

4.4 Products and Quotient Spaces

So far every new vector space in this book has been found **inside** an old one, as a subspace. This section builds new spaces in two other ways. The first glues spaces together side by side, making a product; it is the “external” direct sum promised in Chapter 1. The second, the quotient space V/U , is subtler and more important: it collapses a subspace U to a single point and keeps only what V knows beyond U . Quotients let us state precisely what a linear map remembers and what it forgets, in the three Isomorphism Theorems.

4.4.1 Products of vector spaces

In Chapter 1 we wrote $V = U \oplus W$ for subspaces U and W of one space V (Definition 1.7.2). The two pieces already lived in a common space, and “direct” was a property of how they sat there. But often we want to combine two spaces that have nothing to do with each other, such as $\mathbb{R}^2$ and $\mathbb{R}[x]_{\leq 1}$, into one space in which each keeps its identity. There is no common space to add inside, so we build one.

The model is F^2 . A vector of F^2 is a pair (a, b) of scalars, and we add and scale pairs entry by entry. Nothing in that recipe needs the entries to be scalars: the entries could be vectors, taken from any vector spaces over F .

The product of vector spaces is the space of lists with one entry from each space, added and scaled entry by entry.

Definition 4.4.1. Product of vector spaces

Let $V_1, \dots, V_k$ be vector spaces over the **same** field F . Their **product** (or **external direct sum**) is the Cartesian product (Definition 0.3.10)

$$V_1 \times \cdots \times V_k = \{(v_1, \dots, v_k) : v_i \in V_i \text{ for each } i\},$$

with addition and scalar multiplication defined **entrywise**: for $a \in F$,

$$\begin{aligned} (v_1, \dots, v_k) + (w_1, \dots, w_k) &:= (v_1 + w_1, \dots, v_k + w_k), \\ a(v_1, \dots, v_k) &:= (av_1, \dots, av_k). \end{aligned}$$

In words: the i -th entry of a sum is the sum of the i -th entries, computed **in** V_i , and similarly for scalar multiples. Two elements are equal exactly when all their entries are equal. Every operation happens inside one V_i at a time, which is why all the V_i must share the field F : the same scalar a has to act on every entry.

It is a vector space. Both operations land in $V_1 \times \cdots \times V_k$, since $v_i + w_i \in V_i$ and $av_i \in V_i$. Each axiom of Definition 1.1.1 is an equation between lists, so it holds when it holds in each entry, and in each entry it is the same axiom in V_i . For example, (VS6) reads

$$\begin{aligned} (a + b)(v_1, \dots, v_k) &= ((a + b)v_1, \dots, (a + b)v_k) \\ &= (av_1 + bv_1, \dots, av_k + bv_k) \\ &= a(v_1, \dots, v_k) + b(v_1, \dots, v_k), \end{aligned}$$

using (VS6) in each V_i . The zero vector is $(0, \dots, 0)$, with the zero of V_i in the i -th entry, and the additive inverse of $(v_1, \dots, v_k)$ is $(-v_1, \dots, -v_k)$. The other axioms are checked in exactly the same way.

Examples.

- **The standard example.** $F \times \cdots \times F$ (n factors) is F^n with its usual operations.
- **Unrelated spaces.** An element of $\mathbb{R}^2 \times \mathbb{R}[x]_{\leq 1}$ is a pair $((a, b), c + dx)$, and $((1, 0), x) + ((2, 5), 3 - x) = ((3, 5), 3)$.
- **Augmented data.** A linear system $\mathbf{Ax} = \mathbf{b}$ with m equations in n unknowns is a single element $(\mathbf{A}, \mathbf{b})$ of $M_{m \times n}(F) \times F^m$.
- **Degenerate cases.** For $k = 1$ the product is V_1 itself, with entries written in parentheses. And $V \times \{0\}$ consists of the pairs $(v, 0)$; the map $v \mapsto (v, 0)$ is a linear bijection $V \rightarrow V \times \{0\}$, so a zero factor adds nothing.

Nonexample by minimal change. Take $V_1 = \mathbb{C}$ as a vector space over $\mathbb{C}$ and $V_2 = \mathbb{R}$ over $\mathbb{R}$. The set $\mathbb{C} \times \mathbb{R}$ is fine, but the entrywise rule $i \cdot (z, t) = (iz, it)$ fails: $it \notin \mathbb{R}$ for $t \neq 0$. The clause that fails is “over the **same** field”. Restricting the scalars of $\mathbb{C}$ to $\mathbb{R}$ (Definition 1.9.1) repairs it, and $\mathbb{C} \times \mathbb{R}$ over $\mathbb{R}$ is a vector space of dimension 3.

Bases of the factors combine into a basis of the product. For $v \in V_i$, write $\iota_i(v)$ for the list with v in the i -th entry and 0 in every other entry. The map $\iota_i: V_i \rightarrow V_1 \times \cdots \times V_k$ is linear, because the operations are entrywise.

Theorem 4.4.1. Dimension of a Product

Let $V_1, \dots, V_k$ be finite-dimensional vector spaces over F , and let $(b_{i1}, \dots, b_{in_i})$ be a basis of V_i for each i . Then the list of all vectors $\iota_i(b_{ij})$, for $i = 1, \dots, k$ and $j = 1, \dots, n_i$, is a basis of $V_1 \times \cdots \times V_k$. In particular,

$$\dim(V_1 \times \cdots \times V_k) = \dim V_1 + \cdots + \dim V_k.$$

Proof. Spanning. Let $(v_1, \dots, v_k)$ be in the product. Since the operations are entrywise, $(v_1, \dots, v_k) = \iota_1(v_1) + \cdots + \iota_k(v_k)$. Each v_i is a combination $\sum_j c_{ij} b_{ij}$, and ι_i is linear, so $\iota_i(v_i) = \sum_j c_{ij} \iota_i(b_{ij})$ (Theorem 2.1.4). Hence $(v_1, \dots, v_k)$ is a combination of the vectors $\iota_i(b_{ij})$.

Independence. Let $\sum_{i=1}^k \sum_{j=1}^{n_i} c_{ij} \iota_i(b_{ij}) = 0$. The i -th entry of the left side is $\sum_j c_{ij} b_{ij}$, because every $\iota_{i'}(b_{i'j})$ with $i' \neq i$ has 0 there. So $\sum_j c_{ij} b_{ij} = 0$ in V_i for each i , and the independence of the basis of V_i gives $c_{ij} = 0$ for all j .

The list has $n_1 + \cdots + n_k$ vectors, which gives the dimension (Definition 1.6.1). ■

For instance, $\dim(\mathbb{R}^2 \times \mathbb{R}[x]_{\leq 1}) = 2 + 2 = 4$, so this product is isomorphic to $\mathbb{R}^4$ by Theorem 2.5.4.

How does the external product relate to the internal direct sum of Chapter 1? If $U_1, \dots, U_k$ are subspaces of V , there is a natural linear map from the product to V : add up the entries. The sum is direct exactly when this map loses no information.

Theorem 4.4.2. Internal and External Direct Sums

(a) Let $U_1, \dots, U_k$ be subspaces of a vector space V , and let

$$S: U_1 \times \cdots \times U_k \rightarrow V, \quad S(u_1, \dots, u_k) = u_1 + \cdots + u_k.$$

Then S is linear with $\text{im } S = U_1 + \cdots + U_k$. It is injective if and only if the sum $U_1 + \cdots + U_k$ is direct. In that case S is an isomorphism from $U_1 \times \cdots \times U_k$ onto $U_1 \oplus \cdots \oplus U_k$.

- (b) Conversely, let $V_1, \dots, V_k$ be vector spaces over F . Then the images $\iota_i(V_i)$ are subspaces of $V_1 \times \cdots \times V_k$, each isomorphic to V_i via ι_i , and

$$V_1 \times \cdots \times V_k = \iota_1(V_1) \oplus \cdots \oplus \iota_k(V_k).$$

Proof.

- (a) For lists (u_i) , (u'_i) , and $a \in F$, regrouping the finite sums gives $S((u_i) + (u'_i)) = \sum_i (u_i + u'_i) = S((u_i)) + S((u'_i))$ and $S(a(u_i)) = \sum_i au_i = aS((u_i))$, so S is linear. Its image is $\{u_1 + \cdots + u_k : u_i \in U_i\}$, which is $U_1 + \cdots + U_k$ by Definition 1.7.1. Its kernel consists of the lists with $u_1 + \cdots + u_k = 0$. By Theorem 2.2.3, S is injective if and only if the only such list is $(0, \dots, 0)$, which is condition (b) of Theorem 1.7.5, equivalent to the sum being direct. In that case S , viewed as a map onto its image, is a linear bijection, hence an isomorphism (Definition 2.5.2).
- (b) Each ι_i is linear and injective (its kernel is $\{0\}$, since $\iota_i(v)$ has v as its i -th entry), so $\iota_i(V_i)$ is a subspace (Theorem 2.2.2) and ι_i is a bijection onto it. Every list satisfies $(v_1, \dots, v_k) = \iota_1(v_1) + \cdots + \iota_k(v_k)$, so the product is the sum of the subspaces $\iota_i(V_i)$. If $\iota_1(v_1) + \cdots + \iota_k(v_k) = 0$, then $(v_1, \dots, v_k) = (0, \dots, 0)$, so every piece $\iota_i(v_i)$ is 0. By Theorem 1.7.5 ((b) $\Rightarrow$ (a)) the sum is direct. ■

So internal and external direct sums are the same thing seen from two sides. An internal direct sum is a product that happens to live inside V ; a product is an internal direct sum of copies of its factors. Many books write $\oplus$ for both. This book keeps $\times$ for the external construction, so that $U \oplus W$ always refers to subspaces of one space.

Quick check. Use Theorem 4.4.2 (a) and Rank–Nullity to explain why, for **finite-dimensional** subspaces U, W of V , the sum $U + W$ is direct if and only if $\dim(U + W) = \dim U + \dim W$.

Solution. The addition map $S: U \times W \rightarrow V$ has image $U + W$, and $\dim(U \times W) = \dim U + \dim W$ by Theorem 4.4.1. By Theorem 2.3.1, $\dim U + \dim W = \dim(U + W) + \dim \ker S$. So $\dim(U + W) = \dim U + \dim W$ exactly when $\ker S = \{0\}$, that is, when S is injective, which by the theorem means the sum is direct. This recovers Theorem 1.7.4 (a) $\Leftrightarrow$ (c).

4.4.2 Cosets

The second construction starts from a picture you have already seen. In Chapter 1 we compared the line $U = \{x + 2y = 0\}$ through the origin with the parallel line $L = \{x + 2y = 3\}$, which is not a subspace. In Chapter 3 we found that the solutions of a consistent system $\mathbf{Ax} = \mathbf{b}$ form a set $\mathbf{p} + \text{nul}(\mathbf{A})$, a subspace shifted by one particular solution (Theorem 3.1.3). Both are translates of a subspace. They deserve a name.

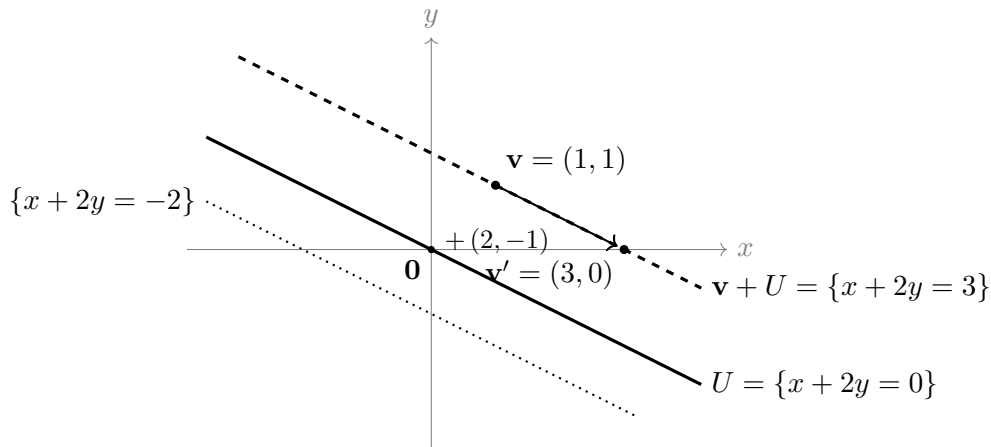
Definition 4.4.2. Coset

Let U be a subspace of a vector space V , and let $v \in V$. The **coset** of U through v is the subset

$$v + U := \{v + u : u \in U\} \subseteq V.$$

Any $w \in v + U$ is called a **representative** of the coset.

In $\mathbb{R}^2$, with $U = \{x + 2y = 0\} = \text{span}((2, -1))$, the coset through $(3, 0)$ is the line $\{x + 2y = 3\}$: a point $(3, 0) + t(2, -1) = (3 + 2t, -t)$ satisfies $x + 2y = 3$, and every solution of $x + 2y = 3$ is of this form with $t = -y$. In general $(a, b) + U = \{x + 2y = a + 2b\}$, so the cosets of U are exactly the lines parallel to U .



The picture shows three cosets of U : U itself (solid), the line through $\mathbf{v} = (1, 1)$ (dashed), and another parallel line (dotted). The points $\mathbf{v} = (1, 1)$ and $\mathbf{v}' = (3, 0)$ lie on the same dashed line, because $\mathbf{v}' - \mathbf{v} = (2, -1) \in U$; so they are two representatives of **one** coset. That observation is the whole theory of cosets.

Lemma 4.4.3. When two cosets are equal

Let U be a subspace of V . For $v, w \in V$, write $v \sim w$ if $v - w \in U$.

- (a) $\sim$ is an equivalence relation on V , and the equivalence class of v is the coset $v + U$.
- (b) $v + U = w + U$ if and only if $v - w \in U$.
- (c) Two cosets of U are either equal or disjoint, and every vector of V lies in exactly one coset.

Proof.

- (a) We check Definition 0.5.2. (E1) $v - v = 0 \in U$. (E2) If $v - w \in U$, then $w - v = (-1)(v - w) \in U$ by Theorem 1.1.7 and closure under scalar multiplication. (E3) If $v - w \in U$ and $w - x \in U$, then $v - x = (v - w) + (w - x) \in U$ by closure under addition. For the class, $w \sim v$ means $w - v = u$ for some $u \in U$, that is, $w = v + u \in v + U$; so the class $[v]$ of Definition 0.5.3 is $v + U$.
- (b) By (a) and Lemma 0.5.1 (a), $v + U = w + U$ if and only if $v \sim w$, that is, $v - w \in U$.

(c) This is Lemma 0.5.1 (b), together with $v \in v + U$ (take $u = 0$).

■

More examples.

- **Vertical lines.** In $\mathbb{R}^3$, let $U = \text{span}(\mathbf{e}_3)$, the z -axis. The coset $(a, b, c) + U$ is the vertical line through $(a, b, 0)$, and $(a, b, c) + U = (a', b', c') + U$ exactly when $a = a'$ and $b = b'$. A coset is determined by the shadow of any of its points on the xy -plane.
- **Polynomials with a prescribed value.** In $F[x]_{\leq n}$, let $U = \{p : p(0) = 0\}$, a subspace (the kernel of evaluation at 0). Then $p + U = q + U$ exactly when $(p - q)(0) = 0$, that is, $p(0) = q(0)$. So $p + U$ is the set of all polynomials with the same constant term as p .
- **Degenerate cases.** If $U = \{0\}$, each coset $v + U = \{v\}$ is a single vector. If $U = V$, there is only one coset, $v + V = V$, because $v - w \in V$ always.

Nonexample by minimal change. Try to use the line $L = \{x + 2y = 3\}$, which is not a subspace, in place of U . The rule “ $\mathbf{v} \sim \mathbf{w}$ if $\mathbf{v} - \mathbf{w} \in L$ ” is not even reflexive: $\mathbf{v} - \mathbf{v} = \mathbf{0} \notin L$. What fails is the hypothesis that U is a subspace, which (E1), (E2), and (E3) each used.

Warning. A coset has many names, and it is usually not a subspace. In the example above, $(1, 1) + U$ and $(3, 0) + U$ are the **same** set, although $(1, 1) \neq (3, 0)$. Never conclude $v = w$ from $v + U = w + U$; conclude only $v - w \in U$. And $v + U$ is a subspace only when it contains 0, that is, when $v \in U$ and so $v + U = U$. The dashed line above is not a subspace.

4.4.3 The quotient space

Now we want to treat each coset as a single point. The motivation is a kind of deliberate forgetting. Suppose we only care about vectors of $\mathbb{R}^2$ “up to” adding something from U , for instance because we only ever measure the quantity $x + 2y$. Then $(1, 1)$ and $(3, 0)$ carry the same information, and the space of all the information we keep is the set of lines parallel to U . We would like this set to be a vector space, so that we can add and scale lines.

The recipe is the one used for $\mathbb{Z}/n\mathbb{Z}$ in Chapter 0 (Example (Addition modulo n is well defined)): define the operations through representatives, then check that the choice of representatives does not matter.

The quotient space V/U is V with everything in U declared to be zero.

Definition 4.4.3. Quotient space

Let U be a subspace of a vector space V over F . The **quotient space** V/U (read “ V modulo U ”) is the set of all cosets of U ,

$$V/U := \{v + U : v \in V\},$$

with the operations

$$\begin{aligned} (v + U) + (w + U) &:= (v + w) + U, \\ a(v + U) &:= av + U \quad (v, w \in V, a \in F). \end{aligned}$$

The **quotient map** is $\pi: V \rightarrow V/U$, $\pi(v) = v + U$.

In words: an element of V/U is a whole coset, a subset of V . By Lemma 4.4.3 (a), V/U is the quotient set of V by the relation $v \sim w \Leftrightarrow v - w \in U$ (Definition 0.5.5), and π is its quotient map. To add two cosets, pick a representative of each, add them in V , and take the coset of the result. To scale a coset, scale a representative. The operations are written with representatives, so before anything else we must check that they are **well defined**.

The danger is real. By Lemma 4.4.3, $(1, 1) + U$ and $(3, 0) + U$ are one element of $\mathbb{R}^2/U$. If the sum $((1, 1) + U) + (\mathbf{w} + U)$ depended on which of the two names we used, the “definition” would assign two values to one pair of cosets, and it would not define a function at all (Definition 0.4.1). The theorem below rules this out, and then shows that the operations make V/U a vector space.

Theorem 4.4.4. The Quotient Space Is a Vector Space

Let U be a subspace of a vector space V over F .

- (a) The addition and scalar multiplication in Definition 4.4.3 are well defined: the right-hand sides depend only on the cosets $v + U$ and $w + U$, not on the representatives v and w .
- (b) With these operations V/U is a vector space over F . Its zero vector is $0 + U = U$, and the additive inverse of $v + U$ is $(-v) + U$.
- (c) The quotient map $\pi: V \rightarrow V/U$ is linear and surjective, and $\ker \pi = U$.

Idea. Chapter 0’s Theorem 0.5.3 turns a function on V that is constant on classes into a function on V/U . It is stated for functions of one variable, so we use it one variable at a time. ① For scalar multiplication by a fixed a : if $v - v' \in U$, then $av - av' = a(v - v') \in U$. ② For addition, first freeze the second summand w and vary the first; then show that the resulting function on cosets does not change when w is replaced by an equivalent w' . Each check is one line and uses exactly one closure property of U . Once the operations exist, every axiom for cosets is the same axiom in V , passed through π .

Proof. Throughout, $v \sim v'$ means $v - v' \in U$, and by Lemma 4.4.3 (b), $v \sim v'$ exactly when $v + U = v' + U$.

- (a) *Scalar multiplication.* Fix $a \in F$, and let $g_a: V \rightarrow V/U$, $g_a(v) = av + U$. If $v \sim v'$, then $av - av' = a(v - v') \in U$ by (VS5) and closure of U under scalar multiplication, so $g_a(v) = g_a(v')$. By Theorem 0.5.3 (a), there is exactly one function $\bar{g}_a: V/U \rightarrow V/U$ with $\bar{g}_a(v + U) = av + U$ for every $v \in V$. We define $a(v + U) := \bar{g}_a(v + U)$.

Addition, first variable. Fix $w \in V$, and let $f_w: V \rightarrow V/U$, $f_w(v) = (v + w) + U$. If $v \sim v'$, then $(v + w) - (v' + w) = v - v' \in U$, so $f_w(v) = f_w(v')$. By Theorem 0.5.3 (a), there is exactly one function $\bar{f}_w: V/U \rightarrow V/U$ with $\bar{f}_w(v + U) = (v + w) + U$ for every v .

Addition, second variable. Let $\mathcal{F}$ be the set of all functions $V/U \rightarrow V/U$, and let $h: V \rightarrow \mathcal{F}$, $h(w) = \bar{f}_w$. Suppose $w \sim w'$. For every $v \in V$, $(v + w) - (v + w') = w - w' \in U$, so

$$\bar{f}_w(v + U) = (v + w) + U = (v + w') + U = \bar{f}_{w'}(v + U).$$

Since every element of V/U has the form $v + U$, the functions $\bar{f}_w$ and $\bar{f}_{w'}$ are equal, that is, $h(w) = h(w')$. By Theorem 0.5.3 (a) again, there is exactly one function $\bar{h}: V/U \rightarrow \mathcal{F}$ with $\bar{h}(w + U) = \bar{f}_w$. We define $X + Y := \bar{h}(Y)(X)$ for $X, Y \in V/U$. Then

$$(v + U) + (w + U) = \bar{h}(w + U)(v + U) = \bar{f}_w(v + U) = (v + w) + U,$$

which is the formula of Definition 4.4.3, and its value depends only on the two cosets.

- (b) Every element of V/U is $\pi(v) = v + U$ for some v , and by (a) we have $\pi(v) + \pi(w) = \pi(v + w)$ and $a\pi(v) = \pi(av)$. Let $X = \pi(x)$, $Y = \pi(y)$, $Z = \pi(z)$ be in V/U , and $a, b \in F$. Each axiom follows by moving π outside, using the axiom in V , and moving π back:

- (VS1) $X + Y = \pi(x + y) = \pi(y + x) = Y + X$.
- (VS2) $(X + Y) + Z = \pi((x + y) + z) = \pi(x + (y + z)) = X + (Y + Z)$.
- (VS3) $X + \pi(0) = \pi(x + 0) = \pi(x) = X$, so $\pi(0) = 0 + U = U$ is a zero vector.
- (VS4) $X + \pi(-x) = \pi(x + (-x)) = \pi(0)$, so $\pi(-x) = (-x) + U$ is an additive inverse of X .
- (VS5) $a(X + Y) = \pi(a(x + y)) = \pi(ax + ay) = aX + aY$.
- (VS6) $(a + b)X = \pi((a + b)x) = \pi(ax + bx) = aX + bX$.
- (VS7) $a(bX) = \pi(a(bx)) = \pi((ab)x) = (ab)X$.
- (VS8) $1X = \pi(1x) = \pi(x) = X$.

Hence V/U is a vector space over F .

- (c) The identities $\pi(v + w) = \pi(v) + \pi(w)$ and $\pi(av) = a\pi(v)$ from (b) say that π is linear. It is surjective by the definition of V/U . Finally, $v \in \ker \pi$ means $v + U = 0 + U$, which by Lemma 4.4.3 (b) means $v = v - 0 \in U$. So $\ker \pi = U$. ■

Part (c) has a pleasant reading: **every subspace is a kernel**. Given $U \subseteq V$, the quotient map is a linear map whose kernel is exactly U .

Examples.

- **Parallel lines.** In $\mathbb{R}^2/U$ with $U = \{x + 2y = 0\}$, write ℓ_c for the line $\{x + 2y = c\}$. Since $(a, b) \in \ell_{a+2b}$, the definition gives $\ell_c + \ell_d = \ell_{c+d}$ and $t\ell_c = \ell_{tc}$: to add two parallel lines, add their “heights” c and d . The zero vector is $\ell_0 = U$.
- **Constant terms.** In $F[x]_{\leq n}/U$ with $U = \{p : p(0) = 0\}$, the coset $p + U$ is determined by $p(0)$, and $(p + U) + (q + U) = (p + q) + U$ is determined by $p(0) + q(0)$. All that survives of a polynomial is its constant term.
- **Degenerate cases.** $V/\{0\}$ has the cosets $\{v\}$, added exactly as in V : nothing is forgotten. V/V has the single element V , so it is the zero space: everything is forgotten.

Nonexample by minimal change. Replace the line U by the union of the two axes, $S = \text{span}(\mathbf{e}_1) \cup \text{span}(\mathbf{e}_2) \subseteq \mathbb{R}^2$. It still contains $\mathbf{0}$ and is still closed under scalar multiplication, but it is not closed under addition. Then “ $\mathbf{v} - \mathbf{w} \in S$ ” is not transitive: $(1, 1) - (0, 1) = (1, 0) \in S$ and $(0, 1) - (0, 0) = (0, 1) \in S$, but $(1, 1) - (0, 0) = (1, 1) \notin S$. Concretely, the “cosets” $(1, 1) + S$ and $(0, 0) + S = S$ both contain $(0, 1) = (1, 1) + (-1, 0)$, yet they are different sets, since $\mathbf{0} \in S$ but $(1, 1) + \mathbf{s} = \mathbf{0}$ would need $\mathbf{s} = (-1, -1) \notin S$. The sets $\mathbf{v} + S$ do not split $\mathbb{R}^2$ into disjoint pieces, so there is no set “ $\mathbb{R}^2/S$ ” of the kind the definition needs. The clause that fails is “ U is a subspace”, specifically closure under addition.

Why this definition. Each part of the hypothesis that U is a subspace has a job: $0 \in U$

makes $\sim$ reflexive; closure under scalar multiplication makes $\sim$ symmetric and scaling of cosets well defined; closure under addition makes $\sim$ transitive and addition of cosets well defined. The nonexample shows what goes wrong when one of them is missing. The notation V/U and the phrase “modulo U ” come from $\mathbb{Z}/n\mathbb{Z}$, where integers are identified when their difference lies in $n\mathbb{Z}$.

Warning. V/U is **not** a subspace of V . Its elements are subsets of V , not vectors of V , and its addition is a new operation. For example, $\mathbb{R}^3/\text{span}(\mathbf{e}_3)$ is not the xy -plane: its elements are vertical lines. The xy -plane is a **complement** of the z -axis, one of many (Chapter 1), whereas V/U involves no choice at all. The two are isomorphic, as we show below, but they are different objects.

In finite dimension the size of a quotient is exactly what one would guess from the picture: we lose $\dim U$ dimensions.

Theorem 4.4.5. Dimension of a Quotient

Let V be finite-dimensional and U a subspace of V . Then V/U is finite-dimensional and

$$\dim(V/U) = \dim V - \dim U.$$

More precisely, if $(u_1, \dots, u_m, w_1, \dots, w_s)$ is a basis of V such that $(u_1, \dots, u_m)$ is a basis of U , then $(w_1 + U, \dots, w_s + U)$ is a basis of V/U .

Proof. By Theorem 4.4.4 (c), $\pi: V \rightarrow V/U$ is linear, surjective, and has kernel U . By the Rank–Nullity Theorem (Theorem 2.3.1), $\text{im } \pi = V/U$ is finite-dimensional and $\dim V = \dim U + \dim(V/U)$, which is the formula.

For the basis, let $v + U \in V/U$ and write $v = \sum_i a_i u_i + \sum_j b_j w_j$. Since π is linear and $\pi(u_i) = U$ is the zero vector, Theorem 2.1.4 gives $v + U = \sum_j b_j(w_j + U)$. So the list $(w_1 + U, \dots, w_s + U)$ spans V/U . Its length is $s = \dim V - \dim U = \dim(V/U)$, so it is a basis by Theorem 1.6.4 (b). ■

Such a basis always exists: take a basis of U and extend it (Theorem 1.6.5). So to find a basis of a quotient, **extend a basis of U to one of V and keep the cosets of the new vectors.**

Example. A basis of a quotient of $\mathbb{R}^3$

Let $U = \text{span}((1, 1, 1)) \subseteq \mathbb{R}^3$. Find a basis of $\mathbb{R}^3/U$, and write $(2, 5, -1) + U$ in it.

Solution. Extend $(1, 1, 1)$ by $\mathbf{e}_1$ and $\mathbf{e}_2$. The list $((1, 1, 1), \mathbf{e}_1, \mathbf{e}_2)$ is independent: in $a(1, 1, 1) + b\mathbf{e}_1 + c\mathbf{e}_2 = (a + b, a + c, a) = \mathbf{0}$, the third entry gives $a = 0$, and then $b = c = 0$. It has length 3, so it is a basis of $\mathbb{R}^3$ (Theorem 1.6.4). By Theorem 4.4.5, $(\mathbf{e}_1 + U, \mathbf{e}_2 + U)$ is a basis of $\mathbb{R}^3/U$, which has dimension $3 - 1 = 2$.

To write $(2, 5, -1) + U$, solve $(2, 5, -1) = a(1, 1, 1) + b\mathbf{e}_1 + c\mathbf{e}_2$: the third entry gives $a = -1$,

then $b = 3$ and $c = 6$. Hence

$$(2, 5, -1) + U = 3(\mathbf{e}_1 + U) + 6(\mathbf{e}_2 + U),$$

and indeed $(2, 5, -1) - (3, 6, 0) = (-1, -1, -1) \in U$. The coordinates $(3, 6)$ are $(x - z, y - z)$ evaluated at $(2, 5, -1)$: they are exactly the quantities that do not change when a multiple of $(1, 1, 1)$ is added.

The warning above said that a quotient is not a complement. The next result says how they are related: any complement is a concrete model of the quotient.

Proposition 4.4.6. A complement is a model of the quotient

Let U be a subspace of V , and let W be a subspace with $V = U \oplus W$. Then the restriction of the quotient map, $W \rightarrow V/U$, $w \mapsto w + U$, is an isomorphism.

Proof. The restriction of the linear map π to W is linear. *Injective:* if $w + U = U$ with $w \in W$, then $w \in U$ by Lemma 4.4.3 (b), so $w \in U \cap W = \{0\}$ (Theorem 1.7.4); by Theorem 2.2.3 the restriction is injective. *Surjective:* given $v + U$, write $v = u + w$ with $u \in U$, $w \in W$; then $v - w = u \in U$, so $v + U = w + U$. A linear bijection is an isomorphism (Definition 2.5.2). ■

So every complement of U is isomorphic to V/U , which gives a second proof that all complements have the same dimension. The quotient is the choice-free object that all the complements imitate.

Quick check. In $\mathbb{R}^2/U$ with $U = \text{span}((1, 1))$, is $(2, 3) + U = (0, 1) + U$? Is $(1, 0) + U = (0, 1) + U$? What is $\dim(\mathbb{R}^2/U)$?

Solution. By Lemma 4.4.3 (b): $(2, 3) - (0, 1) = (2, 2) \in U$, so the first two cosets are equal. $(1, 0) - (0, 1) = (1, -1) \notin U$, so the second pair are different cosets. By Theorem 4.4.5, $\dim(\mathbb{R}^2/U) = 2 - 1 = 1$.

4.4.4 The First Isomorphism Theorem

A linear map $T: V \rightarrow W$ forgets exactly the information in its kernel: $Tv = Tv'$ if and only if $T(v - v') = 0$, that is, $v - v' \in \ker T$. In the language of this section, T takes the same value at two vectors exactly when they lie in the same coset of $\ker T$. So T is really a function of the coset, and on cosets it forgets nothing.

Theorem 4.4.7. First Isomorphism Theorem

Let $T: V \rightarrow W$ be a linear map, and let $K = \ker T$. Then the rule

$$\bar{T}: V/K \rightarrow W, \quad \bar{T}(v + K) = Tv$$

defines a linear map, which is injective with $\text{im } \bar{T} = \text{im } T$, and satisfies $\bar{T} \circ \pi = T$. Hence

$$V/\ker T \cong \text{im } T.$$

Proof. Well defined. K is a subspace by Theorem 2.2.1, so V/K is a vector space. If $v + K = v' + K$, then $v - v' \in K$ by Lemma 4.4.3 (b), so $Tv - Tv' = T(v - v') = 0$. Hence T is constant on each coset of K , and by Theorem 0.5.3 (a) there is exactly one function $\bar{T}: V/K \rightarrow W$ with $\bar{T}(v + K) = Tv$, that is, $\bar{T} \circ \pi = T$.

Linear. For $v, w \in V$ and $a \in F$, by Definition 4.4.3 and the linearity of T ,

$$\begin{aligned} \bar{T}((v + K) + (w + K)) &= \bar{T}((v + w) + K) = T(v + w) \\ &= Tv + Tw = \bar{T}(v + K) + \bar{T}(w + K), \end{aligned}$$

and similarly $\bar{T}(a(v + K)) = \bar{T}(av + K) = T(av) = a\bar{T}(v + K)$.

Injective. If $\bar{T}(v + K) = 0$, then $Tv = 0$, so $v \in K$ and $v + K = K$, the zero of V/K . By Theorem 2.2.3, $\bar{T}$ is injective.

Image. The values of $\bar{T}$ are the vectors Tv , $v \in V$, so $\text{im } \bar{T} = \text{im } T$. Regarded as a map $V/K \rightarrow \text{im } T$, $\bar{T}$ is a linear bijection, hence an isomorphism (Definition 2.5.2). ■

So every linear map factors as “collapse the kernel, then embed”: $T = \bar{T} \circ \pi$, with π surjective and $\bar{T}$ injective. When V is finite-dimensional, taking dimensions with Theorem 4.4.5 gives $\dim V - \dim \ker T = \dim \text{im } T$, which is Rank–Nullity again. The theorem is stronger than the count, because it names the isomorphism, and it holds in infinite dimension too.

The practical use runs the other way: to identify a quotient V/U , find a surjective linear map out of V whose kernel is exactly U .

Example. Identifying quotients

Use the First Isomorphism Theorem to show:

- (a) $\mathbb{R}^3/L \cong \mathbb{R}^2$, where $L = \text{span}((1, 1, 1))$;
- (b) $\mathbb{R}[x]_{\leq n}/U \cong \mathbb{R}$, where $U = \{p : p(0) = 0\}$.

Solution.

- (a) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T(x, y, z) = (x - z, y - z)$, which is linear because each entry is a linear expression. It is surjective, since $T(a, b, 0) = (a, b)$. Its kernel is $\{x = z, y = z\} = \{(t, t, t)\} = L$. By Theorem 4.4.7, $(x, y, z) + L \mapsto (x - z, y - z)$ is an isomorphism $\mathbb{R}^3/L \rightarrow \mathbb{R}^2$. This explains the coordinates found in Example (A basis of a quotient of $\mathbb{R}^3$). The check that it is well defined is visible: adding (t, t, t) to (x, y, z) does not change $x - z$ or $y - z$.
- (b) Let $\text{ev}_0: \mathbb{R}[x]_{\leq n} \rightarrow \mathbb{R}$, $p \mapsto p(0)$. It is linear by Theorem 0.8.5, surjective because the constant polynomial c maps to c , and its kernel is U by definition. By Theorem 4.4.7, $p + U \mapsto p(0)$ is an isomorphism $\mathbb{R}[x]_{\leq n}/U \rightarrow \mathbb{R}$. In particular $\dim(\mathbb{R}[x]_{\leq n}/U) = 1$ by

Theorem 2.5.4, so $\dim U = (n + 1) - 1 = n$ by Theorem 4.4.5.

4.4.5 The Second and Third Isomorphism Theorems

Two more identifications come from the same method: write down a surjective map, compute its kernel, and apply the First Isomorphism Theorem. The first compares a sum of subspaces with an intersection.

Theorem 4.4.8. Second Isomorphism Theorem

Let U and W be subspaces of a vector space V . Then $U \cap W$ is a subspace of U , W is a subspace of $U + W$, and

$$U/(U \cap W) \rightarrow (U + W)/W, \quad u + (U \cap W) \mapsto u + W$$

is a well-defined isomorphism. Hence $(U + W)/W \cong U/(U \cap W)$.

Idea. Picture two planes U and W in $\mathbb{R}^3$ meeting in a line. Modulo W , a vector of $U + W$ is represented by its U -part alone, since the W -part is forgotten. The only vectors of U that become zero are those already in W . So the map “ $u \mapsto u + W$ ” from U is surjective, with kernel $U \cap W$.

Proof. $U \cap W$ is a subspace (Theorem 1.2.2) contained in U , and W is a subspace contained in $U + W$ (Theorem 1.7.1), so both quotients exist. Let $\pi_W: U + W \rightarrow (U + W)/W$ be the quotient map, and let $S: U \rightarrow (U + W)/W$, $Su = u + W$, be its restriction to U ; S is linear because π_W is (Theorem 4.4.4).

S is surjective. An element of $(U + W)/W$ is $(u + w) + W$ with $u \in U$, $w \in W$. Since $(u + w) - u = w \in W$, Lemma 4.4.3 (b) gives $(u + w) + W = u + W = Su$.

$\ker S = U \cap W$. For $u \in U$, $Su = W$ means $u \in W$ (Lemma 4.4.3 (b)), that is, $u \in U \cap W$.

By Theorem 4.4.7 applied to S , the rule $u + (U \cap W) \mapsto Su = u + W$ is a well-defined injective linear map with image $\text{im } S = (U + W)/W$, hence an isomorphism. (Directly: if $u - u' \in U \cap W$, then $u - u' \in W$, so $u + W = u' + W$, and the value does not depend on the representative.)

■

If U and W are finite-dimensional, then so is $U + W$ (it is spanned by the union of bases, Proposition 1.7.2), and Theorem 4.4.5 turns the theorem into $\dim(U + W) - \dim W = \dim U - \dim(U \cap W)$. That is the dimension formula for sums (Theorem 1.7.3), now as a consequence of an isomorphism rather than of a basis computation. An exercise below asks for the details.

The third theorem says that quotients can be taken in stages.

Theorem 4.4.9. Third Isomorphism Theorem

Let $U \subseteq W \subseteq V$ be subspaces. Then $W/U = \{w + U : w \in W\}$ is a subspace of V/U , and

$$(V/U)/(W/U) \rightarrow V/W, \quad (v + U) + W/U \mapsto v + W$$

is a well-defined isomorphism. Hence $(V/U)/(W/U) \cong V/W$.

Proof. Let $S: V/U \rightarrow V/W$, $S(v + U) = v + W$.

Well defined. If $v + U = v' + U$, then $v - v' \in U \subseteq W$, so $v + W = v' + W$ by Lemma 4.4.3 (b). By Theorem 0.5.3 (a), applied to $V \rightarrow V/W$, $v \mapsto v + W$, the rule defines a function on V/U .

Linear and surjective.

$$\begin{aligned} S((v + U) + (w + U)) &= S((v + w) + U) = (v + w) + W \\ &= S(v + U) + S(w + U), \end{aligned}$$

and similarly for scalars, by Definition 4.4.3 in both quotients. Every $v + W$ equals $S(v + U)$.

Kernel. $S(v + U) = W$ means $v \in W$. If $v \in W$, then $v + U \in W/U$. Conversely, if $v + U = w + U$ with $w \in W$, then $v - w \in U \subseteq W$, so $v \in W$. Hence $\ker S = W/U$. In particular W/U is a subspace of V/U , by Theorem 2.2.1.

By Theorem 4.4.7 applied to S , $(V/U)/(W/U) \cong \operatorname{im} S = V/W$, via $(v + U) + W/U \mapsto S(v + U) = v + W$. ■

The notation W/U is used with two meanings, and they agree: the quotient space of W by U , and the set of cosets $w + U$ with $w \in W$, sitting inside V/U . The coset $w + U$ is the same subset of V in both readings, and the operations agree because both are computed with representatives. For example, with $V = \mathbb{R}^3$, U the z -axis, and W the xz -plane, $V/W \cong \mathbb{R}$ through the y -coordinate, and the theorem says that first forgetting z and then forgetting x leaves the same thing as forgetting both at once.

4.4.6 The universal property of the quotient

The proofs of the three theorems repeated one move: a linear map that vanishes on U passes to V/U . We record that move once, in its general form. It is the standard way to **define** linear maps out of a quotient.

Theorem 4.4.10. Universal Property of the Quotient

Let U be a subspace of V , $\pi: V \rightarrow V/U$ the quotient map, and $T: V \rightarrow W$ a linear map.

- (a) If $U \subseteq \ker T$, then there is **exactly one** linear map $\bar{T}: V/U \rightarrow W$ with $\bar{T} \circ \pi = T$, namely $\bar{T}(v + U) = Tv$. Its image is $\operatorname{im} T$, and it is injective if and only if $U = \ker T$.
- (b) Conversely, if some function $\bar{T}: V/U \rightarrow W$ satisfies $\bar{T} \circ \pi = T$, then $U \subseteq \ker T$.

Proof.

- (a) If $v - v' \in U \subseteq \ker T$, then $Tv = Tv'$. By Theorem 0.5.3 (a) there is exactly one function $\bar{T}$ with $\bar{T}(v+U) = Tv$, and it is linear by the computation in the proof of Theorem 4.4.7, which used only this formula. Any linear $\bar{T}$ with $\bar{T} \circ \pi = T$ satisfies this formula, so it is this one. The image is $\{Tv\} = \operatorname{im} T$. The kernel of $\bar{T}$ is $\{v+U : v \in \ker T\}$; it is $\{U\}$, the zero subspace, exactly when every $v \in \ker T$ lies in U , that is, when $\ker T \subseteq U$, which together with $U \subseteq \ker T$ means $U = \ker T$. Apply Theorem 2.2.3.
- (b) Let $u \in U$. Then $\pi(u) = U = \pi(0)$, so $Tu = \bar{T}(\pi(u)) = \bar{T}(\pi(0)) = T(0) = 0$ by Theorem 2.1.1. Hence $U \subseteq \ker T$.

■

In words: **linear maps out of V/U are the same as linear maps out of V that kill U .** The First Isomorphism Theorem is the case $U = \ker T$. For example, the map $T: \mathbb{R}^3 \rightarrow \mathbb{R}$, $T(x, y, z) = x$, kills the z -axis U , so it induces $\bar{T}: \mathbb{R}^3/U \rightarrow \mathbb{R}$, $(x, y, z) + U \mapsto x$. Since $\ker T$ is the yz -plane, strictly larger than U , this $\bar{T}$ is not injective: it also kills the coset $\mathbf{e}_2 + U$.

Quotients will return in Chapter 9, where an operator T with $T(U) \subseteq U$ induces an operator on V/U , and this is the engine of induction arguments on the dimension.

4.4.7 Exercises

4.4.7.1 A. Check your understanding

Exercise. A1

- (a) Define the quotient space V/U and its two operations.
- (b) State the condition, in terms of $v - w$, for $v + U = w + U$.
- (c) True or false: V/U is a subspace of V . Justify your answer.
- (d) State the First Isomorphism Theorem.
- (e) If U is a 2-dimensional subspace of $\mathbb{R}^5$, what is $\dim(\mathbb{R}^5/U)$? What is $\dim(\mathbb{R}^5 \times U)$?
- (f) What must be checked before the formula $(v + U) + (w + U) = (v + w) + U$ can be used as a definition, and which property of U makes the check work?

Solution.

- (a) For a subspace U of V , $V/U = \{v + U : v \in V\}$, with $(v + U) + (w + U) = (v + w) + U$ and $a(v + U) = av + U$ (Definition 4.4.3).
- (b) $v + U = w + U$ if and only if $v - w \in U$ (Lemma 4.4.3).
- (c) False. The elements of V/U are cosets, which are subsets of V , not vectors of V .
- (d) If $T: V \rightarrow W$ is linear, then $v + \ker T \mapsto Tv$ is a well-defined isomorphism $V/\ker T \rightarrow \operatorname{im} T$ (Theorem 4.4.7).
- (e) $5 - 2 = 3$ by Theorem 4.4.5, and $5 + 2 = 7$ by Theorem 4.4.1.
- (f) That the right-hand side does not depend on the representatives: if $v - v' \in U$ and $w - w' \in U$, then $(v + w) + U = (v' + w') + U$. This works because $(v + w) - (v' + w') = (v - v') + (w - w')$ and U is closed under addition.

4.4.7.2 B. Practice

Exercise. B1: A basis of a quotient of $\mathbb{R}^4$

Let $U = \text{span}((1, 0, 1, 0), (0, 1, 0, 1)) \subseteq \mathbb{R}^4$.

- (a) Find $\dim(\mathbb{R}^4/U)$ and a basis of $\mathbb{R}^4/U$.
- (b) Write $(3, 1, 4, 1) + U$ in your basis.
- (c) Decide whether $(1, 2, 3, 4) + U = (3, 4, 1, 2) + U$.

Solution.

- (a) The two spanning vectors are independent (look at the first two entries), so $\dim U = 2$ and $\dim(\mathbb{R}^4/U) = 4 - 2 = 2$ by [Theorem 4.4.5](#). Extend by $\mathbf{e}_1, \mathbf{e}_2$. If $a(1, 0, 1, 0) + b(0, 1, 0, 1) + c\mathbf{e}_1 + d\mathbf{e}_2 = (a + c, b + d, a, b) = \mathbf{0}$, then $a = b = 0$ from the last two entries and $c = d = 0$ from the first two. So the four vectors are independent, hence a basis of $\mathbb{R}^4$ (Theorem 1.6.4), and by [Theorem 4.4.5](#) $(\mathbf{e}_1 + U, \mathbf{e}_2 + U)$ is a basis of $\mathbb{R}^4/U$.
- (b) Solve $(3, 1, 4, 1) = (a + c, b + d, a, b)$: $a = 4, b = 1, c = -1, d = 0$. Since the U -part becomes zero in the quotient, $(3, 1, 4, 1) + U = -1(\mathbf{e}_1 + U) + 0(\mathbf{e}_2 + U)$. Check: $(3, 1, 4, 1) + \mathbf{e}_1 = (4, 1, 4, 1) = 4(1, 0, 1, 0) + (0, 1, 0, 1) \in U$.
- (c) The difference is $(1, 2, 3, 4) - (3, 4, 1, 2) = (-2, -2, 2, 2)$. A vector (a, b, a, b) of U has equal first and third entries, but $-2 \neq 2$. So the difference is not in U , and the cosets are different by [Lemma 4.4.3](#).

Exercise. B2: A map that factors through a quotient

Let $T: \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}^2$, $T(p) = (p(1), p'(1))$, and $U = \text{span}((x - 1)^2)$.

- (a) Show that $U \subseteq \ker T$, so that T induces a linear map $\bar{T}: \mathbb{R}[x]_{\leq 2}/U \rightarrow \mathbb{R}^2$ with $\bar{T}(p + U) = T(p)$.
- (b) Prove that $\bar{T}$ is an isomorphism.
- (c) Compute $\bar{T}(x^2 + U)$, and find the coset $\bar{T}^{-1}(0, 1)$.

Solution.

- (a) T is linear, since evaluation and differentiation are. For $q = (x - 1)^2$, $q(1) = 0$ and $q'(1) = 2(1 - 1) = 0$, so $T(q) = \mathbf{0}$, and hence $T(cq) = \mathbf{0}$ for all c . So $U \subseteq \ker T$, and [Theorem 4.4.10](#) (a) gives the linear map $\bar{T}$.
- (b) $T(1) = (1, 0)$ and $T(x - 1) = (0, 1)$, so T is surjective and $\text{rank } T = 2$. By Theorem 2.3.1, $\dim \ker T = 3 - 2 = 1 = \dim U$, so $\ker T = U$ by Theorem 1.6.8. By [Theorem 4.4.10](#) (a), $\bar{T}$ is injective with image $\text{im } T = \mathbb{R}^2$. Hence $\bar{T}$ is a linear bijection, an isomorphism.
- (c) $\bar{T}(x^2 + U) = T(x^2) = (1, 2)$. Since $T(x - 1) = (0, 1)$, $\bar{T}^{-1}(0, 1) = (x - 1) + U$, the set of all $x - 1 + c(x - 1)^2$.

Exercise. B3: Matrices modulo scalars

Let $n \geq 1$ and $S = \{c\mathbf{I}_n : c \in F\} \subseteq M_n(F)$.

- (a) Show that S is a subspace and find $\dim(M_n(F)/S)$.
- (b) For $n = 2$, find a basis of $M_2(F)/S$.
- (c) Show that $\mathbf{A} + S = \mathbf{B} + S$ if and only if $\mathbf{A} - \mathbf{B}$ is a scalar matrix, and decide whether

$$\begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix} + S = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + S.$$

Solution.

- (a) $S = \text{span}(\mathbf{I}_n)$ is a subspace by Theorem 1.3.1, of dimension 1 since $\mathbf{I}_n \neq \mathbf{O}$. By Theorem 4.4.5, $\dim(M_n(F)/S) = n^2 - 1$.
- (b) Extend $(\mathbf{I}_2)$ by $\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}$. If $a\mathbf{I}_2 + b\mathbf{E}_{11} + c\mathbf{E}_{12} + d\mathbf{E}_{21} = \begin{pmatrix} a+b & c \\ d & a \end{pmatrix} = \mathbf{O}$, then $a = c = d = 0$ and then $b = 0$. So the four matrices are independent, hence a basis of $M_2(F)$ (Theorem 1.6.4), and $(\mathbf{E}_{11} + S, \mathbf{E}_{12} + S, \mathbf{E}_{21} + S)$ is a basis of $M_2(F)/S$ by Theorem 4.4.5.
- (c) This is Lemma 4.4.3 (b) for the subspace S . Here the difference is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbf{I}_2 \in S$, so the two cosets are equal.

4.4.7.3 C. Going deeper

Exercise. C1: The dimension formula from isomorphisms

Let U and W be finite-dimensional subspaces of a vector space V .

- (a) Using Theorem 4.4.8 and Theorem 4.4.5, prove that $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.
- (b) Give a second proof: apply Rank–Nullity to the addition map $S: U \times W \rightarrow V$, $S(u, w) = u + w$, after showing that $u \mapsto (u, -u)$ is an isomorphism from $U \cap W$ onto $\ker S$.

Solution.

- (a) $U + W$ is finite-dimensional, being spanned by a basis of U followed by a basis of W (Proposition 1.7.2), and $U \cap W \subseteq U$ is finite-dimensional (Theorem 1.6.7). By Theorem 4.4.8, $(U + W)/W \cong U/(U \cap W)$, and isomorphic finite-dimensional spaces have equal dimension (Theorem 2.5.4). Applying Theorem 4.4.5 to both sides,

$$\dim(U + W) - \dim W = \dim U - \dim(U \cap W),$$

which rearranges to the formula.

- (b) The map S is linear with image $U + W$ by Theorem 4.4.2 (a). Let $J: U \cap W \rightarrow U \times W$, $Ju = (u, -u)$; it is well defined because $u \in U$ and $-u \in W$, and linear because the operations are entrywise. It is injective, since $(u, -u) = (0, 0)$ forces $u = 0$. Its image lies in $\ker S$, since $u + (-u) = 0$. Conversely, if $(u, w) \in \ker S$, then $w = -u$; so $u = -w \in W$, hence $u \in U \cap W$ and $(u, w) = Ju$. Thus J is an isomorphism onto

$\ker S$, and $\dim \ker S = \dim(U \cap W)$. By [Theorem 4.4.1](#) and [Theorem 2.3.1](#),

$$\dim U + \dim W = \dim(U \times W) = \dim(U + W) + \dim(U \cap W).$$

Exercise. C2: The operator induced on a quotient

Let V be finite-dimensional, $T \in \mathcal{L}(V)$, and U a subspace with $Tu \in U$ for every $u \in U$.

- (a) Prove that $\bar{T}(v + U) := Tv + U$ is a well-defined linear operator on V/U .
 (b) Let $(u_1, \dots, u_m)$ be a basis of U , extended to a basis $\mathcal{B} = (u_1, \dots, u_m, w_1, \dots, w_s)$ of V . Show that

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{O} & D \end{pmatrix},$$

where $\mathbf{A} \in M_m(F)$ is the matrix of the restriction $T|_U: U \rightarrow U$ in $(u_1, \dots, u_m)$, $D \in M_s(F)$ is the matrix of $\bar{T}$ in $(w_1 + U, \dots, w_s + U)$, and $\mathbf{O}$ is the $s \times m$ zero matrix.

- (c) Check (b) when T is differentiation on $\mathbb{R}[x]_{\leq 2}$, with $U = \mathbb{R}[x]_{\leq 1}$ and $\mathcal{B} = (1, x, x^2)$.

Hint: for (a), apply [Theorem 4.4.10](#) to $\pi \circ T$.

Solution.

- (a) The map $\pi \circ T: V \rightarrow V/U$, $v \mapsto Tv + U$, is linear as a composition of linear maps ([Theorem 2.4.2](#)). For $u \in U$, $Tu \in U$, so $(\pi \circ T)(u) = U$, the zero of V/U ; thus $U \subseteq \ker(\pi \circ T)$. By [Theorem 4.4.10](#) (a) there is exactly one linear map $\bar{T}: V/U \rightarrow V/U$ with $\bar{T}(v + U) = Tv + U$.
 (b) For $j \leq m$, $Tu_j \in U$, so it is a combination of $u_1, \dots, u_m$ only, and its $\mathcal{B}$ -coordinates are 0 in the last s places. These first m columns, restricted to their first m entries, are the columns of the matrix $\mathbf{A}$ of $T|_U$ (which maps U to U by hypothesis and is linear). For the remaining columns, write $Tw_k = \sum_i b_{ik}u_i + \sum_l d_{lk}w_l$. Passing to V/U , where $u_i + U = U$ is zero, gives $\bar{T}(w_k + U) = \sum_l d_{lk}(w_l + U)$. So the $s \times s$ matrix (d_{lk}) is the matrix D of $\bar{T}$ in the basis $(w_1 + U, \dots, w_s + U)$ of [Theorem 4.4.5](#), and the bottom-right block of $[T]_{\mathcal{B}}^{\mathcal{B}}$ is D .

- (c) $T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x$, so $[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Differentiation maps $U = \mathbb{R}[x]_{\leq 1}$ into itself. The block $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the matrix of T on U in $(1, x)$, the lower-left block is $\begin{pmatrix} 0 & 0 \end{pmatrix}$, and the block D is the 1×1 matrix $\mathbf{O}_1$: indeed $\bar{T}(x^2 + U) = 2x + U = U$, the zero coset, since $2x \in U$.

Exercise. C3: Counting cosets over a finite field

Let $F = \mathbb{F}_q$ be a finite field with q elements, V a vector space over F of dimension n , and U a subspace of dimension m .

- (a) Show that $|V| = q^n$.

- (b) Show that every coset $v + U$ has exactly $|U|$ elements, and deduce $|V/U| = q^{n-m}$ by counting.
- (c) Check that (b) agrees with [Theorem 4.4.5](#).

Solution.

- (a) Choose a basis $(v_1, \dots, v_n)$. By Theorem 1.5.1, $(a_1, \dots, a_n) \mapsto a_1v_1 + \dots + a_nv_n$ is a bijection $F^n \rightarrow V$, and F^n has q^n elements.
- (b) The map $U \rightarrow v + U$, $u \mapsto v + u$, is surjective by [Definition 4.4.2](#) and injective because $v + u = v + u'$ implies $u = u'$. So $|v + U| = |U| = q^m$ by (a) applied to U . By [Lemma 4.4.3](#) (c), the cosets partition V into $|V/U|$ disjoint sets of size q^m , so $q^n = |V/U| \cdot q^m$, and $|V/U| = q^{n-m}$.
- (c) By [Theorem 4.4.5](#), V/U is a vector space over $\mathbb{F}_q$ of dimension $n - m$, so by (a) it has q^{n-m} elements, in agreement with (b). For example, $\mathbb{F}_2^3$ modulo a line has $8/2 = 4$ cosets.