

## 2.2 Kernel and Image

For any function, Chapter 0 asked two questions: is it injective, and is it surjective? For a function between sets, answering them means comparing outputs pair by pair and hunting for preimages point by point. For a linear map both questions reduce to a single subspace each. This section defines the two subspaces, the kernel and the image, shows how to compute them, and proves the test that makes injectivity cheap: a linear map is injective exactly when its kernel is zero.

### 2.2.1 Two questions, two subspaces

Let  $T: V \rightarrow W$  be linear. Start with injectivity. Suppose two inputs collide,  $T(u) = T(v)$ . By Theorem 2.1.2 this is the same as

$$T(u - v) = T(u) - T(v) = 0.$$

So a collision between  $u$  and  $v$  is the same thing as their difference being sent to 0. To detect every collision at once, it is enough to know **which vectors are sent to 0**. For a general function this trick is unavailable, because there is no subtraction; linearity turns a question about all pairs into a question about one set.

Surjectivity asks whether every  $w \in W$  is an output, so the set to know is **the set of all outputs**. For the matrix map  $T_{\mathbf{A}}(\mathbf{x}) = \mathbf{Ax}$  both sets are concrete: the outputs are the vectors  $\mathbf{b}$  for which  $\mathbf{Ax} = \mathbf{b}$  can be solved, and the inputs sent to  $\mathbf{0}$  are the solutions of  $\mathbf{Ax} = \mathbf{0}$ . Chapter 3 computes both. The definitions below give the two sets a name for every linear map. They are defined for what they measure.

*The kernel is everything  $T$  sends to zero; the image is everything  $T$  reaches.*

#### Definition 2.2.1. Kernel

Let  $T: V \rightarrow W$  be a linear map over  $F$ . The **kernel** of  $T$  is the subset of the **domain**

$$\ker T := \{v \in V : T(v) = 0_W\} \subseteq V.$$

#### Definition 2.2.2. Image

Let  $T: V \rightarrow W$  be a linear map over  $F$ . The **image** of  $T$  is the subset of the **codomain**

$$\operatorname{im} T := \{T(v) : v \in V\} = \{w \in W : w = T(v) \text{ for some } v \in V\} \subseteq W.$$

In words: a vector  $v$  is in  $\ker T$  when  $v$  lives in  $V$  **and**  $T$  sends it to the zero vector of  $W$ . A vector  $w$  is in  $\operatorname{im} T$  when  $w$  lives in  $W$  **and** there **exists** at least one input with output  $w$ ; the input need not be unique. In the language of Definition 0.4.2,  $\ker T = T^{-1}(\{0\})$  is the preimage of the zero vector, and  $\operatorname{im} T = T(V)$  is the image of the function  $T$ . The kernel is also called the **null space** of  $T$ , and the image its **range**. For group homomorphisms, the kernel was defined the same way (Definition 0.10.3), with 0 playing the role of the identity element.

There is nothing to check for well-definedness: both sets are described by conditions that either hold or fail. What does need checking is that they are **subspaces**, not just subsets.

**Theorem 2.2.1. The Kernel Is a Subspace**

Let  $T: V \rightarrow W$  be linear. Then  $\ker T$  is a subspace of  $V$ .

*Proof.* We use the Subspace Test (Theorem 1.2.1).

- (1) By Theorem 2.1.1,  $T(0_V) = 0_W$ , so  $0_V \in \ker T$ .
- (2) Let  $u, v \in \ker T$ , so  $T(u) = T(v) = 0_W$ . By (LT1),  $T(u+v) = T(u) + T(v) = 0_W + 0_W = 0_W$ . Therefore  $u + v \in \ker T$ .
- (3) Let  $v \in \ker T$  and  $c \in F$ . By (LT2),  $T(cv) = cT(v) = c0_W = 0_W$ , using Theorem 1.1.6. Therefore  $cv \in \ker T$ .

■

**Theorem 2.2.2. The Image Is a Subspace**

Let  $T: V \rightarrow W$  be linear. Then:

- (a)  $\operatorname{im} T$  is a subspace of  $W$ ;
- (b)  $T$  is surjective if and only if  $\operatorname{im} T = W$ .

*Proof.*

- (a) We use the Subspace Test (Theorem 1.2.1). (1)  $0_W = T(0_V)$  by Theorem 2.1.1, so  $0_W \in \operatorname{im} T$ . (2) Let  $w_1, w_2 \in \operatorname{im} T$ , say  $w_1 = T(v_1)$  and  $w_2 = T(v_2)$  with  $v_1, v_2 \in V$ . By (LT1),  $w_1 + w_2 = T(v_1 + v_2)$ , and  $v_1 + v_2 \in V$ , so  $w_1 + w_2 \in \operatorname{im} T$ . (3) Let  $w = T(v) \in \operatorname{im} T$  and  $c \in F$ . By (LT2),  $cw = T(cv)$  with  $cv \in V$ , so  $cw \in \operatorname{im} T$ .
- (b) The inclusion  $\operatorname{im} T \subseteq W$  always holds. By Definition 0.4.3,  $T$  is surjective exactly when every  $w \in W$  equals  $T(v)$  for some  $v$ , that is, when  $W \subseteq \operatorname{im} T$ . So  $T$  is surjective if and only if  $\operatorname{im} T = W$ .

■

This pays off a pattern from Chapter 1, §2. The null space of a matrix (Example (Solutions of a Homogeneous System)), the symmetric matrices (Example (Symmetric Matrices)) and the trace-free matrices (Example (Trace-Free Matrices)) were each shown to be subspaces by a separate check, and that section promised a single theorem. Each is the kernel of a linear map ( $T_{\mathbf{A}}, \mathbf{A} \mapsto \mathbf{A} - \mathbf{A}^T$ , and the trace), so [Theorem 2.2.1](#) covers them all at once.

**First examples.** Start with the maps that have the most extreme kernels and images.

**Example. The Zero Map and the Identity**

Find the kernel and image of the zero map  $0: V \rightarrow W$  and of the identity  $\operatorname{id}_V: V \rightarrow V$ .

*Solution.* The zero map sends every vector to  $0_W$ , so  $\ker 0 = V$  and  $\operatorname{im} 0 = \{0_W\}$ . The identity sends  $v$  to  $v$ , which is 0 only when  $v = 0$ , so  $\ker \operatorname{id}_V = \{0\}$ . Every  $v$  is its own image,

so  $\text{im id}_V = V$ .

These two are the ends of the scale. The kernel is always between  $\{0\}$  and  $V$ , and the image between  $\{0\}$  and  $W$ . The zero map kills everything and reaches nothing but 0; the identity kills nothing and reaches everything.

**Example. Projection onto an Axis**

Find the kernel and image of the projection  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $T(a_1, a_2) = (a_1, 0)$ , of Example (Projection onto an Axis).

*Solution.*  $T(a_1, a_2) = (0, 0)$  exactly when  $a_1 = 0$ . So  $\ker T = \{(0, a_2) : a_2 \in \mathbb{R}\}$ , the  $y$ -axis. Every output has second entry 0, and every  $(a_1, 0)$  is an output, namely  $T(a_1, 0)$ . So  $\text{im } T$  is the  $x$ -axis. The kernel is the direction the light shines along; the image is the screen the shadow falls on.

**A nonexample by minimal change.** Replace  $\mathbf{0}$  by another vector: for the projection,  $\{\mathbf{v} : T(\mathbf{v}) = (1, 0)\} = \{(1, a_2) : a_2 \in \mathbb{R}\}$ . This is the vertical line through  $(1, 0)$ . It does not contain  $\mathbf{0}$ , and it is not closed under addition, since  $(1, 0) + (1, 0) = (2, 0)$  lies off the line. So it is **not** a subspace. It is the kernel shifted by  $(1, 0)$ . The same thing happens for systems in Chapter 3: the solution set of  $\mathbf{A}\mathbf{x} = \mathbf{b}$  is one particular solution plus the set of solutions of  $\mathbf{A}\mathbf{x} = \mathbf{0}$ , which is  $\ker T_{\mathbf{A}}$ .

**Why the preimage of 0?** Because 0 is the one output that every linear map is guaranteed to hit, so its preimage is never empty, and because all other preimages are translates of it. If  $T(p) = w$ , then  $T(v) = w$  exactly when  $T(v - p) = 0$ , that is, when  $v \in p + \ker T$ . The kernel describes the shape of every nonempty preimage at once.

*Warning. Keep track of where each subspace lives.* The kernel sits inside the **domain**  $V$ ; the image sits inside the **codomain**  $W$ . For  $T_{\mathbf{A}}: F^n \rightarrow F^m$ , the kernel consists of columns with  $n$  entries and the image of columns with  $m$  entries, so “ $\ker T = \text{im } T$ ” does not even make sense unless  $V = W$ . And the image is **not** the codomain in general: the differentiation map  $D: \mathbb{R}[x]_{\leq 3} \rightarrow \mathbb{R}[x]_{\leq 3}$  has codomain  $\mathbb{R}[x]_{\leq 3}$ , but  $x^3$  is not the derivative of any cubic, so  $\text{im } D \neq \mathbb{R}[x]_{\leq 3}$ .

## 2.2.2 Injectivity via the kernel

The computation that opened the section now becomes a theorem. It is the reason to prefer the kernel over the definition of injectivity whenever a map is linear.

**Theorem 2.2.3. Kernel Test for Injectivity**

Let  $T: V \rightarrow W$  be linear. Then  $T$  is injective **if and only if**  $\ker T = \{0\}$ .

*Proof.*  $(\Rightarrow)$  Suppose  $T$  is injective, and let  $v \in \ker T$ . Then  $T(v) = 0_W = T(0_V)$  by Theorem 2.1.1. Injectivity gives  $v = 0_V$ . Hence  $\ker T \subseteq \{0\}$ , and the reverse inclusion holds

because  $\ker T$  is a subspace (Theorem 2.2.1).

( $\Leftarrow$ ) Suppose  $\ker T = \{0\}$ , and let  $T(u) = T(v)$ . By Theorem 2.1.2,  $T(u-v) = T(u) - T(v) = 0$ , so  $u - v \in \ker T = \{0\}$ . Hence  $u = v$ , and  $T$  is injective. ■

Compare the work. The definition asks: for **all pairs**  $u, v$ , does  $T(u) = T(v)$  force  $u = v$ ? The theorem asks only: for **each**  $v$ , does  $T(v) = 0$  force  $v = 0$ ? The second question is usually a homogeneous linear system or a coefficient comparison. For  $T_{\mathbf{A}}$ , the theorem says that  $\mathbf{x} \mapsto \mathbf{Ax}$  is injective exactly when  $\mathbf{Ax} = \mathbf{0}$  has only the trivial solution; Chapter 3 turns that into a test that elimination can run.

## 2.2.3 Computing the image

The image is defined by an existence condition, which is hard to check directly. When  $V$  has a spanning list, applying  $T$  to it gives a spanning list of  $\operatorname{im} T$ .

### Theorem 2.2.4. The Image Is Spanned by the Images of a Spanning List

Let  $T: V \rightarrow W$  be linear, and suppose  $(v_1, \dots, v_k)$  spans  $V$ . Then

$$\operatorname{im} T = \operatorname{span}(T(v_1), \dots, T(v_k)).$$

In particular, if  $V$  is finite-dimensional, then so is  $\operatorname{im} T$ , even when  $W$  is not.

*Proof.* ( $\supseteq$ ) Each  $T(v_i)$  lies in  $\operatorname{im} T$ , which is a subspace by Theorem 2.2.2. By Theorem 1.3.1 (3), the span of these vectors is contained in  $\operatorname{im} T$ .

( $\subseteq$ ) Let  $w \in \operatorname{im} T$ , say  $w = T(v)$  with  $v \in V$ . Since  $(v_1, \dots, v_k)$  spans  $V$ ,  $v = a_1 v_1 + \dots + a_k v_k$  for some  $a_i \in F$ . By Theorem 2.1.4,  $w = a_1 T(v_1) + \dots + a_k T(v_k) \in \operatorname{span}(T(v_1), \dots, T(v_k))$ .

If  $V$  is finite-dimensional, some finite list spans  $V$  (Definition 1.3.4), and its images form a finite list spanning  $\operatorname{im} T$ . ■

So the recipe for the image is: apply  $T$  to a basis (or any spanning list) of  $V$ , then sift the results to a basis if one is wanted (Theorem 1.5.4). The images of a basis need not be independent; the projection sends the basis  $(\mathbf{e}_1, \mathbf{e}_2)$  to  $(\mathbf{e}_1, \mathbf{0})$ .

## 2.2.4 Examples

We now compute kernels and images across the families: polynomials, matrices, and sequences. In each case we then read off injectivity from the kernel and surjectivity from the image.

### Example. Differentiation

Find the kernel and image of  $D: \mathbb{R}[x]_{\leq 3} \rightarrow \mathbb{R}[x]_{\leq 2}$ ,  $D(p) = p'$ .

*Solution. Kernel.* Let  $p = a_0 + a_1x + a_2x^2 + a_3x^3$ . Then  $D(p) = a_1 + 2a_2x + 3a_3x^2$ , which is the zero polynomial exactly when  $a_1 = 2a_2 = 3a_3 = 0$ . Since  $2 \neq 0$  and  $3 \neq 0$  in  $\mathbb{R}$ , this means  $a_1 = a_2 = a_3 = 0$ . So  $\ker D = \{a_0 : a_0 \in \mathbb{R}\} = \text{span}(1)$ , the constants.

*Image.* By [Theorem 2.2.4](#) applied to the basis  $(1, x, x^2, x^3)$ ,

$$\begin{aligned}\text{im } D &= \text{span}(D(1), D(x), D(x^2), D(x^3)) = \text{span}(0, 1, 2x, 3x^2) \\ &= \text{span}(1, x, x^2) = \mathbb{R}[x]_{\leq 2}.\end{aligned}$$

So  $D$  is surjective but not injective:  $p$  and  $p + 5$  have the same derivative. Note  $\dim \ker D + \dim \text{im } D = 1 + 3 = 4 = \dim \mathbb{R}[x]_{\leq 3}$ .

### Example. Trace

Find the kernel and image of  $\text{tr}: M_2(\mathbb{R}) \rightarrow \mathbb{R}$ .

*Solution. Kernel.* A matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  has trace 0 exactly when  $d = -a$ , that is, when it equals

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\mathbf{E}_{12} + c\mathbf{E}_{21}.$$

So  $\mathbf{E}_{11} - \mathbf{E}_{22}, \mathbf{E}_{12}, \mathbf{E}_{21}$  span  $\ker \text{tr}$ . They are independent: the combination above is the zero matrix only if  $a = b = c = 0$ . Hence they form a basis, and  $\dim \ker \text{tr} = 3$ .

*Image.*  $\text{im tr}$  is a subspace of  $\mathbb{R}$ , and it contains  $\text{tr } \mathbf{E}_{11} = 1$ . The lazy preimage  $r\mathbf{E}_{11}$  of any  $r \in \mathbb{R}$  shows  $\text{im tr} = \mathbb{R}$ . So the trace is surjective but not injective. Again  $3 + 1 = 4 = \dim M_2(\mathbb{R})$ .

The next two examples live in infinite-dimensional spaces, and the pattern “one property but not the other” goes the opposite way.

### Example. Integration

Find the kernel and image of  $J: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ ,  $J(p)(x) = \int_0^x p(t) dt$ , of Example (Integration).

*Solution. Kernel.* For  $p = a_0 + a_1x + \cdots + a_nx^n$ ,  $J(p) = a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$ . This is the zero polynomial only if every  $\frac{a_k}{k+1} = 0$ , that is, every  $a_k = 0$ . So  $\ker J = \{0\}$ , and  $J$  is injective by [Theorem 2.2.3](#).

*Image.* Let  $U = \{q \in \mathbb{R}[x] : q(0) = 0\}$ , the polynomials with constant coefficient 0. Every  $J(p)$  has constant coefficient 0, so  $\text{im } J \subseteq U$ . Conversely, if  $q = b_1x + b_2x^2 + \cdots + b_mx^m \in U$ , then  $p = b_1 + 2b_2x + \cdots + mb_mx^{m-1}$  satisfies  $J(p) = q$ . Hence  $\text{im } J = U$ . Since  $1 \notin U$ ,  $J$  is **not** surjective.

**Example. The Shifts: Injective but Not Surjective, and Conversely**

Find the kernels and images of the right and left shifts  $R, L: F^{\mathbb{N}} \rightarrow F^{\mathbb{N}}$  of Example (The Shift Maps on Sequences).

*Solution. Right shift.*  $R(s_0, s_1, \dots) = (0, s_0, s_1, \dots)$  is the zero sequence only if every  $s_k = 0$ . So  $\ker R = \{0\}$ , and  $R$  is injective. Every output has first entry 0; conversely,  $(0, t_1, t_2, \dots) = R(t_1, t_2, \dots)$ . So  $\operatorname{im} R = \{t \in F^{\mathbb{N}} : t_0 = 0\} \neq F^{\mathbb{N}}$ , and  $R$  is not surjective:  $e_0 = (1, 0, 0, \dots)$  is never an output.

*Left shift.*  $L(s_0, s_1, s_2, \dots) = (s_1, s_2, \dots)$  is zero exactly when  $s_1 = s_2 = \dots = 0$ , with  $s_0$  arbitrary. So  $\ker L = \operatorname{span}(e_0) \neq \{0\}$ , and  $L$  is not injective. Every sequence is an output, since  $L(R(t)) = t$ . So  $\operatorname{im} L = F^{\mathbb{N}}$ , and  $L$  is surjective.

This pays off the pointer from Chapter 1, §8:  $R$  matches  $F^{\mathbb{N}}$  one-to-one with its proper subspace  $\operatorname{im} R$ . For a map  $V \rightarrow V$  on a **finite** set, injective and surjective are equivalent (Theorem 0.4.9). The shifts, and the pair  $J$  and  $D$  on  $\mathbb{R}[x]$  (where  $D(J(p)) = p$  makes  $D$  surjective, while  $D(1) = 0$  makes it not injective), show that for linear maps on an infinite-dimensional space this equivalence fails. The next section proves that it holds again when  $V$  is finite-dimensional.

*Quick check.* Find  $\ker T$  and  $\operatorname{im} T$  for the map  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $T(x, y) = (2x - y, x + y, x - y)$ , of Example (Finding a Formula from Values on a Basis). Is  $T$  injective? Surjective?

*Solution.* If  $T(x, y) = \mathbf{0}$ , then  $x + y = 0$  and  $x - y = 0$ , so  $x = y = 0$ . Hence  $\ker T = \{\mathbf{0}\}$ , and  $T$  is injective. By Theorem 2.2.4,  $\operatorname{im} T = \operatorname{span}(T(\mathbf{e}_1), T(\mathbf{e}_2)) = \operatorname{span}((2, 1, 1), (-1, 1, -1))$ , a plane in  $\mathbb{R}^3$ . A span of two vectors has dimension at most  $2 < 3$ , so  $\operatorname{im} T \neq \mathbb{R}^3$  and  $T$  is not surjective.

## 2.2.5 What injective and surjective maps do to lists

Injectivity and surjectivity are statements about vectors. They translate into statements about lists: injective maps keep independent lists independent, and surjective maps keep spanning lists spanning. On a basis, the translation goes both ways.

### Theorem 2.2.5. Injective Maps Preserve Independence, Surjective Maps Preserve Spanning

Let  $T: V \rightarrow W$  be linear.

- (a) If  $T$  is injective and  $(v_1, \dots, v_k)$  is linearly independent in  $V$ , then  $(T(v_1), \dots, T(v_k))$  is linearly independent in  $W$ .
- (b) If  $T$  is surjective and  $(v_1, \dots, v_k)$  spans  $V$ , then  $(T(v_1), \dots, T(v_k))$  spans  $W$ .

- (c) If  $(v_1, \dots, v_n)$  is a basis of  $V$ , then  $T$  is injective if and only if  $(T(v_1), \dots, T(v_n))$  is linearly independent, and  $T$  is surjective if and only if  $(T(v_1), \dots, T(v_n))$  spans  $W$ .

*Proof.*

- (a) Let  $a_1T(v_1) + \dots + a_kT(v_k) = 0$ . By Theorem 2.1.4,  $T(a_1v_1 + \dots + a_kv_k) = 0$ , so  $a_1v_1 + \dots + a_kv_k \in \ker T$ . Since  $T$  is injective,  $\ker T = \{0\}$  (Theorem 2.2.3), so  $a_1v_1 + \dots + a_kv_k = 0$ . The independence of  $(v_1, \dots, v_k)$  gives  $a_1 = \dots = a_k = 0$ .
- (b) By Theorem 2.2.4,  $\text{span}(T(v_1), \dots, T(v_k)) = \text{im } T$ , and  $\text{im } T = W$  because  $T$  is surjective (Theorem 2.2.2).
- (c) *Injectivity.*  $(\Rightarrow)$  is (a).  $(\Leftarrow)$  Suppose the images are independent, and let  $v \in \ker T$ . Write  $v = a_1v_1 + \dots + a_nv_n$ . By Theorem 2.1.4,  $0 = T(v) = a_1T(v_1) + \dots + a_nT(v_n)$ , so all  $a_i = 0$  and  $v = 0$ . Hence  $\ker T = \{0\}$ , and  $T$  is injective by Theorem 2.2.3.

*Surjectivity.* By Theorem 2.2.4,  $\text{im } T = \text{span}(T(v_1), \dots, T(v_n))$ . By Theorem 2.2.2,  $T$  is surjective exactly when this span is  $W$ . ■

**Warning. Without injectivity, independence can be lost.** The projection  $T(a_1, a_2) = (a_1, 0)$  sends the independent list  $((1, 0), (1, 1))$  to  $((1, 0), (1, 0))$ , which is dependent. The kernel is to blame:  $(1, 1) - (1, 0) = (0, 1)$  lies in  $\ker T$ . Likewise, a map that is not surjective sends **no** list to a spanning list of  $W$ , since all the images lie in  $\text{im } T \neq W$ : the inclusion  $\mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 3}$  sends the basis  $(1, x, x^2)$  to a list that misses  $x^3$ .

Part (c) turns a question about a map into a question about  $n$  vectors in  $W$ , which for vectors in  $F^m$  is a question Chapter 3 settles by elimination. In the next section, counting dimensions will let us check only **one** of injectivity and surjectivity when  $\dim V = \dim W$ .

## 2.2.6 Exercises

### 2.2.6.1 A. Check your understanding

#### Exercise. A1

- (a) Define the kernel and the image of a linear map  $T: V \rightarrow W$ . Of which spaces are they subspaces?
- (b) State the kernel test for injectivity.
- (c) True or false: if  $T(u) = T(v)$ , then  $u - v \in \ker T$ . Justify your answer.
- (d) True or false: for  $D: \mathbb{R}[x]_{\leq 3} \rightarrow \mathbb{R}[x]_{\leq 3}$ ,  $D(p) = p'$ , we have  $\text{im } D = \mathbb{R}[x]_{\leq 3}$ . Justify your answer.
- (e) True or false: for a linear  $T: V \rightarrow W$  and **any**  $w \in W$ , the set  $\{v \in V : T(v) = w\}$  is a subspace of  $V$ . Justify your answer.

*Solution.*

- (a) See [Definition 2.2.1](#) and [Definition 2.2.2](#):  $\ker T = \{v \in V : T(v) = 0\}$ , a subspace of  $V$  ([Theorem 2.2.1](#));  $\operatorname{im} T = \{T(v) : v \in V\}$ , a subspace of  $W$  ([Theorem 2.2.2](#)).
- (b) See [Theorem 2.2.3](#): a linear map is injective if and only if its kernel is  $\{0\}$ .
- (c) True. By Theorem 2.1.2,  $T(u - v) = T(u) - T(v) = 0$ .
- (d) False. Every derivative of a polynomial of degree at most 3 has degree at most 2, so  $x^3 \notin \operatorname{im} D$ .
- (e) False. If  $w \neq 0$ , the set does not contain 0, since  $T(0) = 0 \neq w$ . For example, for the projection of [Example \(Projection onto an Axis\)](#) and  $w = (1, 0)$ , the set is the line  $\{(1, a_2)\}$ . It is a subspace exactly when  $w = 0$ , in which case it is  $\ker T$ .

### 2.2.6.2 B. Practice

#### Exercise. B1: Kernel and Image from the Definitions

For each map, find the kernel and the image, give a basis of each, and decide whether the map is injective and whether it is surjective.

- (a)  $D^2: \mathbb{R}[x]_{\leq 3} \rightarrow \mathbb{R}[x]_{\leq 3}$ , where  $D$  is differentiation.
- (b)  $T: M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ ,  $T(\mathbf{A}) = \mathbf{A} - \mathbf{A}^\top$ .

*Solution.*

- (a) Write  $p = a_0 + a_1x + a_2x^2 + a_3x^3$ . Then  $D^2p = 2a_2 + 6a_3x$ . *Kernel:*  $D^2p = 0$  forces  $a_2 = a_3 = 0$ , so  $\ker D^2 = \mathbb{R}[x]_{\leq 1}$ , with basis  $(1, x)$ . *Image:* the formula shows every  $D^2p$  lies in  $\mathbb{R}[x]_{\leq 1}$ , and conversely  $b_0 + b_1x = D^2(\frac{b_0}{2}x^2 + \frac{b_1}{6}x^3)$ , so  $\operatorname{im} D^2 = \mathbb{R}[x]_{\leq 1}$ , again with basis  $(1, x)$ . Since  $\ker D^2 \neq \{0\}$ ,  $D^2$  is not injective ([Theorem 2.2.3](#)); since  $\operatorname{im} D^2 \neq \mathbb{R}[x]_{\leq 3}$  (it omits  $x^2$ ), it is not surjective.
- (b)  $T$  is linear because the transpose is:  $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$  and  $(c\mathbf{A})^\top = c\mathbf{A}^\top$  ([Theorem 0.9.5](#)). *Kernel:*  $\mathbf{A} - \mathbf{A}^\top = \mathbf{O}$  says  $\mathbf{A} = \mathbf{A}^\top$ , so  $\ker T$  is the space of symmetric matrices, with basis  $(\mathbf{E}_{11}, \mathbf{E}_{22}, \mathbf{E}_{12} + \mathbf{E}_{21})$ . *Image:* each  $\mathbf{A} - \mathbf{A}^\top$  is skew-symmetric, since  $(\mathbf{A} - \mathbf{A}^\top)^\top = \mathbf{A}^\top - \mathbf{A} = -(\mathbf{A} - \mathbf{A}^\top)$ ; and every skew-symmetric  $\mathbf{B} \in M_2(\mathbb{R})$  is a multiple of  $\mathbf{E}_{12} - \mathbf{E}_{21}$ , with  $\mathbf{B} = T(\frac{1}{2}\mathbf{B})$ . So  $\operatorname{im} T = \operatorname{span}(\mathbf{E}_{12} - \mathbf{E}_{21})$ , with basis  $(\mathbf{E}_{12} - \mathbf{E}_{21})$ .  $T$  is neither injective (the kernel is nonzero) nor surjective (the image is not all of  $M_2(\mathbb{R})$ ).

#### Exercise. B2: Kernels and Images Beyond $F^n$

For each map, find a basis of the kernel and a basis of the image, and decide whether the map is injective and whether it is surjective.

- (a)  $T: \mathbb{R}[x]_{\leq 2} \rightarrow \mathbb{R}[x]_{\leq 2}$ ,  $T(p) = p' + p$ .
- (b)  $S: M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ ,  $S(\mathbf{A}) = \mathbf{A} - \mathbf{A}^\top$ .

*Solution.* Both maps are linear. For  $c \in \mathbb{R}$ , by linearity of differentiation ([Example \(Differentiation\)](#)),  $T(cp + q) = (cp' + q') + (cp + q) = cT(p) + T(q)$ . By linearity of the transpose ([Example \(Trace and Transpose\)](#)),  $S(c\mathbf{A} + \mathbf{B}) = c\mathbf{A} + \mathbf{B} - c\mathbf{A}^\top - \mathbf{B}^\top = cS(\mathbf{A}) + S(\mathbf{B})$ . Both are linear by [Theorem 2.1.3](#).

- (a) *Kernel.* Let  $p = a + bx + cx^2$ . Then  $T(p) = (b + 2cx) + (a + bx + cx^2) = (a + b) + (b + 2c)x + cx^2$ . This is zero exactly when  $c = 0$ ,  $b + 2c = 0$ , and  $a + b = 0$ , that is,  $c = b = a = 0$ . So  $\ker T = \{0\}$ , with the empty list as basis, and  $T$  is injective.

*Image.* We show  $T$  is surjective. Given  $q \in \mathbb{R}[x]_{\leq 2}$ , let  $p = q - q' + q''$ . Then  $p \in \mathbb{R}[x]_{\leq 2}$ , and since  $q''' = 0$ ,

$$T(p) = (q' - q'' + q''') + (q - q' + q'') = q.$$

So  $\operatorname{im} T = \mathbb{R}[x]_{\leq 2}$ , with basis  $(1, x, x^2)$ , and  $T$  is surjective.

- (b) *Kernel.*  $S(\mathbf{A}) = \mathbf{O}$  exactly when  $\mathbf{A}^\top = \mathbf{A}$ . A matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is symmetric exactly when  $b = c$ , that is, when it equals  $a\mathbf{E}_{11} + d\mathbf{E}_{22} + b(\mathbf{E}_{12} + \mathbf{E}_{21})$ . These three matrices are independent (compare entries), so  $(\mathbf{E}_{11}, \mathbf{E}_{22}, \mathbf{E}_{12} + \mathbf{E}_{21})$  is a basis of  $\ker S$ .  $S$  is not injective.

*Image.* By Theorem 2.2.4 with the basis  $(\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22})$ ,  $\operatorname{im} S$  is spanned by  $S(\mathbf{E}_{11}) = \mathbf{O}$ ,  $S(\mathbf{E}_{12}) = \mathbf{E}_{12} - \mathbf{E}_{21}$ ,  $S(\mathbf{E}_{21}) = \mathbf{E}_{21} - \mathbf{E}_{12}$ , and  $S(\mathbf{E}_{22}) = \mathbf{O}$ . So  $\operatorname{im} S = \operatorname{span}(\mathbf{E}_{12} - \mathbf{E}_{21})$ , with basis  $(\mathbf{E}_{12} - \mathbf{E}_{21})$ , the skew-symmetric matrices. Since  $\mathbf{I}_2 \notin \operatorname{im} S$ ,  $S$  is not surjective.

### 2.2.6.3 C. Going deeper

In these exercises, for linear maps  $T: U \rightarrow V$  and  $S: V \rightarrow W$  we write  $ST$  for the composite  $S \circ T: U \rightarrow W$ , which applies  $T$  first. It is linear:  $ST(cu + u') = S(cT(u) + T(u')) = cST(u) + ST(u')$ , by Theorem 2.1.3 for  $T$  and then for  $S$ .

#### Exercise. C1: Kernels and Images of a Composite

Let  $T: U \rightarrow V$  and  $S: V \rightarrow W$  be linear.

- Prove that  $\ker T \subseteq \ker(ST)$  and  $\operatorname{im}(ST) \subseteq \operatorname{im} S$ .
- Give an example, with  $U = V = W = \mathbb{R}^2$ , where  $\ker T \neq \ker(ST)$ .
- Deduce that if  $ST$  is injective, then  $T$  is injective, and if  $ST$  is surjective, then  $S$  is surjective.

*Solution.*

- Let  $u \in \ker T$ . Then  $ST(u) = S(0) = 0$  by Theorem 2.1.1, so  $u \in \ker(ST)$ . Let  $w \in \operatorname{im}(ST)$ , say  $w = S(T(u))$ . Then  $w = S(v)$  with  $v = T(u) \in V$ , so  $w \in \operatorname{im} S$ .
- Let  $T = \operatorname{id}_{\mathbb{R}^2}$  and let  $S$  be the projection  $S(a_1, a_2) = (a_1, 0)$ . Then  $\ker T = \{0\}$ , while  $\ker(ST) = \ker S$  is the  $y$ -axis (Example (Projection onto an Axis)).
- If  $ST$  is injective, then  $\ker(ST) = \{0\}$  (Theorem 2.2.3), so  $\ker T = \{0\}$  by (a), and  $T$  is injective. If  $ST$  is surjective, then  $W = \operatorname{im}(ST) \subseteq \operatorname{im} S \subseteq W$  by (a), so  $\operatorname{im} S = W$  and  $S$  is surjective (Theorem 2.2.2).

#### Exercise. C2: When Is a Composite Injective?

Let  $T: U \rightarrow V$  and  $S: V \rightarrow W$  be linear, and suppose  $T$  is injective. Prove that  $ST$  is injective if and only if  $\ker S \cap \operatorname{im} T = \{0\}$ , that is, if and only if  $S$  is injective on  $\operatorname{im} T$ . Give an example where  $S$  is not injective but  $ST$  is.

*Solution.* ( $\Rightarrow$ ) Suppose  $ST$  is injective, and let  $v \in \ker S \cap \operatorname{im} T$ . Then  $v = T(u)$  for some  $u \in U$ , and  $ST(u) = S(v) = 0$ . So  $u \in \ker(ST) = \{0\}$  (Theorem 2.2.3), and  $v = T(0) = 0$ .

( $\Leftarrow$ ) Suppose  $\ker S \cap \operatorname{im} T = \{0\}$ , and let  $u \in \ker(ST)$ . Then  $T(u) \in \operatorname{im} T$  and  $S(T(u)) = 0$ , so  $T(u) \in \ker S \cap \operatorname{im} T = \{0\}$ . Since  $T$  is injective,  $u = 0$ . Hence  $\ker(ST) = \{0\}$ , and  $ST$  is injective.

Example: let  $T: \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $T(a) = (a, 0)$ , which is injective with  $\operatorname{im} T$  the  $x$ -axis, and let  $S: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $S(a_1, a_2) = a_1$ . Then  $\ker S$  is the  $y$ -axis, so  $S$  is not injective, but  $\ker S \cap \operatorname{im} T = \{0\}$ , and indeed  $ST(a) = a$  is injective.

### Exercise. C3: Differentiation over $\mathbb{F}_2$

Let  $D: \mathbb{F}_2[x]_{\leq 3} \rightarrow \mathbb{F}_2[x]_{\leq 3}$  be the formal derivative (Example (Differentiation)). Find bases of  $\ker D$  and  $\operatorname{im} D$ . Which step of Example (Differentiation) fails over  $\mathbb{F}_2$ ?

*Solution.* For  $p = a_0 + a_1x + a_2x^2 + a_3x^3$ ,  $D(p) = a_1 + 2a_2x + 3a_3x^2$ . In  $\mathbb{F}_2$ ,  $2 = 0$  and  $3 = 1$ , so  $D(p) = a_1 + a_3x^2$ .

*Kernel.*  $D(p) = 0$  exactly when  $a_1 = a_3 = 0$ , so  $\ker D = \{a_0 + a_2x^2\}$ , with basis  $(1, x^2)$ . The polynomials  $1$  and  $x^2$  are independent by Theorem 1.4.4.

*Image.* By Theorem 2.2.4,  $\operatorname{im} D = \operatorname{span}(D(1), D(x), D(x^2), D(x^3)) = \operatorname{span}(0, 1, 0, x^2)$ , with basis  $(1, x^2)$ .

The failing step is “ $2a_2 = 0$  forces  $a_2 = 0$ ”, which divides by 2; over  $\mathbb{F}_2$ ,  $2 = 0$ , and  $D(x^2) = 2x = 0$  although  $x^2$  is not constant. So over  $\mathbb{F}_2$  the kernel of differentiation is bigger than the constants, and here  $\ker D = \operatorname{im} D$ .