

1.3 The Least Upper Bound Axiom

We now know exactly what goes wrong in the rationals. The set of positive rationals whose square is less than 2 is nonempty and bounded above, and yet it has no supremum in $\mathbb{Q}$ (Theorem 1.2.2). Every failure of calculus over $\mathbb{Q}$ traces back to gaps of this kind. This section adds one axiom to the ordered-field axioms, stating that such gaps do not occur in $\mathbb{R}$. Then it draws the first consequence: infima exist too, for free.

1.3.1 The axiom

Up to now, “ $\mathbb{R}$ is an ordered field containing $\mathbb{Q}$ ” has been our only assumption about the real numbers (Example (The rationals and the reals)). That assumption holds for $\mathbb{Q}$ itself, so it cannot be what makes $\mathbb{R}$ different. We add exactly one more.

Every set of reals that is bounded above has a least upper bound, unless it is empty.

Axiom 1.3.1. Least Upper Bound Axiom

Every **nonempty** subset of $\mathbb{R}$ that is **bounded above** has a supremum in $\mathbb{R}$.

In symbols: for every $A \subseteq \mathbb{R}$,

$$\left(A \neq \emptyset \text{ and } \exists M \in \mathbb{R} \forall a \in A : a \leq M \right) \Rightarrow \exists s \in \mathbb{R} : s = \sup A.$$

Standing assumption. From now on, $\mathbb{R}$ is an ordered field (Definition 1.1.1), containing $\mathbb{Q}$, that satisfies [Axiom 1.3.1](#). Every theorem about real numbers in this book is proved from these assumptions, together with the logic and set theory of *Logic and Proof* and *Sets, Relations, and Cardinality* (including, where it is flagged, a choice principle).

In words, the axiom has two hypotheses and one conclusion. The hypotheses are about the set A : it has at least one element, and at least one upper bound. The conclusion is about $\mathbb{R}$: among the upper bounds of A there is a least one, and it is a real number. By Theorem 1.2.1 it is unique, so the notation $\sup A$ is justified whenever the two hypotheses hold. The axiom is also called the **completeness axiom**; “complete” will acquire a second, related meaning in Chapter 2, and the two are reconciled in Chapter 11.

Each hypothesis is there because without it the conclusion is false.

- **Nonempty.** The empty set is bounded above, by every real number, and it has no supremum (shown in the section on suprema). Without “nonempty”, the axiom would be false.
- **Bounded above.** The interval $[0, \infty)$ contains 0, but no real number sits above all of it, so there is no least such number either. The set $\mathbb{N}$ is a subtler example of the same failure; that it is not bounded above in $\mathbb{R}$ is proved later in this chapter, from this very axiom.

The axiom is a statement about $\mathbb{R}$ and not about ordered fields in general. It fails in $\mathbb{Q}$: some nonempty subset of $\mathbb{Q}$, bounded above in $\mathbb{Q}$, has no supremum in $\mathbb{Q}$. Which result of the previous section shows this, and whether it needs adapting, is [Exercise \(C1: The axiom in other ordered fields\)](#) (a). So the axiom is exactly the property that separates the two fields we have been comparing.

Remark. Why is this an axiom and not a theorem? Because we have not said what a real number is. One can **construct** $\mathbb{R}$ from $\mathbb{Q}$ and then prove the least upper bound property as a theorem. *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$,* Chapters 5 and 6 take this route with Cauchy sequences of rationals, and show that any two ordered fields satisfying the axiom are the same up to relabeling, so the axiom pins $\mathbb{R}$ down completely; the optional *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$,* Chapter 7 builds $\mathbb{R}$ by Dedekind cuts instead. This volume does not use those chapters: everything we need follows from the axiom.

Example. A set of reals that must have a supremum

Let $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\}$. Show that $\sup S$ exists in $\mathbb{R}$. What can be said about it so far?

Solution. We check the two hypotheses of [Axiom 1.3.1](#). S is nonempty, since $1 > 0$ and $1^2 < 2$. S is bounded above by 2: if $x \in S$ had $x > 2$, then $x^2 > 4$ by Theorem 1.1.2 (g), contradicting $x^2 < 2$. So the axiom gives a real number $s = \sup S$.

So far we know only what (S1) and (S2) give: $1 \leq s \leq 2$, since $1 \in S$ and 2 is an upper bound. Compare this with Theorem 1.2.2. The rational version of this set has no supremum in $\mathbb{Q}$, and the proof of that theorem showed that a rational candidate s with $s^2 < 2$ or $s^2 > 2$ cannot be the supremum. Neither case used rationality except to keep $s \pm h$ rational. So we can guess the answer: $s^2 = 2$. Proving it, and so proving that $\sqrt{2}$ exists, is done later in this chapter.

The example shows the pattern of every use of the axiom. First check the two hypotheses, which is usually routine. Then the axiom hands over a number s , and all we know about s is (S1) and (S2). Everything else must be extracted from those two properties.

Warning. Do not forget “nonempty”. The empty set is bounded above and has no supremum, so an argument that applies the axiom to a set without first producing an element of that set is incomplete. The failure is easy to commit when the set is defined by a condition, such as $\{x \in \mathbb{R} : x^2 < c\}$: for $c \leq 0$ that set is empty.

Warning. The axiom gives existence, not a formula. It says that $\sup A$ exists; it does not say what it is, and it is usually **not** the upper bound M used to check the hypothesis. For S above, the hypothesis was checked with $M = 2$, and $\sup S \neq 2$: the smaller number $\frac{3}{2}$ is also an upper bound, since any $x > \frac{3}{2}$ has $x^2 > \frac{9}{4} > 2$ by Theorem 1.1.2 (g). To identify a supremum we must still verify (S1) and (S2) for a specific candidate.

Quick check. Let $A = \{n \in \mathbb{Z} : n \leq 3.5\} = \{\dots, 1, 2, 3\}$. Does A satisfy the hypotheses of [Axiom 1.3.1](#)? What is $\sup A$, and is it a maximum?

Solution. Yes: $3 \in A$, so A is nonempty, and 3.5 is an upper bound by definition of A . So $\sup A$ exists. It is 3, not 3.5: the element 3 is an upper bound, because an integer greater than 3 is at least 4, which exceeds 3.5 (arithmetic of the integers, *Sets, Relations, and Cardinality*, Chapter 2), and it belongs to A , so it is the maximum, and a maximum is the supremum (Exercise (B1: Maxima are suprema)). The bound used to check the hypothesis was not the supremum.

1.3.2 Infima come free

The axiom is stated for suprema only. We do not need a second axiom for infima: reflecting the line through 0 turns lower bounds into upper bounds and “greatest” into “least”. To say this precisely we need the reflected set.

For $A \subseteq \mathbb{R}$, let

$$-A := \{-a : a \in A\}.$$

So $x \in -A$ if and only if $-x \in A$, and $-(-A) = A$. For example, $-(0, 1] = [-1, 0)$, and $-\{-1, 3\} = \{1, -3\}$. The definition makes sense in any ordered field, and the next theorem holds in any ordered field; the axiom enters only afterwards.

Theorem 1.3.2. Infimum by Reflection

Let F be an ordered field, $A \subseteq F$, and $m, t \in F$.

- (a) m is a lower bound of A if and only if $-m$ is an upper bound of $-A$.
- (b) t is an infimum of A if and only if $-t$ is a supremum of $-A$.

In particular, if $-A$ has a supremum, then A has an infimum, and $\inf A = -\sup(-A)$.

Idea. Reflection $x \mapsto -x$ reverses every inequality (Theorem 1.1.2 (a)), so it should carry each property of an infimum to the matching property of a supremum. The proof checks this property by property: (I1) against (S1), and (I2) against (S2), which is the “split into two properties” move for suprema.

Proof.

- (a) By Theorem 1.1.2 (a), $m \leq a$ if and only if $-a \leq -m$. So $m \leq a$ for every $a \in A$ if and only if $-a \leq -m$ for every $a \in A$. The elements $-a$ with $a \in A$ are exactly the elements of $-A$, so the last statement says that $-m$ is an upper bound of $-A$.
- (b) $(\Rightarrow)$ Suppose t is an infimum of A . By (I1), t is a lower bound of A , so by (a), $-t$ is an upper bound of $-A$: this is (S1) for $-t$. For (S2), let M be an upper bound of $-A$. By (a) applied to $m = -M$, the element $-M$ is a lower bound of A . By (I2), $-M \leq t$, and so $-t \leq M$ by Theorem 1.1.2 (a). Hence $-t$ is a supremum of $-A$.
- $(\Leftarrow)$ Suppose $-t$ is a supremum of $-A$. By (S1) and (a), t is a lower bound of A : this is (I1). For (I2), let m be a lower bound of A . By (a), $-m$ is an upper bound of $-A$, so (S2) gives $-t \leq -m$, that is, $m \leq t$. Hence t is an infimum of A .

The final sentence follows from (b) with $t := -\sup(-A)$, since $-t = \sup(-A)$.

Now the axiom can be applied to $-A$.

Theorem 1.3.3. Greatest Lower Bound Property

Every **nonempty** subset A of $\mathbb{R}$ that is **bounded below** has an infimum in $\mathbb{R}$, and

$$\inf A = -\sup(-A).$$

Proof. Since A is nonempty, so is $-A$. Let m be a lower bound of A . By Theorem 1.3.2 (a), $-m$ is an upper bound of $-A$, so $-A$ is bounded above. By Axiom 1.3.1, $-A$ has a supremum, and by Theorem 1.3.2, A has the infimum $-\sup(-A)$. ■

Alternatively, one can take the supremum of the lower bounds. Let L be the set of lower bounds of A . It is nonempty, because A is bounded below, and any element of A , of which there is at least one because A is nonempty, sits above all of L . So $\ell := \sup L$ exists by Axiom 1.3.1. Each $a \in A$ is an upper bound of L , so $\ell \leq a$ by (S2); hence ℓ is a lower bound of A . And every lower bound of A lies in L , so it is at most ℓ by (S1). Thus $\ell = \inf A$. In this route the two hypotheses on A swap roles: “bounded below” makes L nonempty, and “nonempty” makes L bounded above.

The hypotheses of Theorem 1.3.3 are needed for the same reasons as those of the axiom. The empty set is bounded below and has no infimum, since every real number is a lower bound of it and $t+1$ beats any candidate t . The interval $(-\infty, 0]$ is nonempty but not bounded below, and has no infimum.

So in $\mathbb{R}$, suprema and infima exist under mirror-image hypotheses, and each can be computed from the other. The mirror identity $\sup A = -\inf(-A)$, for A nonempty and bounded above, is Exercise (B1: The mirror identity).

1.3.3 Unbounded and empty sets

It is sometimes convenient to give $\sup A$ a meaning even when the axiom does not apply. For a nonempty set A that is not bounded above one writes $\sup A = +\infty$, and one sets $\sup \emptyset = -\infty$; symmetrically, $\inf A = -\infty$ for a nonempty set not bounded below, and $\inf \emptyset = +\infty$. The strange-looking value $\sup \emptyset = -\infty$ is chosen so that “ $A \subseteq B$ implies $\sup A \leq \sup B$ ” stays true when $A = \emptyset$.

Remark. These are conventions of notation, and $\pm\infty$ are **not** real numbers: they obey none of the field axioms, and no rule of Theorem 1.1.2 applies to them. The series uses them from *Measure and Integration* on, where measure theory needs suprema of arbitrary sets, and announces them again there. Until then, whenever we write $\sup A$, the set A is nonempty and bounded above, and $\sup A$ is a real number.

1.3.4 What the axiom will do

With the axiom in place, the rest of the chapter extracts its first consequences: a practical test for recognizing a supremum, the Archimedean property, the density of the rationals, the existence of roots, and the nested interval property. The first, the test, holds in every ordered field; it is the tool for the others. Each of the rest fails in $\mathbb{Q}$ or in some other ordered field. Three of them, the Archimedean property, the roots, and the nested interval property, are proved by the same pattern: define a set, check the two hypotheses, take its supremum, and extract what is needed from (S1) and (S2). The density of the rationals then follows from the Archimedean property, with no new supremum.

1.3.5 Exercises

1.3.5.1 A. Check your understanding

Exercise. A1

For each set, decide whether it satisfies both hypotheses of [Axiom 1.3.1](#). If a hypothesis fails, name it; if both hold, find the supremum.

- (a) $\{x \in \mathbb{R} : x^2 < 0\}$.
- (b) $\{x^2 : x \in \mathbb{R}\}$.
- (c) $(-\infty, 0)$.
- (d) $\{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, regarded as a subset of the ordered field $\mathbb{Q}$.

Solution.

- (a) The set is empty, since $x^2 \geq 0$ for every x (Theorem 1.1.2 (e)). “Nonempty” fails.
- (b) “Bounded above” fails. Given $M \in \mathbb{R}$, let $x := |M| + 1$. Then $x \geq 1$, so $x^2 \geq x$ (multiply $x \geq 1$ by $x > 0$), and $x > |M| \geq M$. Hence $x^2 > M$, and M is not an upper bound.
- (c) Both hold: -1 is an element and 0 is an upper bound. The supremum is 0 : it is an upper bound, and if $M < 0$ were an upper bound, then $M \geq -1$ and the midpoint $\frac{M}{2}$ would satisfy $M < \frac{M}{2} < 0$, so it lies in the set and exceeds M .
- (d) Both hypotheses hold in $\mathbb{Q}$ (Theorem 1.2.2), but the axiom is a statement about $\mathbb{R}$, not about $\mathbb{Q}$. In $\mathbb{Q}$ the conclusion fails: the set has no supremum in $\mathbb{Q}$, by the same theorem. This is exactly why $\mathbb{Q}$ does not satisfy the axiom.

1.3.5.2 B. Practice

Exercise. B1: The mirror identity

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Show that $-A$ has a lower bound, and that $\sup A = -\inf(-A)$.

Solution. Since A is nonempty, so is $-A$. Let M be an upper bound of A . For every $x \in -A$ we have $-x \in A$, so $-x \leq M$, and hence $-M \leq x$ by Theorem 1.1.2 (a). So $-M$ is a lower bound of $-A$, and $-A$ is bounded below. By [Theorem 1.3.3](#), $\inf(-A)$ exists

and equals $-\sup(-(-A)) = -\sup A$, because $-(-A) = A$. Negating both sides gives $\sup A = -\inf(-A)$.

Exercise. B2: The infimum is below the supremum

Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Prove that $\inf A \leq \sup A$, with equality if and only if A has exactly one element.

Solution. Both exist, by [Axiom 1.3.1](#) and [Theorem 1.3.3](#). Since A is nonempty, choose $a \in A$. By (II), $\inf A \leq a$, and by (S1), $a \leq \sup A$. Hence $\inf A \leq \sup A$.

If $A = \{a\}$, then a is both the maximum and the minimum of A , so $\sup A = a = \inf A$ by Exercise (B1: Maxima are suprema). Conversely, suppose $\inf A = \sup A = c$. For every $a \in A$, $c \leq a \leq c$, so $a = c$ by trichotomy. Hence $A \subseteq \{c\}$, and since A is nonempty, $A = \{c\}$.

1.3.5.3 C. Going deeper

Exercise. C1: The axiom in other ordered fields

- Say what it would mean for the ordered field $\mathbb{Q}$ to satisfy the least upper bound property, and show that it does not. Is Theorem 1.2.2 exactly a witness, or does it need to be adapted?
- Say that an ordered field F has the **greatest lower bound property** if every nonempty subset of F that is bounded below has an infimum in F . Prove that if F has the greatest lower bound property, then it has the least upper bound property.
- Conclude that, for an ordered field, the two properties are equivalent.

Solution.

- $\mathbb{Q}$ would satisfy it if every nonempty subset of $\mathbb{Q}$ that is bounded above in $\mathbb{Q}$ had a supremum in $\mathbb{Q}$. Theorem 1.2.2 is exactly a witness that it does not: its set is a subset of $\mathbb{Q}$, nonempty, bounded above by the rational 2, and has no supremum in $\mathbb{Q}$. Nothing needs adapting, because the theorem was stated and proved inside the ordered field $\mathbb{Q}$, with the upper bounds in (S2) ranging over $\mathbb{Q}$.
- Let $A \subseteq F$ be nonempty and bounded above, with upper bound M . Then $-A$ is nonempty, and it is bounded below by $-M$, by [Theorem 1.3.2](#) (a) applied with $m = -M$ and the set $-A$ in place of A (so that $-(-A) = A$). By the greatest lower bound property, $-A$ has an infimum t in F . By [Theorem 1.3.2](#) (b), applied to the set $-A$, $-t$ is a supremum of $-(-A) = A$. So A has a supremum in F .
- The converse direction is the proof of [Theorem 1.3.3](#), which used only [Theorem 1.3.2](#) (valid in every ordered field) and the least upper bound property. So in any ordered field each property implies the other.

Exercise. C2: The upper bounds form a ray

- (a) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $U(A)$ be its set of upper bounds. Prove that $U(A) = [\sup A, \infty)$.
- (b) Let $B = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, and let U be its set of upper bounds in $\mathbb{Q}$. Prove that there is no rational q with $U = \{u \in \mathbb{Q} : u \geq q\}$.
- (c) Explain in one sentence how (a) and (b) restate the difference between $\mathbb{R}$ and $\mathbb{Q}$.

Solution.

- (a) ($\subseteq$) If $M \in U(A)$, then $\sup A \leq M$ by (S2), so $M \in [\sup A, \infty)$. ($\supseteq$) If $M \geq \sup A$, then for every $a \in A$, $a \leq \sup A \leq M$ by (S1), so $M \in U(A)$.
- (b) Suppose $U = \{u \in \mathbb{Q} : u \geq q\}$ for some rational q . Then $q \in U$, and $q \leq u$ for every $u \in U$. So q is the minimum of U , that is, a rational upper bound of B below every other rational upper bound. That makes q a supremum of B in $\mathbb{Q}$, which contradicts Theorem 1.2.2.
- (c) In $\mathbb{R}$ the upper bounds of a nonempty bounded-above set always form a closed ray that starts at a point, while in $\mathbb{Q}$ they can form a ray whose starting point is missing.