

5.1 Linear Functionals and the Dual Space

Chapter 2 studied linear maps $V \rightarrow W$ into arbitrary targets. The smallest nonzero target is the field F itself, and maps into it are everywhere: a coordinate, the value of a polynomial at a point, an integral, the trace. This section names these maps and shows that they form a vector space V^* , of the same dimension as V when V is finite-dimensional. It also proves the fact that makes them useful: every nonzero vector is detected by at least one of them.

5.1.1 Measuring a vector by a number

After studying vector spaces and the maps between them, we look at the simplest possible target. Here are four maps we have already met, from four different families.

- **A coordinate.** Fix a basis $\mathcal{B} = (v_1, \dots, v_n)$ of V . The map $V \rightarrow F$ sending v to the first entry of $[v]_{\mathcal{B}}$ respects sums and multiples, by Theorem 1.5.2.
- **Evaluation.** The map $F[x] \rightarrow F$, $p \mapsto p(3)$, satisfies $(p + q)(3) = p(3) + q(3)$ and $(cp)(3) = cp(3)$, by Theorem 0.8.5 (a constant c is a polynomial with value c).
- **Integration.** The map $\mathbb{R}[x] \rightarrow \mathbb{R}$, $p \mapsto \int_0^1 p(t) dt$, is linear by the linearity of the integral from calculus.
- **Trace.** The map $\text{tr}: M_n(F) \rightarrow F$ is linear (Example (Trace and Transpose)).

The inputs are columns, polynomials, and matrices. The outputs are always single scalars. Each map is a way of **measuring** a vector by one number, and each measurement respects sums and scalar multiples. The same expression keeps recurring, “a linear map whose target is F ”, and we will see that these maps form a space with a structure of its own. So we give them a name.

A linear functional is a linear measurement: it assigns a scalar to each vector, and the measurement of a linear combination is the same combination of the measurements.

Definition 5.1.1. Linear Functional

Let V be a vector space over F , and regard F as a vector space over itself. A **linear functional** on V is a linear map $\varphi: V \rightarrow F$. That is, **for all** $u, v \in V$ and **all** $c \in F$,

$$\varphi(u + v) = \varphi(u) + \varphi(v), \quad \varphi(cv) = c\varphi(v).$$

Definition 5.1.2. Dual Space

Let V be a vector space over F . The **dual space** of V is the set of **all** linear functionals on V ,

$$V^* := \mathcal{L}(V, F),$$

with the pointwise operations $(\varphi + \psi)(v) := \varphi(v) + \psi(v)$ and $(c\varphi)(v) := c\varphi(v)$ for $\varphi, \psi \in V^*$, $c \in F$, and $v \in V$.

In words: a linear functional is a linear map (Definition 2.1.1) whose codomain is the one-dimensional space F . Its values lie in the **same** field F over which V is a vector space. The first condition says that measuring a sum gives the sum of the measurements, and the second says that scaling a vector scales its measurement. The dual space collects **every** such measurement, and it adds them the natural way: the measurement $\varphi + \psi$ reports the sum of the two readings. We write functionals with Greek letters φ, ψ , which keeps them apart from vectors of V .

Well-definedness. Two things need checking: that $\varphi + \psi$ and $c\varphi$ are again linear, and that the vector space axioms hold. Both are the case $W = F$ of Theorem 2.4.1. So V^* is a vector space over F . Its zero vector is the **zero functional** $v \mapsto 0$, and the negative of φ is $v \mapsto -\varphi(v)$. Note that V^* is a set of **functions on V** , not a subset of V .

Example. Four Families of Functionals

Show that each map is a linear functional.

- (a) $\varepsilon_c: F[x] \rightarrow F, p \mapsto p(c)$, for a fixed $c \in F$ (**evaluation at c**).
- (b) $\iota: \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}, p \mapsto \int_0^1 p(t) dt$.
- (c) $\text{tr}: M_n(F) \rightarrow F$.
- (d) The zero functional on any V . Also find the dual space of the zero space $\{0\}$.

Solution.

- (a) By Theorem 0.8.5, $(p + q)(c) = p(c) + q(c)$, and, regarding ap as the product of the constant polynomial a with p , $(ap)(c) = a(c)p(c) = ap(c)$, because the constant polynomial a has value a at c . These are the two conditions of Definition 5.1.1.
- (b) By linearity of the integral, $\int_0^1 (p + q) = \int_0^1 p + \int_0^1 q$ and $\int_0^1 ap = a \int_0^1 p$. On coefficients, $\iota(a_0 + a_1x + a_2x^2) = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$.
- (c) This is Theorem 0.9.6 (1): $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr } \mathbf{A} + \text{tr } \mathbf{B}$ and $\text{tr}(c\mathbf{A}) = c \text{tr } \mathbf{A}$.
- (d) $0 = 0 + 0$ and $0 = c \cdot 0$ in F , so the zero functional is linear. On $\{0\}$, a linear functional must send 0 to 0 (Theorem 2.1.1), and there is only one vector. So the zero functional is the only element, and $\{0\}^* = \{0\}$. This degenerate case matters: it is the case $n = 0$ of the count $\dim V^* = \dim V$ proved below.

The family F^n can be described completely. A row vector times a column is a scalar, and every functional on F^n arises this way.

Theorem 5.1.1. Functionals on F^n Are Row Vectors

For $\mathbf{a} = (a_1, \dots, a_n) \in F^n$, define $\varphi_{\mathbf{a}}: F^n \rightarrow F$ by

$$\varphi_{\mathbf{a}}(\mathbf{x}) := \mathbf{a}^\top \mathbf{x} = a_1x_1 + \dots + a_nx_n.$$

Then $\varphi_{\mathbf{a}} \in (F^n)^*$. Every $\varphi \in (F^n)^*$ equals $\varphi_{\mathbf{a}}$ for **exactly one** $\mathbf{a}$, namely $(\varphi(\mathbf{e}_1), \dots, \varphi(\mathbf{e}_n))$. The map $F^n \rightarrow (F^n)^*, \mathbf{a} \mapsto \varphi_{\mathbf{a}}$, is an isomorphism.

Proof. The map $\varphi_{\mathbf{a}}$ is $T_{\mathbf{a}^\top}$ for the $1 \times n$ matrix $\mathbf{a}^\top$, with a 1×1 matrix read as a scalar, so it is linear by Example (Multiplication by a Matrix). Let $\varphi \in (F^n)^*$ and put $a_i := \varphi(\mathbf{e}_i)$. Then $\varphi_{\mathbf{a}}(\mathbf{e}_i) = a_i = \varphi(\mathbf{e}_i)$ for each i . Two linear maps that agree on the basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ are equal by the uniqueness part of Theorem 2.1.5, so $\varphi = \varphi_{\mathbf{a}}$. If $\varphi_{\mathbf{a}} = \varphi_{\mathbf{b}}$, then $a_i = \varphi_{\mathbf{a}}(\mathbf{e}_i) = \varphi_{\mathbf{b}}(\mathbf{e}_i) = b_i$ for every i , so $\mathbf{a} = \mathbf{b}$.

Hence $\mathbf{a} \mapsto \varphi_{\mathbf{a}}$ is a bijection $F^n \rightarrow (F^n)^*$. It is linear, since $\varphi_{\mathbf{a}+c\mathbf{b}}(\mathbf{x}) = \sum_i (a_i + cb_i)x_i =$

$\varphi_{\mathbf{a}}(\mathbf{x}) + c\varphi_{\mathbf{b}}(\mathbf{x})$ for every $\mathbf{x}$. By Theorem 2.5.1 it is an isomorphism. ■

So in F^n a functional **is** a row vector acting by matrix multiplication, and its $1 \times n$ matrix in the standard bases is $(\varphi(\mathbf{e}_1) \cdots \varphi(\mathbf{e}_n))$. This is how we compute with functionals: rows act on columns.

Nonexamples by minimal change. Each of the following changes one of the examples slightly. In each case something still works, and we name a condition that fails.

- On $\mathbb{R}^2$, the length $(x, y) \mapsto \sqrt{x^2 + y^2}$ (Example (Length)). It sends $\mathbf{0}$ to 0 and respects **positive** scalars. But with $c = -1$ and $\mathbf{v} = (1, 0)$ it gives 1, not -1 , so the scaling condition fails.
- On $\mathbb{R}[x]$, squaring the evaluation at 0: $p \mapsto p(0)^2$. The outputs are still scalars. But for $p = q = 1$ we get $(1 + 1)^2 = 4 \neq 1 + 1$, so additivity fails.
- On $M_2(F)$, replace the trace $a + d$ of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $ad - bc$. Then $\mathbf{E}_{11} \mapsto 0$ and $\mathbf{E}_{22} \mapsto 0$, but $\mathbf{E}_{11} + \mathbf{E}_{22} = \mathbf{I}_2 \mapsto 1$, so additivity fails. The expression $ad - bc$ is linear in each **row** separately, which is a different property.

Why this definition. Why single out the target F ? Because nothing is lost. A linear map $T: V \rightarrow F^m$ is the same thing as m functionals: if $\varphi_i(v)$ denotes the i -th entry of Tv , then each φ_i is linear and $Tv = (\varphi_1(v), \dots, \varphi_m(v))$; conversely any m functionals stacked this way give a linear map. So functionals are the “rows” of linear maps, and understanding them one at a time is a way of understanding all maps into F^m . The name comes from analysis, where the vectors were themselves functions and a map such as $f \mapsto \int_0^1 f$ was called a “functional”, a function of a function. The word “dual” will explain itself in this chapter: every statement about V has a mirror statement about V^* .

Warning. **The values must lie in the field of scalars of V .** Regard $\mathbb{C}$ as a vector space over $\mathbb{C}$. The real part $z \mapsto \operatorname{Re} z$ is additive, and it respects **real** scalars, but it is **not** a linear functional on $\mathbb{C}$ over $\mathbb{C}$: with $c = i$ and $z = 1$, $\operatorname{Re}(i \cdot 1) = 0 \neq i \cdot \operatorname{Re}(1) = i$. It is a functional on $\mathbb{C}$ regarded as a vector space over $\mathbb{R}$. The dual space depends on the field.

The first payoff is a count that holds for **every** nonzero functional: it hits every scalar, and in finite dimension it kills exactly one dimension.

Proposition 5.1.2. A Non-Zero Functional Is Surjective

Let V be a vector space over F , and let $\varphi \in V^*$ be **nonzero**. Then φ is surjective. If V is finite-dimensional, then $\dim \ker \varphi = \dim V - 1$.

Proof. Since $\varphi \neq 0$, there is $u \in V$ with $c := \varphi(u) \neq 0$. For every $a \in F$, linearity gives $\varphi(ac^{-1}u) = ac^{-1}c = a$, so φ is surjective. Hence $\operatorname{im} \varphi = F$ and $\operatorname{rank} \varphi = 1$. If V is finite-dimensional, the Rank–Nullity Theorem (Theorem 2.3.1) gives $\dim \ker \varphi = \dim V - 1$. ■

For example, $\ker \text{tr}$ in $M_n(F)$ has dimension $n^2 - 1$, and the polynomials in $F[x]_{\leq n}$ with $p(c) = 0$ form a subspace of dimension n . Every nonzero functional cuts out a subspace one dimension smaller than the whole space. Later in this chapter these kernels, the hyperplanes, become the building blocks of every subspace.

Quick check. Which of the following are linear functionals? (a) $p \mapsto p(0) + 1$ on $\mathbb{R}[x]$. (b) $p \mapsto p(1) - 2p'(0)$ on $\mathbb{R}[x]$. (c) $\mathbf{A} \mapsto a_{12}$ on $M_2(F)$.

Solution.

- (a) No. It sends the zero polynomial to 1, but a linear map sends 0 to 0 (Theorem 2.1.1).
 (b) Yes. Evaluation at 1, differentiation, and evaluation at 0 are linear, and V^* is closed under linear combinations and composition with linear maps (Theorem 2.4.1, Theorem 2.4.2).
 (c) Yes. It is the coordinate of $\mathbf{A}$ at $\mathbf{E}_{12}$ in the standard basis of Example (Standard Bases), and coordinates are linear (Theorem 1.5.2).

5.1.2 The dual basis

Return to the coordinate functionals. Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V , and let $\varphi_i(v)$ be the i -th entry of $[v]_{\mathcal{B}}$. The basis vector v_j has coordinate vector $\mathbf{e}_j$, so

$$\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

with the Kronecker delta of Definition 0.9.3. Each φ_i reads off one coordinate and ignores the others. These n functionals are the natural candidates for a basis of V^* , and the theorem says they are one.

Theorem 5.1.3. The Dual Basis

Let V be a finite-dimensional vector space over F , and let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V .

- (a) There are **unique** $\varphi_1, \dots, \varphi_n \in V^*$ with $\varphi_i(v_j) = \delta_{ij}$ for all i, j . For every $v \in V$, $\varphi_i(v)$ is the i -th entry of $[v]_{\mathcal{B}}$.
 (b) The list $(\varphi_1, \dots, \varphi_n)$ is a basis of V^* .
 (c) For every $v \in V$ and every $\varphi \in V^*$,

$$v = \sum_{i=1}^n \varphi_i(v) v_i, \quad \varphi = \sum_{i=1}^n \varphi(v_i) \varphi_i.$$

Idea. ① A functional is a linear map into F , and a linear map may be prescribed freely on a basis, so existence and uniqueness in (a) are one citation. ② Independence: apply a vanishing combination $\sum c_i \varphi_i$ to v_j ; the delta kills every term but c_j . ③ Spanning: we cannot count, because $\dim V^*$ is exactly what we do not know yet. So we guess the coefficients. If $\varphi = \sum c_i \varphi_i$, then applying both sides to v_j forces $c_j = \varphi(v_j)$. With this guess, the two sides are linear maps that agree on a basis, hence equal.

Proof.

- (a) For each i , apply Theorem 2.1.5 with $W = F$ and the prescribed values $\delta_{i1}, \dots, \delta_{in} \in F$. It gives exactly one linear map $\varphi_i: V \rightarrow F$ with $\varphi_i(v_j) = \delta_{ij}$ for all j . If $v = a_1v_1 + \dots + a_nv_n$, then by Theorem 2.1.4

$$\varphi_i(v) = \sum_{j=1}^n a_j \varphi_i(v_j) = \sum_{j=1}^n a_j \delta_{ij} = a_i,$$

the i -th entry of $[v]_{\mathcal{B}}$ (Definition 1.5.2).

- (b) *Independence.* Let $c_1\varphi_1 + \dots + c_n\varphi_n = 0$ in V^* . Applying both sides to v_j and using the pointwise operations of Definition 5.1.2,

$$0 = \sum_{i=1}^n c_i \varphi_i(v_j) = \sum_{i=1}^n c_i \delta_{ij} = c_j$$

for each j . Hence $(\varphi_1, \dots, \varphi_n)$ is linearly independent.

Spanning. Let $\varphi \in V^*$, and put $\psi := \sum_{i=1}^n \varphi(v_i) \varphi_i \in V^*$. For each j , $\psi(v_j) = \sum_i \varphi(v_i) \delta_{ij} = \varphi(v_j)$. So ψ and φ are linear maps $V \rightarrow F$ that agree on the basis $\mathcal{B}$, and $\psi = \varphi$ by the uniqueness part of Theorem 2.1.5. Hence $\varphi \in \text{span}(\varphi_1, \dots, \varphi_n)$, and $(\varphi_1, \dots, \varphi_n)$ is a basis of V^* .

- (c) The second formula was proved in the spanning step. The first is the definition of coordinates, since $\varphi_i(v)$ is the i -th coordinate of v by (a). This proves the theorem. ■

The two formulas in (c) are mirror images. A vector is recovered from its measurements by the dual basis; a functional is recovered from its values on the basis.

Definition 5.1.3. Dual Basis

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of a finite-dimensional vector space V over F . The basis $(\varphi_1, \dots, \varphi_n)$ of V^* given by Theorem 5.1.3, characterized by $\varphi_i(v_j) = \delta_{ij}$, is **the dual basis** of $\mathcal{B}$. We write it $\mathcal{B}^*$.

Counting the dual basis gives the dimension of the dual space at once.

Corollary 5.1.4. Dimension of the Dual Space

Let V be a finite-dimensional vector space over F . Then V^* is finite-dimensional and $\dim V^* = \dim V$. In particular $V^* \cong V$.

Proof. Let $n = \dim V$ and choose a basis of V (Corollary 1.5.5). By Theorem 5.1.3 (b), its dual basis is a basis of V^* of length n . Hence $\dim V^* = n$, and $V^* \cong V$ by Theorem 2.5.4. ■

For instance $\dim M_{m \times n}(F)^* = mn$ and $\dim F[x]_{\leq n}^* = n + 1$.

Remark. Many books write v_i^* for the dual basis functional φ_i . We avoid that notation for two reasons. It suggests that φ_i is determined by v_i alone, which is false (see the warning below). And in this book the star on a map is reserved for adjoints, which come much later. We write φ_i , and speak of “the dual basis of $\mathcal{B}$ ”.

How do we find a dual basis in practice? In F^n , functionals are rows, and the conditions $\varphi_i(v_j) = \delta_{ij}$ turn into a single matrix equation.

Example. The Dual Basis Is Read Off the Inverse Matrix

- (a) Find the dual basis of $\mathcal{B} = ((1, 1), (1, -1))$ in $\mathbb{R}^2$.
- (b) More generally, let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a basis of F^n , and let $\mathbf{P} = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n)$. Show that $\mathbf{P}$ is invertible and that the dual basis of $\mathcal{B}$ is $\varphi_i(\mathbf{x}) = \mathbf{r}_i \mathbf{x}$, where $\mathbf{r}_i$ is the i -th **row** of $\mathbf{P}^{-1}$.

Solution.

- (a) By [Theorem 5.1.1](#), $\varphi_1(x, y) = px + qy$ for some $p, q \in \mathbb{R}$, that is, $\varphi_1(\mathbf{x}) = \begin{pmatrix} p & q \end{pmatrix} \mathbf{x}$. The conditions $\varphi_1(1, 1) = 1$ and $\varphi_1(1, -1) = 0$ say $\begin{pmatrix} p & q \end{pmatrix} \mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ with $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Likewise the row $\mathbf{r}_2$ of φ_2 satisfies $\mathbf{r}_2 \mathbf{P} = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Stacking the two rows into a matrix $\mathbf{R}$, the four conditions become $\mathbf{R} \mathbf{P} = \mathbf{I}_2$, so $\mathbf{R} = \mathbf{P}^{-1}$. By [Theorem 0.9.8](#), with $ad - bc = -2$,

$$\mathbf{P}^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Hence

$$\varphi_1(x, y) = \frac{x+y}{2}, \quad \varphi_2(x, y) = \frac{x-y}{2}.$$

Check: $\varphi_1(1, 1) = 1$, $\varphi_1(1, -1) = 0$, $\varphi_2(1, 1) = 0$, $\varphi_2(1, -1) = 1$. And the first formula of [Theorem 5.1.3](#) (c) for $\mathbf{v} = (3, 1)$ reads $(3, 1) = 2(1, 1) + 1(1, -1)$, which is true. The second formula expresses a functional in the dual basis through its values on $\mathcal{B}$: for $\psi(x, y) = x$, the values are $\psi(1, 1) = 1$ and $\psi(1, -1) = 1$, so $\psi = \varphi_1 + \varphi_2$, and indeed $\frac{x+y}{2} + \frac{x-y}{2} = x$.

- (b) The columns of $\mathbf{P}$ form a basis of F^n , so $\mathbf{P}$ is invertible by [Theorem 3.4.4](#). Let $\mathbf{R} \in M_n(F)$ have rows $\mathbf{r}_1, \dots, \mathbf{r}_n$, and let $\psi_i(\mathbf{x}) := \mathbf{r}_i \mathbf{x}$. By the entry formula for a product ([Theorem 0.9.2](#)), $\psi_i(\mathbf{v}_j) = \mathbf{r}_i \mathbf{v}_j = (\mathbf{R} \mathbf{P})_{ij}$. Hence $\psi_i(\mathbf{v}_j) = \delta_{ij}$ for all i, j if and only if $\mathbf{R} \mathbf{P} = \mathbf{I}_n$, if and only if $\mathbf{R} = \mathbf{P}^{-1}$ (multiply by $\mathbf{P}^{-1}$ on the right). Now take $\mathbf{R} = \mathbf{P}^{-1}$: the functionals $\psi_i(\mathbf{x}) = \mathbf{r}_i \mathbf{x}$ are linear ([Theorem 5.1.1](#)) and satisfy $\psi_i(\mathbf{v}_j) = \delta_{ij}$, so by the uniqueness in [Theorem 5.1.3](#) (a) they are the dual basis.

This agrees with Chapter 2: $\mathbf{P} = [\text{id}]_{\mathcal{B}}^{\mathcal{E}}$ and $\mathbf{P}^{-1} = [\text{id}]_{\mathcal{E}}^{\mathcal{B}}$ ([Theorem 4.2.1](#)), so $\mathbf{P}^{-1} \mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, and row i of $\mathbf{P}^{-1}$ computes the i -th coordinate, as [Theorem 5.1.3](#) (a) predicts. The columns of $\mathbf{P}$ are the basis; the rows of $\mathbf{P}^{-1}$ are the dual basis.

Warning. A dual basis functional depends on the whole basis, not only on its own vector. In $\mathbb{R}^2$, the basis $((1, 0), (0, 1))$ has first dual functional $\varphi_1(x, y) = x$. The basis

$((1,0), (1,1))$ has the same first vector, but $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, so its first dual functional is $\varphi_1(x, y) = x - y$. Changing $\mathbf{v}_2$ changed φ_1 , because φ_1 must vanish on $\mathbf{v}_2$.

The same method works away from F^n , once we recognize the coordinates.

Example. Dual Bases in $\mathbb{R}[x]_{\leq 2}$

- (a) Show that the dual basis of $(1, x, x^2)$ in $\mathbb{R}[x]_{\leq 2}$ is $(\varphi_0, \varphi_1, \varphi_2)$ with $\varphi_0(p) = p(0)$, $\varphi_1(p) = p'(0)$, and $\varphi_2(p) = \frac{1}{2}p''(0)$. (We index from 0 to match the exponents.)
- (b) Let $\ell_{-1} = \frac{1}{2}x(x-1)$, $\ell_0 = 1 - x^2$, and $\ell_1 = \frac{1}{2}x(x+1)$. Show that $(\ell_{-1}, \ell_0, \ell_1)$ is a basis of $\mathbb{R}[x]_{\leq 2}$ whose dual basis is $(\varepsilon_{-1}, \varepsilon_0, \varepsilon_1)$, the evaluations at $-1, 0, 1$. Deduce that $p = p(-1)\ell_{-1} + p(0)\ell_0 + p(1)\ell_1$ for every $p \in \mathbb{R}[x]_{\leq 2}$.

Solution.

- (a) Each φ_k is linear, as a composition of differentiation (Example (Differentiation)) with evaluation at 0, scaled. For $p = a_0 + a_1x + a_2x^2$, we have $p(0) = a_0$, $p'(0) = a_1$, and $\frac{1}{2}p''(0) = \frac{1}{2} \cdot 2a_2 = a_2$. So $\varphi_k(x^j) = \delta_{kj}$ for $j, k \in \{0, 1, 2\}$, and by the uniqueness in [Theorem 5.1.3](#) (a) these are the dual basis. Over a general field F , the dual basis of $(1, x, x^2)$ is still “the coefficient of x^k ”, but the formula $\frac{1}{2}p''(0)$ needs $2 \neq 0$ in F .
- (b) The values of the ℓ ’s at the three points are

$$\begin{aligned}\ell_{-1}(-1) &= 1, \ell_{-1}(0) = 0, \ell_{-1}(1) = 0; \\ \ell_0(-1) &= 0, \ell_0(0) = 1, \ell_0(1) = 0; \\ \ell_1(-1) &= 0, \ell_1(0) = 0, \ell_1(1) = 1.\end{aligned}$$

That is, $\varepsilon_i(\ell_j) = \delta_{ij}$ for $i, j \in \{-1, 0, 1\}$. *Basis.* Let $c_{-1}\ell_{-1} + c_0\ell_0 + c_1\ell_1 = 0$. Evaluating at $j \in \{-1, 0, 1\}$ gives $c_j = 0$. So the list is independent, of length $3 = \dim \mathbb{R}[x]_{\leq 2}$, hence a basis by Theorem 1.6.4. *Dual basis.* The evaluations are linear ([Example \(Four Families of Functionals\)](#)) and satisfy $\varepsilon_i(\ell_j) = \delta_{ij}$, so they are the dual basis by the uniqueness in [Theorem 5.1.3](#) (a). The first formula of [Theorem 5.1.3](#) (c) now reads $p = \sum_i \varepsilon_i(p) \ell_i = p(-1)\ell_{-1} + p(0)\ell_0 + p(1)\ell_1$.

The last formula rebuilds a polynomial from three of its values. It is a preview of the Lagrange interpolation formula of Chapter 6, and it explains where the ℓ ’s come from: they are the basis dual to the evaluations.

Quick check. Let (v_1, v_2, v_3) be a basis of V with dual basis $(\varphi_1, \varphi_2, \varphi_3)$. Compute $\varphi_2(3v_1 - v_2 + 5v_3)$ and $(2\varphi_1 + \varphi_3)(v_3)$.

Solution. $\varphi_2(3v_1 - v_2 + 5v_3) = 3 \cdot 0 - 1 \cdot 1 + 5 \cdot 0 = -1$, the second coordinate. $(2\varphi_1 + \varphi_3)(v_3) = 2 \cdot 0 + 1 = 1$.

Warning. $V \cong V^*$ is true in finite dimension, but no isomorphism is singled out. Corollary 5.1.4 gives an isomorphism by sending $v_i \mapsto \varphi_i$, and that depends on the basis. In $\mathbb{R}^2$ with the standard basis, it sends $(1, 0)$ to the functional $(x, y) \mapsto x$. With the basis $((1, 0), (1, 1))$, it sends the **same** vector $(1, 0)$ to $(x, y) \mapsto x - y$, by the warning above. A vector does not come with “its” functional. Later in this chapter we make precise the sense in which V^* needs a choice to be identified with V , while the double dual V^{**} does not.

5.1.3 Testing against functionals

In F^n a vector is zero exactly when all its entries are zero, and the entries are the values of the coordinate functionals. The same holds in any finite-dimensional space, and it gives a flexible way to prove that a vector is zero.

Theorem 5.1.5. Functionals Separate Vectors

Let V be a finite-dimensional vector space over F .

- (a) If $v \in V$ and $v \neq 0$, then there is $\varphi \in V^*$ with $\varphi(v) = 1$.
- (b) If $v \in V$ satisfies $\varphi(v) = 0$ for **every** $\varphi \in V^*$, then $v = 0$. It suffices that $\varphi_i(v) = 0$ for the functionals φ_i of **one** dual basis.
- (c) For $u, w \in V$: $u = w$ if and only if $\varphi(u) = \varphi(w)$ for every $\varphi \in V^*$.

Idea. We need a functional that does not vanish at v , and the only functionals we can build at will are dual basis functionals. The first one takes the value 1 on the first basis vector. So make v the first vector of a basis: this is possible exactly because $v \neq 0$.

Proof.

- (a) Since $v \neq 0$, the list (v) is linearly independent: if $av = 0$, then $a = 0$ by Theorem 1.1.9. By the Basis Extension Theorem (Theorem 1.6.5), there is a basis $(v, v_2, \dots, v_n)$ of V . Let $(\varphi_1, \dots, \varphi_n)$ be its dual basis (Theorem 5.1.3). Then $\varphi_1(v) = 1$.
- (b) The first statement is the contrapositive of (a). For the second, if $(\varphi_1, \dots, \varphi_n)$ is the dual basis of a basis $(v_1, \dots, v_n)$ and every $\varphi_i(v) = 0$, then $v = \sum_i \varphi_i(v)v_i = 0$ by Theorem 5.1.3 (c).
- (c) If $u = w$, the values agree. Conversely, if $\varphi(u) = \varphi(w)$ for every φ , then $\varphi(u - w) = \varphi(u) - \varphi(w) = 0$ for every φ , so $u - w = 0$ by (b). ■

This is the signature move of the chapter: **to show a vector is zero, test it against every functional, and a dual basis suffices.** It turns one equation in V into scalar equations, which are often easier to reach.

Remark. Parts (a), (b) (first sentence) and (c) hold for **every** vector space V ; only (a) needs a new construction. By Theorem 1.8.2, $\{v\}$ extends to a basis B of V . Every $w \in V$ is a finite combination of distinct vectors of B , with coefficients that are unique because B is

independent. Let $\varphi(w)$ be the coefficient of v in this expansion (and 0 if v does not occur). Adding or scaling expansions gives expansions, so by uniqueness coefficients add and scale, and φ is linear with $\varphi(v) = 1$. The price is Zorn's Lemma (Theorem 1.8.1): for $\mathbb{R}$ over $\mathbb{Q}$, no one has an explicit formula for such a functional.

Example. A Matrix Killed by Every Trace

Let $\mathbf{A} \in M_n(F)$ satisfy $\text{tr}(\mathbf{A}\mathbf{X}) = 0$ for every $\mathbf{X} \in M_n(F)$. Show that $\mathbf{A} = \mathbf{O}$. Deduce that every linear functional on $M_n(F)$ is $\mathbf{X} \mapsto \text{tr}(\mathbf{A}\mathbf{X})$ for exactly one $\mathbf{A} \in M_n(F)$.

Solution. The matrix units $\mathbf{E}_{ij}$ form a basis of $M_n(F)$ (Example (Standard Bases)), and $\mathbf{A} = \sum_{i,j} a_{ij} \mathbf{E}_{ij}$. By Theorem 5.1.3 (a), its dual basis consists of the coordinate functionals $\varepsilon_{ij}(\mathbf{A}) = a_{ij}$. We compute $\varepsilon_{ij}(\mathbf{A})$ as a trace. By Definition 0.9.9, $(\mathbf{A}\mathbf{E}_{ji})_{kk} = \sum_l a_{kl} (\mathbf{E}_{ji})_{lk}$, and $(\mathbf{E}_{ji})_{lk} = 1$ only for $l = j$ and $k = i$. So the only nonzero diagonal entry of $\mathbf{A}\mathbf{E}_{ji}$ is $(\mathbf{A}\mathbf{E}_{ji})_{ii} = a_{ij}$, and

$$\text{tr}(\mathbf{A}\mathbf{E}_{ji}) = a_{ij} = \varepsilon_{ij}(\mathbf{A}).$$

Taking $\mathbf{X} = \mathbf{E}_{ji}$ in the hypothesis gives $\varepsilon_{ij}(\mathbf{A}) = 0$ for all i, j . By Theorem 5.1.5 (b), $\mathbf{A} = \mathbf{O}$.

For the deduction, let $\Theta: M_n(F) \rightarrow M_n(F)^*$, $\Theta(\mathbf{A})(\mathbf{X}) := \text{tr}(\mathbf{A}\mathbf{X})$. Each $\Theta(\mathbf{A})$ is linear in $\mathbf{X}$, and Θ is linear in $\mathbf{A}$, both by Theorem 0.9.6 (1) and the distributive laws of Theorem 0.9.4. We just showed $\ker \Theta = \{\mathbf{O}\}$, so Θ is injective (Theorem 2.2.3). Since $\dim M_n(F)^* = n^2 = \dim M_n(F)$ (Corollary 5.1.4), Θ is bijective by Corollary 2.3.2 (e). This is the matrix analogue of Theorem 5.1.1.

5.1.4 When V is infinite-dimensional

The main results of the last two subsections assumed finite dimension, or used a finite basis. That hypothesis does real work. Consider $F[x]$, with its basis $\{1, x, x^2, \dots\}$ (Example (A Basis of $F[x]$)). For $k \in \mathbb{N}$ let

$$\varphi_k(p) := \text{the coefficient of } x^k \text{ in } p.$$

These are the coordinate functionals, and $\varphi_k(x^j) = \delta_{kj}$, exactly as in Theorem 5.1.3. But they are not a basis of the dual space.

Proposition 5.1.6. The Coordinate Functionals Do Not Span $F[x]^*$

The functionals φ_k ($k \in \mathbb{N}$) lie in $F[x]^*$ and form a linearly independent set. But they do **not** span $F[x]^*$: the evaluation $\varepsilon_1: p \mapsto p(1)$ is **not** a linear combination of finitely many φ_k .

Proof. Polynomials are added and scaled coefficient by coefficient (Definition 0.8.1), so each φ_k is linear. *Independence.* Let $k_1, \dots, k_m$ be distinct and $c_1\varphi_{k_1} + \dots + c_m\varphi_{k_m} = 0$. Applying both sides to x^{k_j} gives $c_j = 0$, since $\varphi_{k_i}(x^{k_j}) = \delta_{k_i k_j}$.

Not spanning. The evaluation ε_1 is linear by Example (Four Families of Functionals) (a), and $\varepsilon_1(x^N) = 1^N = 1$ for every $N \in \mathbb{N}$. Suppose, for a contradiction, that $\varepsilon_1 = c_1\varphi_{k_1} + \dots + c_m\varphi_{k_m}$ for some $m \in \mathbb{N}$, distinct k_j , and $c_j \in F$. Choose $N \in \mathbb{N}$ larger than every k_j . Then $\varphi_{k_j}(x^N) = 0$ for every j , so the right-hand side sends x^N to 0, while $\varepsilon_1(x^N) = 1$. This

contradiction shows $\varepsilon_1 \notin \text{span}\{\varphi_k : k \in \mathbb{N}\}$. ■

Where did the proof of [Theorem 5.1.3](#) break? In the spanning step. The candidate $\sum_i \varphi(v_i)\varphi_i$ for $\varphi = \varepsilon_1$ would be $\varphi_0 + \varphi_1 + \varphi_2 + \cdots$, an infinite sum, and infinite sums mean nothing in a vector space (the warning in Chapter 1, §8). A functional on $F[x]$ may take nonzero values on infinitely many basis vectors, while a finite combination of the φ_k cannot.

Remark. More is true, and we state it without proof. For every infinite-dimensional V , a basis of V^* is **strictly larger**, as an infinite set, than a basis of V ; in particular $V^* \not\cong V$. So “ $\dim V^* = \dim V$ ” is a genuinely finite-dimensional fact, and this is why finite dimension appears in the hypotheses of most theorems in this chapter. A first step towards seeing how large $F[x]^*$ is appears in Exercise C4 below.

5.1.5 Exercises

5.1.5.1 A. Check your understanding

Exercise. A1

- (a) Define a linear functional on a vector space V over F , and the dual space V^* .
- (b) True or false: $p \mapsto p(1)^2$ is a linear functional on $\mathbb{R}[x]$. Justify your answer.
- (c) True or false: if (v_1, v_2) and (w_1, w_2) are bases of V , then the first functionals of their dual bases are equal. Justify your answer.
- (d) What is $\dim M_{2 \times 3}(F)^*$?
- (e) State the formula that expresses $\varphi \in V^*$ in the dual basis of $(v_1, \dots, v_n)$.
- (f) Name the move you would use to prove that a vector v of a finite-dimensional space is 0 without computing v itself.

Solution.

- (a) See [Definition 5.1.1](#) and [Definition 5.1.2](#): a linear functional is a linear map $V \rightarrow F$, and $V^* = \mathcal{L}(V, F)$ with pointwise addition and scaling.
- (b) False. For $p = 1$ and $c = 2$, the polynomial $cp = 2$ is sent to $2^2 = 4$, while c times the image of p is $2 \cdot 1^2 = 2$. So the scaling condition fails.
- (c) False. By the warning after [Example \(The Dual Basis Is Read Off the Inverse Matrix\)](#), in $\mathbb{R}^2$ the bases $((1, 0), (0, 1))$ and $((1, 0), (1, 1))$ have first dual functionals x and $x - y$.
- (d) 6, by [Corollary 5.1.4](#), since $\dim M_{2 \times 3}(F) = 6$.
- (e) $\varphi = \sum_{i=1}^n \varphi(v_i)\varphi_i$, by [Theorem 5.1.3](#) (c).
- (f) Test against functionals: show $\varphi(v) = 0$ for every $\varphi \in V^*$, or only for the functionals of one dual basis, and apply [Theorem 5.1.5](#) (b).

5.1.5.2 B. Practice

Exercise. B1: Finding dual bases

- (a) Find the dual basis of $((1, 0, 1), (0, 1, 1), (1, 1, 1))$ in $\mathbb{R}^3$.
- (b) Show that the dual basis of $(1, x-1, (x-1)^2)$ in $\mathbb{R}[x]_{\leq 2}$ is $p \mapsto p(1)$, $p \mapsto p'(1)$, $p \mapsto \frac{1}{2}p''(1)$. Hence write $p = 2 + 3x - x^2$ in this basis.

Solution.

- (a) Let $\mathbf{P}$ have the three vectors as columns. Row reduction of $[\mathbf{P} \mid \mathbf{I}_3]$ (Theorem 3.4.6) gives

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

and one checks $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}_3$. By [Example \(The Dual Basis Is Read Off the Inverse Matrix\)](#) (b), the dual basis is

$$\varphi_1(x, y, z) = -y + z, \quad \varphi_2(x, y, z) = -x + z, \quad \varphi_3(x, y, z) = x + y - z.$$

For instance $\varphi_3(1, 0, 1) = 0$, $\varphi_3(0, 1, 1) = 0$, and $\varphi_3(1, 1, 1) = 1$.

- (b) The three maps are linear, as combinations of differentiation and evaluation at 1. Call them ψ_1, ψ_2, ψ_3 . On the basis:

$$\psi_1(1) = 1, \quad \psi_1(x-1) = 0, \quad \psi_1((x-1)^2) = 0;$$

$$\psi_2(1) = 0, \quad \psi_2(x-1) = 1, \quad \psi_2((x-1)^2) = 2(1-1) = 0;$$

$$\psi_3(1) = 0, \quad \psi_3(x-1) = 0, \quad \psi_3((x-1)^2) = \frac{1}{2} \cdot 2 = 1.$$

So ψ_i takes the value δ_{ij} on the j -th basis vector, and by the uniqueness in [Theorem 5.1.3](#) (a) they are the dual basis. (The list $(1, x-1, (x-1)^2)$ is a basis: it has distinct degrees 0, 1, 2, so it is independent by Theorem 1.4.4, and it has length $3 = \dim \mathbb{R}[x]_{\leq 2}$, so Theorem 1.6.4 applies.) For $p = 2 + 3x - x^2$: $p(1) = 4$, $p'(1) = 3 - 2 = 1$, $\frac{1}{2}p''(1) = -1$. By [Theorem 5.1.3](#) (c),

$$p = 4 + (x-1) - (x-1)^2.$$

Check: $4 + x - 1 - (x^2 - 2x + 1) = 2 + 3x - x^2$.

Exercise. B2: Which maps are functionals?

Determine which of the following are linear functionals. Justify your answer.

- (a) $\mathbb{R}^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto 2x - z$.
- (b) $\mathbb{R}^3 \rightarrow \mathbb{R}$, $(x, y, z) \mapsto xy$.
- (c) $M_2(\mathbb{R}) \rightarrow \mathbb{R}$, $\mathbf{A} \mapsto a_{12} - 3a_{21}$.
- (d) $\mathbb{R}[x] \rightarrow \mathbb{R}$, $p \mapsto \int_0^1 t p(t) dt$.
- (e) $\mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}$, $p \mapsto p(0) + p'(0) + 1$.
- (f) $\mathbb{C}^2 \rightarrow \mathbb{C}$, $(z, w) \mapsto z + \overline{w}$, where $\mathbb{C}^2$ is a vector space over $\mathbb{C}$.

Solution.

- (a) Yes: it is $\varphi_{\mathbf{a}}$ with $\mathbf{a} = (2, 0, -1)$, linear by [Theorem 5.1.1](#).
- (b) No. $(1, 1, 0) \mapsto 1$, but $2(1, 1, 0) = (2, 2, 0) \mapsto 4 \neq 2$, so the scaling condition fails.
- (c) Yes: it is $\varepsilon_{12} - 3\varepsilon_{21}$, a combination of coordinate functionals for the standard basis, and V^* is a vector space ([Definition 5.1.2](#)).
- (d) Yes. By linearity of the integral, $\int_0^1 t(p+q)(t) dt = \int_0^1 tp(t) dt + \int_0^1 tq(t) dt$ and $\int_0^1 t(cp)(t) dt = c \int_0^1 tp(t) dt$.
- (e) No. The zero polynomial is sent to $1 \neq 0$, contradicting Theorem 2.1.1.
- (f) No. With $c = i$ and $(z, w) = (0, 1)$: $(0, i) \mapsto \bar{i} = -i$, while $i \cdot (0 + \bar{1}) = i$. The map is additive and respects real scalars, but it is not linear over $\mathbb{C}$.

Exercise. B3: Coordinates of a functional

- (a) Let (φ_1, φ_2) be the dual basis of $((1, 1), (1, -1))$ from [Example \(The Dual Basis Is Read Off the Inverse Matrix\)](#). Write $\psi(x, y) = 3x + 5y$ as a combination of φ_1 and φ_2 .
- (b) Let $(\varphi_0, \varphi_1, \varphi_2)$ be the dual basis of $(1, x, x^2)$ from [Example \(Dual Bases in \$\mathbb{R}\[x\]_{\leq 2}\$ \)](#). Write $\iota(p) = \int_0^1 p(t) dt$ as a combination of them.

Solution.

- (a) By [Theorem 5.1.3](#) (c), $\psi = \psi(1, 1)\varphi_1 + \psi(1, -1)\varphi_2 = 8\varphi_1 - 2\varphi_2$. Check: $8 \cdot \frac{x+y}{2} - 2 \cdot \frac{x-y}{2} = 4x + 4y - x + y = 3x + 5y$.
- (b) By [Theorem 5.1.3](#) (c), $\iota = \iota(1)\varphi_0 + \iota(x)\varphi_1 + \iota(x^2)\varphi_2 = \varphi_0 + \frac{1}{2}\varphi_1 + \frac{1}{3}\varphi_2$. Check: for $p = a_0 + a_1x + a_2x^2$, the right-hand side is $a_0 + \frac{a_1}{2} + \frac{a_2}{3} = \int_0^1 p$.

5.1.5.3 C. Going deeper

Exercise. C1: Functionals with the same kernel

Let V be **any** vector space over F , and let $\varphi, \psi \in V^*$ with $\ker \varphi = \ker \psi$. Prove that $\psi = c\varphi$ for some $c \in F$. Is the converse true?

Hint: if $\varphi \neq 0$, fix u with $\varphi(u) = 1$ and consider $v - \varphi(v)u$.

Solution. If $\varphi = 0$, then $\ker \psi = \ker \varphi = V$, so $\psi = 0 = 1 \cdot \varphi$. Suppose $\varphi \neq 0$. By [Proposition 5.1.2](#) there is $u \in V$ with $\varphi(u) = 1$. Let $v \in V$. Then

$$\varphi(v - \varphi(v)u) = \varphi(v) - \varphi(v)\varphi(u) = 0,$$

so $v - \varphi(v)u \in \ker \varphi = \ker \psi$. Therefore $0 = \psi(v - \varphi(v)u) = \psi(v) - \varphi(v)\psi(u)$, that is, $\psi(v) = \psi(u)\varphi(v)$. Since v was arbitrary, $\psi = c\varphi$ with $c = \psi(u)$.

The converse holds only for $c \neq 0$: if $\psi = c\varphi$ with $c \neq 0$, then $\psi(v) = 0 \iff \varphi(v) = 0$. For $c = 0$ and $\varphi \neq 0$, $\ker \psi = V \neq \ker \varphi$.

Exercise. C2: Functionals vanishing where others vanish

Let V be a vector space over F , and let $\varphi_1, \dots, \varphi_k, \psi \in V^*$.

- (a) Prove that if $\ker \varphi_1 \cap \dots \cap \ker \varphi_k \subseteq \ker \psi$, then $\psi \in \text{span}(\varphi_1, \dots, \varphi_k)$.
- (b) Deduce that if V is finite-dimensional, then $(\varphi_1, \dots, \varphi_k)$ spans V^* if and only if $\ker \varphi_1 \cap \dots \cap \ker \varphi_k = \{0\}$.

Hint: for (a), consider the linear map $T: V \rightarrow F^k$, $v \mapsto (\varphi_1(v), \dots, \varphi_k(v))$.

Solution.

- (a) Let T be as in the hint; it is linear because each φ_i is. Its kernel is $\ker \varphi_1 \cap \dots \cap \ker \varphi_k$. Define $S: \text{im } T \rightarrow F$ by $S(Tv) := \psi(v)$. This is well defined: if $Tv = Tv'$, then $v - v' \in \ker T \subseteq \ker \psi$, so $\psi(v) = \psi(v')$. It is linear, since $S(cTv + Tv') = S(T(cv + v')) = \psi(cv + v') = c\psi(v) + \psi(v') = cS(Tv) + S(Tv')$.

Now extend S to F^k . The subspace $\text{im } T \subseteq F^k$ has a basis $(\mathbf{y}_1, \dots, \mathbf{y}_r)$ (Theorem 1.6.7, Corollary 1.5.5), which extends to a basis $(\mathbf{y}_1, \dots, \mathbf{y}_k)$ of F^k by Theorem 1.6.5. By Theorem 2.1.5 there is $\tilde{S} \in (F^k)^*$ with $\tilde{S}(\mathbf{y}_j) = S(\mathbf{y}_j)$ for $j \leq r$ and $\tilde{S}(\mathbf{y}_j) = 0$ for $j > r$. Then $\tilde{S}$ and S agree on a basis of $\text{im } T$, so $\tilde{S} = S$ on $\text{im } T$. By Theorem 5.1.1, $\tilde{S}(z_1, \dots, z_k) = a_1 z_1 + \dots + a_k z_k$ for some $a_i \in F$. Hence for every $v \in V$,

$$\psi(v) = S(Tv) = \tilde{S}(\varphi_1(v), \dots, \varphi_k(v)) = a_1 \varphi_1(v) + \dots + a_k \varphi_k(v),$$

so $\psi = a_1 \varphi_1 + \dots + a_k \varphi_k$. Note that V was arbitrary; only F^k needed to be finite-dimensional.

- (b) ($\Leftarrow$) If the intersection is $\{0\}$, it lies in $\ker \psi$ for every $\psi \in V^*$, so every ψ is in the span by (a). ($\Rightarrow$) Suppose the φ_i span V^* , and let v lie in every $\ker \varphi_i$. Every $\psi \in V^*$ is a combination $\sum c_i \varphi_i$, so $\psi(v) = \sum c_i \varphi_i(v) = 0$. By Theorem 5.1.5 (b), $v = 0$.

Exercise. C3: Evaluations as a dual basis

Let $n \in \mathbb{N}$.

- (a) Let $x_0, \dots, x_n \in F$ be **distinct**. Prove that the evaluations $(\varepsilon_{x_0}, \dots, \varepsilon_{x_n})$ form a basis of $F[x]_{\leq n}^*$, and that they are the dual basis of some basis of $F[x]_{\leq n}$.
- (b) Over $\mathbb{F}_2$ with $n = 2$, show that the evaluations at the elements of $\mathbb{F}_2$ do **not** span $\mathbb{F}_2[x]_{\leq 2}^*$.

Hint: for (a), use Theorem 3.7.3 to build the basis first.

Solution.

- (a) For each j , Theorem 3.7.3 gives $\ell_j \in F[x]_{\leq n}$ with $\ell_j(x_i) = \delta_{ij}$ for $i = 0, \dots, n$, because the x_i are distinct. If $\sum_j c_j \ell_j = 0$, evaluating at x_i gives $c_i = 0$. So $(\ell_0, \dots, \ell_n)$ is independent of length $n + 1 = \dim F[x]_{\leq n}$, hence a basis by Theorem 1.6.4. The evaluations are linear (Example (Four Families of Functionals)) and $\varepsilon_{x_i}(\ell_j) = \delta_{ij}$, so by the uniqueness in Theorem 5.1.3 (a) they are the dual basis of $(\ell_0, \dots, \ell_n)$, and by Theorem 5.1.3 (b) a basis of $F[x]_{\leq n}^*$.

- (b) $\mathbb{F}_2 = \{0, 1\}$ has only two elements, so only ε_0 and ε_1 are available, and $\dim \mathbb{F}_2[x]_{\leq 2}^* = 3$ by [Corollary 5.1.4](#). Two functionals cannot span a 3-dimensional space (Theorem 1.6.3). Concretely, $p = x^2 + x$ is nonzero but $p(0) = 0$ and $p(1) = 1 + 1 = 0$; the coefficient functional of x^2 takes the value 1 on p , so it is not a combination of $\varepsilon_0, \varepsilon_1$. The hypothesis “distinct points” in (a) needs $n + 1$ distinct elements of F , which a small field may not have.

Exercise. C4: How large is $F[x]^*$?

Let $F^{\mathbb{N}}$ be the space of sequences with entries in F .

- (a) Prove that $\Phi: F[x]^* \rightarrow F^{\mathbb{N}}, \varphi \mapsto (\varphi(1), \varphi(x), \varphi(x^2), \dots)$, is an isomorphism.
 (b) Identify $\Phi(\varphi_k)$ for the coordinate functionals φ_k of [Proposition 5.1.6](#), and $\Phi(\varepsilon_1)$. Explain how [Proposition 5.1.6](#) matches Example (The Vectors e_i Do Not Span $F^{\mathbb{N}}$).

Hint: for surjectivity, remember that a polynomial has only finitely many nonzero coefficients.

Solution.

- (a) *Linear.* The k -th entry of $\Phi(c\varphi + \psi)$ is $(c\varphi + \psi)(x^k) = c\varphi(x^k) + \psi(x^k)$, by the pointwise operations of [Definition 5.1.2](#). *Injective.* If $\Phi(\varphi) = 0$, then φ vanishes on the basis $\{x^k\}$ of Example (A Basis of $F[x]$). Every polynomial is a finite combination of the x^k , so $\varphi = 0$ by Theorem 2.1.4; hence Φ is injective by Theorem 2.2.3. *Surjective.* Given $(a_0, a_1, \dots) \in F^{\mathbb{N}}$, define $\varphi(b_0 + b_1x + \dots + b_Nx^N) := a_0b_0 + \dots + a_Nb_N$. This is a finite sum, it is well defined because coefficients are unique, and it is linear because coefficients add and scale; it is the linear map with $\varphi(x^k) = a_k$ given by Theorem 2.1.6. Then $\Phi(\varphi) = (a_0, a_1, \dots)$. By Theorem 2.5.1, Φ is an isomorphism.
 (b) $\Phi(\varphi_k) = (\delta_{k0}, \delta_{k1}, \dots) = e_k$, and $\Phi(\varepsilon_1) = (1, 1, 1, \dots)$. An isomorphism carries spans to spans, so $\varepsilon_1 \in \text{span}\{\varphi_k\}$ would give $(1, 1, 1, \dots) \in \text{span}\{e_k\}$. Example (The Vectors e_i Do Not Span $F^{\mathbb{N}}$) showed this is false, because finite combinations of the e_k have only finitely many nonzero entries. So $F[x]$, with its basis $\{x^k\}$ indexed by $\mathbb{N}$, has a dual as large as the whole sequence space $F^{\mathbb{N}}$.