#)).

5.5.6.3 C. Going deeper

Exercise. C1: Subspaces of finite codimension are intersections of hyperplanes

Let V be a vector space over F , of any dimension, and let U be a subspace of finite codimension $k \geq 1$.

- (a) Prove that there are nonzero $\varphi_1, \dots, \varphi_k \in U^0$ with $U = \ker \varphi_1 \cap \dots \cap \ker \varphi_k$. Deduce that U is an intersection of k hyperplanes.
- (b) Prove that U is not an intersection of fewer than k hyperplanes.

Hint: for (a), take a dual basis of $(V/U)^*$; for (b), consider the map $V \rightarrow F^m$, $v \mapsto (\varphi_1(v), \dots, \varphi_m(v))$.

Solution.

- (a) Let $\pi: V \rightarrow V/U$ be the quotient map. Since $\dim(V/U) = k$, choose a basis $(c_1, \dots, c_k)$ of V/U and let $(\psi_1, \dots, \psi_k)$ be its dual basis (Theorem 5.1.3). Put $\varphi_i = \psi_i \circ \pi$. By Theorem 5.5.2, $\varphi_i \in U^0$, and $\varphi_i \neq 0$ because π' is injective and $\psi_i \neq 0$. Since each φ_i lies in U^0 , $U \subseteq \ker \varphi_i$ for each i . Conversely, let $v \in \ker \varphi_1 \cap \dots \cap \ker \varphi_k$, and write $v + U = \sum_j a_j c_j$. Applying ψ_i gives $a_i = \psi_i(v + U) = \varphi_i(v) = 0$ for every i . So $v + U$ is the zero coset, and $v \in U$ by Lemma 4.4.3 (b). This proves $U = \bigcap_i \ker \varphi_i$. Each $\ker \varphi_i$ is a hyperplane by Theorem 5.5.3.
- (b) Suppose $U = H_1 \cap \dots \cap H_m$ with hyperplanes H_i . By Theorem 5.5.3, $H_i = \ker \varphi_i$ for nonzero $\varphi_i \in V^*$. The map $S: V \rightarrow F^m$, $S(v) = (\varphi_1(v), \dots, \varphi_m(v))$, is linear, since each entry is, and $\ker S = \bigcap_i \ker \varphi_i = U$. By Theorem 4.4.7, $V/U \cong \text{im } S$, a subspace of F^m . Hence $k = \dim(V/U) = \dim \text{im } S \leq m$ by Theorem 1.6.7. So at least k hyperplanes are needed.

Exercise. C2: Complements and a choice of copy of U^*

Let V be finite-dimensional, and let U, W be subspaces with $V = U \oplus W$.

- (a) Prove that $V^* = U^0 \oplus W^0$ without using Theorem 5.2.3 (d). (This is Exercise (C3: Dual of a direct sum decomposition) (a), which used it.)
- (b) Deduce that the restriction map $\rho: V^* \rightarrow U^*$ restricts to an isomorphism $W^0 \rightarrow U^*$.
- (c) In $\mathbb{R}^2$, let $U = \text{span}(\mathbf{e}_1)$, $W_1 = \text{span}(\mathbf{e}_2)$, and $W_2 = \text{span}((1, 1))$. Find W_1^0 and W_2^0 , and find the element of each that restricts to the functional $\xi \in U^*$ with $\xi(\mathbf{e}_1) = 1$. Conclude that the copy of U^* inside V^* depends on the complement.

Solution.

- (a) Let $n = \dim V$. If $\varphi \in U^0 \cap W^0$, then φ vanishes on U and on W , hence on every $u + w$, that is, on $U + W = V$; so $\varphi = 0$. By Theorem 5.2.2, $\dim U^0 + \dim W^0 = (n - \dim U) + (n - \dim W) = 2n - n = n$, using $\dim U + \dim W = n$ for a direct sum (Theorem 1.7.4). By Theorem 1.7.3, $\dim(U^0 + W^0) = n - 0 = n = \dim V^*$, so $U^0 + W^0 = V^*$ (Theorem 1.6.8), and the sum is direct because the intersection is $\{0\}$.
- (b) The restriction of ρ to W^0 is linear. Its kernel is $W^0 \cap \ker \rho = W^0 \cap U^0 = \{0\}$ by Theorem 5.5.1 and (a), so it is injective. Its domain has dimension $n - \dim W =$

$\dim U = \dim U^*$. An injective linear map between spaces of the same finite dimension is an isomorphism (Corollary 2.3.2 (e)).

- (c) Write functionals on $\mathbb{R}^2$ as $ax + by$. W_1^0 consists of those with $b = 0$, so $W_1^0 = \text{span}(x)$. W_2^0 consists of those with $a + b = 0$, so $W_2^0 = \text{span}(x - y)$. Restricting to U : $x|_U$ and $(x - y)|_U$ both take the value 1 at $\mathbf{e}_1$, so both equal ξ . The elements x and $x - y$ are different, so the subspaces $W_1^0 \neq W_2^0$ of V^* are two different copies of U^* , one for each complement.