

1.7 Nested Intervals and the Uncountability of the Reals

Each point we have built so far was a single supremum: the supremum of a set that “reaches toward” the point we want. This section builds points from a different kind of data, a sequence of closed intervals, each inside the one before. We prove that such intervals always have a point in common. In $\mathbb{Q}$ this fails, and so it is one more way of saying that $\mathbb{R}$ has no gaps. The payoff is a proof that the real numbers cannot be listed, one that uses no decimal expansions, only the axiom.

1.7.1 Trapping a point

Here is a way to hunt for $\sqrt{2}$ using nothing but rational arithmetic. Start with the interval $[1, 2]$; since $1^2 < 2 < 2^2$, the number we are after is in there. Cut the interval at its midpoint $\frac{3}{2}$. Since $(\frac{3}{2})^2 = \frac{9}{4} > 2$, keep the left half $[1, \frac{3}{2}]$. Cut again at $\frac{5}{4}$: since $(\frac{5}{4})^2 = \frac{25}{16} < 2$, keep the right half. Continuing, we get

$$\begin{aligned} [1, 2] &\supseteq [1, \tfrac{3}{2}] \supseteq [\tfrac{5}{4}, \tfrac{3}{2}] \\ &\supseteq [\tfrac{11}{8}, \tfrac{3}{2}] \supseteq [\tfrac{11}{8}, \tfrac{23}{16}] \supseteq \cdots, \end{aligned}$$

each interval half as long as the one before, each with rational endpoints, and each containing $\sqrt{2}$. Is there a number lying in all of them?

In $\mathbb{R}$ the answer should be “yes, $\sqrt{2}$ ”. In $\mathbb{Q}$ the answer is “no”: the only candidate is $\sqrt{2}$, and it is not rational. We make both answers precise below. First, a name for the shape of this data. A family of sets $I_1, I_2, I_3, \dots$ is **nested** if $I_{n+1} \subseteq I_n$ for every $n \geq 1$.

Theorem 1.7.1. Nested Interval Theorem

For each $n \geq 1$, let $I_n = [a_n, b_n]$ be a closed bounded interval with $a_n \leq b_n$, and suppose the family is nested: $I_{n+1} \subseteq I_n$ for every $n \geq 1$. Then $\bigcap_{n \geq 1} I_n \neq \emptyset$. More precisely, $\alpha := \sup\{a_n : n \geq 1\}$ exists and lies in every I_n .

If moreover for every $\varepsilon > 0$ there is $n \geq 1$ with $b_n - a_n < \varepsilon$, then $\bigcap_{n \geq 1} I_n = \{\alpha\}$.

Idea. Picture the intervals on the line. As n grows, the left endpoints move right (or stay), the right endpoints move left (or stay), and they never cross: every left endpoint lies below every right endpoint, even for different indices. So the set of left endpoints is bounded above by each b_n . Its supremum α is at least every a_n , because it is an upper bound. It is at most every b_n , because it is the *least* upper bound and each b_n is an upper bound. Hence $\alpha \in [a_n, b_n]$ for every n .

For uniqueness: two different common points $x < y$ would force every interval to have length at least $y - x$, and the shrinking hypothesis with $\varepsilon = y - x$ forbids that.

Proof. We first compare endpoints with different indices.

Claim. $a_m \leq b_n$ for all $m, n \geq 1$.

Proof of Claim. From $I_{j+1} \subseteq I_j$ for every j , induction on $k - m$ gives $I_k \subseteq I_m$ whenever

$m \leq k$. Let $m, n \geq 1$ and $k := \max\{m, n\}$. Then $I_k \subseteq I_m$ and $I_k \subseteq I_n$. Since $a_k \in I_k \subseteq I_m$, we have $a_m \leq a_k$. Since $b_k \in I_k \subseteq I_n$, we have $b_k \leq b_n$. Hence $a_m \leq a_k \leq b_k \leq b_n$. ■

Let $A := \{a_n : n \geq 1\}$. It is nonempty, since $a_1 \in A$, and by the claim it is bounded above by b_1 . By Axiom 1.3.1, $\alpha := \sup A$ exists.

Fix $n \geq 1$. Since α is an upper bound of A , $a_n \leq \alpha$. By the claim, b_n is an upper bound of A , so (S2) of Definition 1.2.4 gives $\alpha \leq b_n$. Hence $\alpha \in [a_n, b_n] = I_n$. As n was arbitrary, $\alpha \in \bigcap_n I_n$, and the intersection is nonempty.

Now assume the shrinking hypothesis, and let $x \in \bigcap_n I_n$. Suppose $x \neq \alpha$, and let $u := \min\{x, \alpha\}$ and $v := \max\{x, \alpha\}$, so $u < v$ and both lie in every I_n . For every n , $a_n \leq u < v \leq b_n$, so $b_n - a_n \geq v - u$. But $\varepsilon := v - u > 0$, and by hypothesis some n has $b_n - a_n < v - u$. This is a contradiction, so $x = \alpha$. Hence $\bigcap_n I_n = \{\alpha\}$. ■

Closedness is what lets α belong to I_n . The proof gives only $a_n \leq \alpha \leq b_n$, with nonstrict inequalities, and α may well equal an endpoint. An open interval would exclude it.

Now back to the bisection. Its intervals have length $1/2^{n-1}$, and $2^{n-1} \geq n$ for every $n \geq 1$ (by Logic and Proof, Axiom 4.2.1), so $1/2^{n-1} \leq 1/n$. Given $\varepsilon > 0$, Corollary 1.5.2 gives n with $1/n < \varepsilon$, and then the n -th interval is shorter than ε . The rule keeps $a_n^2 < 2 < b_n^2$ at every step, so $a_n < \sqrt{2} < b_n$ by Theorem 1.1.2 (g), and every interval contains $\sqrt{2}$. By Theorem 1.7.1 the intersection is exactly $\{\sqrt{2}\}$. The bisection traps $\sqrt{2}$ and nothing else.

1.7.2 What each hypothesis does

Theorem 1.7.1 has four hypotheses: the intervals are closed, they are bounded, they are nested, and they live in $\mathbb{R}$. The uniqueness clause adds a fifth, that the lengths shrink below every ε . Each one is needed, and each has a witness.

Hypothesis dropped	Family	Intersection
closed	$(0, 1/n)$	$\emptyset$
bounded	$[n, \infty)$	$\emptyset$
nested	$[n, n+1]$	$\emptyset$
the field is $\mathbb{R}$	the rational bisection intervals, in $\mathbb{Q}$	$\emptyset$
lengths shrink (uniqueness only)	$[-1/n, 1 + 1/n]$	$[0, 1]$

We check each row.

- **Closed.** The intervals $(0, 1/n)$ are nested and bounded. A real $x \leq 0$ is not in $(0, 1)$. For $x > 0$, Corollary 1.5.2 gives n with $1/n < x$, so $x \notin (0, 1/n)$. Hence no real number is in every interval. Here the supremum of the left endpoints is $\alpha = 0$, and it lies in none of the intervals: exactly the step that closedness secured.
- **Bounded.** The intervals $[n, \infty)$ are closed and nested. For any $x \in \mathbb{R}$, Theorem 1.5.1 gives $n > x$, and then $x \notin [n, \infty)$. The proof breaks earlier here: the left endpoints form $\mathbb{Z}_{>0}$, which has no upper bound, so α does not exist.

- **Nested.** The intervals $[n, n + 1]$ are closed and bounded, but $[1, 2] \cap [3, 4] = \emptyset$ already.
- **The field is $\mathbb{R}$.** In $\mathbb{Q}$, a closed interval means $\{q \in \mathbb{Q} : a \leq q \leq b\}$. Take the bisection intervals $[a_n, b_n]$ from the start of this section. They are defined by rational arithmetic alone, because each midpoint c is rational, and $c^2 \neq 2$ by Logic and Proof, Theorem 2.2.1, so the rule “keep the half whose endpoints’ squares straddle 2” always applies. We showed that the real intervals $[a_n, b_n]$ have intersection $\{\sqrt{2}\}$. So their rational parts have intersection $\mathbb{Q} \cap \{\sqrt{2}\} = \emptyset$.
- **Shrinking.** The intervals $[-1/n, 1 + 1/n]$ are closed, bounded, and nested, and their intersection is $[0, 1]$: every point of $[0, 1]$ lies in all of them, and a point outside $[0, 1]$ is excluded from some interval by Corollary 1.5.2 ([Exercise \(B2: An intersection that is not a point\)](#) writes this out). The intersection is nonempty, but it is not a single point. Their lengths $1 + 2/n$ never drop below 1.

Warning. Dropping a hypothesis removes the guarantee. It does not force the conclusion to fail. The open intervals $(-1/n, 1/n)$ are nested, and their intersection is $\{0\}$, which is not empty. A witness that a hypothesis is needed must be a family where the conclusion actually fails, like $(0, 1/n)$.

Quick check. The table’s witness for “closed” uses the open intervals $(0, 1/n)$. Do the half-open intervals $(0, 1/n]$ witness it too? And the half-open intervals $[0, 1/n)$? Give a reason for each.

Solution. The half-open intervals $(0, 1/n]$ are nested and bounded, but no real number lies in all of them: $x \leq 0$ is not in $(0, 1]$, and for $x > 0$ some n has $1/n < x$ (Corollary 1.5.2). The supremum of the left endpoints is 0, which is excluded from every interval. So $(0, 1/n]$ is a witness too. The intervals $[0, 1/n)$ are not: 0 lies in all of them, so their intersection $\{0\}$ is not empty. Closing the left end puts the supremum 0 of the left endpoints back in.

1.7.3 The real numbers cannot be listed

Sets, Relations, and Cardinality, Chapter 3 calls a set **countable** if its elements can be listed as $x_0, x_1, x_2, \dots$, repetitions allowed; for a nonempty set this means that it is the image of some function from $\mathbb{N}$. That chapter shows that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable. It also shows that $\mathbb{R}$ is not, by Cantor’s diagonal argument with decimal expansions. Decimal expansions were taken there as a fact from school: every real number has one, and two different expansions such as $0.4999\dots$ and $0.5000\dots$ can name the same number. Neither fact has been proved from our axioms.

Here is a proof with no decimals, using only [Theorem 1.7.1](#).

Theorem 1.7.2. Uncountability of the Reals

Let $x_1, x_2, x_3, \dots$ be any list of real numbers, that is, any function $n \mapsto x_n$ from $\mathbb{Z}_{>0}$ to $\mathbb{R}$. Then there is $c \in \mathbb{R}$ with $c \neq x_n$ for every $n \geq 1$. Consequently $\mathbb{R}$ is not countable.

Idea. Build nested closed intervals $I_1 \supseteq I_2 \supseteq \cdots$, where I_n is chosen to avoid x_n . A point c common to all of them, which [Theorem 1.7.1](#) provides, avoids every x_n : it lies in I_n , and x_n does not.

To pass from I_n to I_{n+1} , cut I_n into thirds and look at the two outer thirds. They are disjoint closed intervals, so x_{n+1} misses at least one of them; take that one.

Before reading on: why thirds, and not halves?

Proof. We define closed intervals $I_n = [a_n, b_n]$ with $a_n < b_n$ by recursion on n .

Let $I_1 := [x_1 + 1, x_1 + 2]$. Then $x_1 \notin I_1$.

Suppose $I_n = [a_n, b_n]$ with $a_n < b_n$ has been defined. Let $d := b_n - a_n > 0$, and let

$$L := [a_n, a_n + \frac{d}{3}], \quad R := [b_n - \frac{d}{3}, b_n].$$

Both are closed intervals of length $d/3 > 0$ contained in I_n . They are disjoint, because $a_n + d/3 < b_n - d/3$, which is $2d/3 < d$. Define

$$I_{n+1} := \begin{cases} L & \text{if } x_{n+1} \notin L, \\ R & \text{if } x_{n+1} \in L. \end{cases}$$

In the second case $x_{n+1} \in L$, so $x_{n+1} \notin R$, since L and R are disjoint. In both cases $x_{n+1} \notin I_{n+1}$, $I_{n+1} \subseteq I_n$, and I_{n+1} has positive length.

Each I_{n+1} is determined by I_n and x_{n+1} through an explicit rule, so the family is defined by recursion (*Sets, Relations, and Cardinality*, Chapter 2); no choice principle is involved.

The intervals I_n are closed, bounded, and nested. By [Theorem 1.7.1](#), there is $c \in \bigcap_n I_n$. For each $n \geq 1$ we have $c \in I_n$ and $x_n \notin I_n$, so $c \neq x_n$.

Hence no function f from $\mathbb{N}$ to $\mathbb{R}$ has image all of $\mathbb{R}$: the list $x_n := f(n-1)$, for $n \geq 1$, has the same image as f , and it misses c . Since $\mathbb{R}$ is nonempty, it is not countable. This proves the theorem. ■

The answer to the prompt: halves share their midpoint. If x_{n+1} were exactly that midpoint, it would lie in both closed halves, and neither could be chosen to avoid it. Thirds leave a gap between the two outer pieces.

Only the existence half of [Theorem 1.7.1](#) was used; the intervals need not shrink.

Remark. Since $\mathbb{Q}$ is countable, and a union of two countable sets is countable (both from *Sets, Relations, and Cardinality*, Chapter 3), the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable. Otherwise $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be countable. In this sense most real numbers are irrational, although so far we have named only a few of them, such as $\sqrt{2}$ and $\sqrt{2} + q$ for rational q (the proof of Corollary 1.6.5).

The decimal proof of *Sets, Relations, and Cardinality*, Chapter 3 is secretly a nested interval proof too: the first n digits of a number place it in an interval of length 10^{-n} . The version above avoids the digits, and with them the numbers that have two expansions.

1.7.4 What the axiom bought us

Here is the chapter so far in one table. Each row is a property that $\mathbb{R}$ has, checked against the two ordered fields we have used as foils: $\mathbb{Q}$ and the field $\mathbb{R}(x)$ of Example (A non-Archimedean field).

Property	$\mathbb{R}$	$\mathbb{Q}$	$\mathbb{R}(x)$
nonempty sets bounded above have suprema	yes (axiom)	no: positive q with $q^2 < 2$	no: $\mathbb{N}$
Archimedean property	yes	yes	no: $x > n$ for all n
$\mathbb{Q}$ is dense	yes	yes	no: nothing rational in $(x, x + 1)$
positive elements have square roots	yes	no: 2	no: x
nested closed intervals meet	yes	no: bisection toward $\sqrt{2}$	no: $[n, x/n]$
uncountable	yes	no (<i>Sets, Relations, and Cardinality</i> , Chapter 3)	yes: contains $\mathbb{R}$

The entries for $\mathbb{R}$ are this chapter's theorems. For $\mathbb{Q}$, the failures come from Theorem 1.2.2, Logic and Proof, Theorem 2.2.1, and the bisection above; $\mathbb{Q}$ is Archimedean by Exercise (C3: Archimedean without the axiom); and $\mathbb{Q}$ is dense in itself because the average of two rationals is a rational between them. For $\mathbb{R}(x)$: every rational r is below some natural number (Exercise (C3: Archimedean without the axiom)), hence below x , so none lies in $(x, x + 1)$. If $(p/q)^2 = x$ for polynomials p, q , then $p^2 = xq^2$, whose sides have even and odd degree, which is impossible. That the nested intervals $[n, x/n]$ have no common point is proved in [Exercise \(C3: Nested intervals in the rational functions\)](#). And $\mathbb{R}(x)$ contains the constants, a copy of $\mathbb{R}$, so it cannot be countable, since a subset of a countable set is countable (*Sets, Relations, and Cardinality*, Chapter 3).

The $\mathbb{Q}$ column shows that the Archimedean property and density are far weaker than the axiom; the $\mathbb{R}(x)$ column shows an ordered field where even they fail.

Two questions are left open. Is the least upper bound axiom *equivalent* to some of the other rows? And could a different “no gaps” property serve as the axiom instead? Chapter 11 answers both. Over an arbitrary ordered field, the least upper bound axiom is equivalent to each of: the monotone convergence of bounded monotone sequences; the Bolzano–Weierstrass property; Cauchy completeness together with the Archimedean property; and the nested interval property together with the Archimedean property. You can prove one arrow of that last equivalence now ([Exercise \(C1: Exit check: nested intervals and the axiom\)](#)).

1.7.5 Exercises

1.7.5.1 A. Check your understanding

Exercise. A1

For each family $(I_n)_{n \geq 1}$, say which hypothesis of [Theorem 1.7.1](#) fails, and find $\bigcap_n I_n$.

- (a) $I_n = [0, 1/n]$.
- (b) $I_n = [n, n + 1/n]$.
- (c) $I_n = (-1/n, 1/n]$.
- (d) $I_n = [-n, n]$.
- (e) Run the construction in the proof of [Theorem 1.7.2](#) on a list that begins $x_1 = 0$, $x_2 = 1$, $x_3 = 2$: find I_1 , I_2 , and I_3 .
- (f) Where does the proof of [Theorem 1.7.2](#) use that the intervals L and R are **closed**, and what could go wrong with open thirds?

Solution.

- (a) Not closed. The intersection is $\{0\}$: 0 lies in every $[0, 1/n]$; a negative number lies in none; and for $x > 0$, Corollary 1.5.2 gives n with $1/n < x$, so $x \notin [0, 1/n]$. The conclusion holds although a hypothesis fails.
- (b) Not nested: $I_2 = [2, \frac{5}{2}] \not\subseteq [1, 2] = I_1$. The intersection is empty, since already $I_1 \cap I_3 = [1, 2] \cap [3, \frac{10}{3}] = \emptyset$.
- (c) Not closed. The intersection is $\{0\}$: 0 lies in every interval, and for $x \neq 0$, Corollary 1.5.2 gives n with $1/n < |x|$, so $x \notin I_n$.
- (d) Not nested in the required direction: $I_{n+1} \supseteq I_n$, and $I_2 = [-2, 2] \not\subseteq [-1, 1] = I_1$. The intersection is $I_1 = [-1, 1]$, since every I_n contains $[-1, 1]$.
- (e) $I_1 = [x_1 + 1, x_1 + 2] = [1, 2]$. For $n = 1$, $d = 1$, $L = [1, \frac{4}{3}]$, and $R = [\frac{5}{3}, 2]$; since $x_2 = 1 \in L$, $I_2 = R = [\frac{5}{3}, 2]$. For $n = 2$, $d = \frac{1}{3}$, $L = [\frac{5}{3}, \frac{16}{9}]$, and $R = [\frac{17}{9}, 2]$; since $x_3 = 2 \notin L$, $I_3 = L = [\frac{5}{3}, \frac{16}{9}]$. Each I_n misses x_n , and $I_3 \subseteq I_2 \subseteq I_1$.
- (f) Only in the last step, where [Theorem 1.7.1](#) is applied: its hypothesis is that the intervals are closed. With open thirds the family would be nested open intervals, and those can have empty intersection, like $(0, \frac{1}{n})$; then there might be no point c at all. The avoidance of x_{n+1} does not need closedness, only that L and R are disjoint.

1.7.5.2 B. Practice

Exercise. B1: Shrinking by a fixed factor

- (a) Let $I_n = [a_n, b_n]$ be nested closed intervals with $b_n - a_n = 2^{-n}$ for every n . Prove that $\bigcap_n I_n$ consists of exactly one point.
- (b) For $n \geq 1$, let $m_n := \lfloor 10^n \sqrt{2} \rfloor$ (Lemma 1.6.3), and let $a_n := m_n/10^n$ and $b_n := a_n + 10^{-n}$, so that a_n is the n -digit decimal truncation of $\sqrt{2}$. Prove that the intervals $[a_n, b_n]$ are nested, and that their intersection is $\{\sqrt{2}\}$. Compute a_1 and a_2 .

Solution.

(a) By [Theorem 1.7.1](#) it suffices to check the shrinking hypothesis. By induction, $2^n > n$ for every $n \in \mathbb{N}$ (as in Exercise (C2: Dyadic rationals)), so $2^{-n} < 1/n$ for every $n \geq 1$. Given $\varepsilon > 0$, Corollary 1.5.2 gives n with $1/n < \varepsilon$, and then $b_n - a_n = 2^{-n} < \varepsilon$. Hence the intersection is a single point.

(b) By definition of the floor, $m_n \leq 10^n \sqrt{2} < m_n + 1$, so $a_n \leq \sqrt{2} < b_n$ for every n .

Nested. Multiplying $m_n \leq 10^n \sqrt{2}$ by 10 gives $10m_n \leq 10^{n+1} \sqrt{2}$. An integer $k \leq z$ always satisfies $k \leq \lfloor z \rfloor$, since $k \geq \lfloor z \rfloor + 1$ would give $k > z$. Applied to $k = 10m_n$ and $z = 10^{n+1} \sqrt{2}$, this gives $10m_n \leq m_{n+1}$, that is, $a_n \leq a_{n+1}$. Similarly $m_{n+1} \leq 10^{n+1} \sqrt{2} < 10m_n + 10$, so $m_{n+1} < 10m_n + 10$, and since both sides are integers, $m_{n+1} + 1 \leq 10m_n + 10$. Dividing by 10^{n+1} gives $b_{n+1} \leq b_n$. Hence $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$.

The intersection. The lengths are $10^{-n} \leq 2^{-n}$, so by the argument of (a) the intersection is one point. Since $\sqrt{2}$ lies in every interval, that point is $\sqrt{2}$.

The first terms. $14^2 = 196 \leq 200 < 225 = 15^2$, so $14 \leq 10\sqrt{2} < 15$, $m_1 = 14$, and $a_1 = 1.4$. Likewise $141^2 = 19881 \leq 20000 < 20164 = 142^2$, so $m_2 = 141$ and $a_2 = 1.41$.

Exercise. B2: An intersection that is not a point

Prove that $\bigcap_{n \geq 1} [-1/n, 1 + 1/n] = [0, 1]$. Which hypothesis of the uniqueness clause of [Theorem 1.7.1](#) fails?

Solution. ($\supseteq$) Let $x \in [0, 1]$. For every n , $-1/n < 0 \leq x \leq 1 < 1 + 1/n$, so $x \in [-1/n, 1 + 1/n]$.

($\subseteq$) Let x lie in every $[-1/n, 1 + 1/n]$. Suppose $x < 0$. By Corollary 1.5.2 with $\varepsilon = -x > 0$, there is n with $1/n < -x$, that is, $x < -1/n$, so $x \notin [-1/n, 1 + 1/n]$: a contradiction. Suppose $x > 1$. With $\varepsilon = x - 1 > 0$, there is n with $1/n < x - 1$, that is, $x > 1 + 1/n$: again a contradiction. Hence $0 \leq x \leq 1$.

The lengths are $1 + 2/n > 1$, so for $\varepsilon = 1$ no interval is shorter than ε : the shrinking hypothesis fails. The intervals are closed, bounded, and nested, and the existence part applies, as it must.

1.7.5.3 C. Going deeper

Exercise. C1: Exit check: nested intervals and the axiom

(a) Without looking back, prove the existence part of [Theorem 1.7.1](#) from the least upper bound axiom.

(b) Now work in an ordered field F in which the Archimedean property holds (for every $t \in F$ some $n \in \mathbb{N}$ has $n > t$) and every nested family of closed bounded intervals has a common point. Let $A \subseteq F$ be nonempty and bounded above by M , and pick $a \in A$. Define intervals $[a_n, b_n]$ by $a_1 := a$, $b_1 := M$, and, with $c := (a_n + b_n)/2$, let $[a_{n+1}, b_{n+1}] := [a_n, c]$ if c is an upper bound of A , and $[c, b_n]$ otherwise. Prove that for every n : b_n is an upper bound of A ; some element of A is $\geq a_n$; and $b_n - a_n = (M - a)/2^{n-1}$.

- (c) Let s be a common point of all the $[a_n, b_n]$. Prove that $s = \sup A$. Deduce that in an ordered field, the nested interval property together with the Archimedean property implies the least upper bound axiom.

Solution (Hint 1). In (c), prove the two properties of a supremum separately, each by contradiction. Each contradiction comes from making $b_n - a_n$ smaller than a positive gap you are given, which is where the Archimedean property enters.

Solution (Hint 2). For “ s is an upper bound” in (c), argue by contradiction: what does an element $y \in A$ with $y > s$ give you, and what do you know about b_n ?

Solution.

- (a) Let $I_n = [a_n, b_n]$ be nested. For $m \leq k$, $I_k \subseteq I_m$, by induction from $I_{j+1} \subseteq I_j$. Given m, n , put $k := \max\{m, n\}$; then $a_m \leq a_k \leq b_k \leq b_n$. So $A := \{a_n : n \geq 1\}$ is nonempty and bounded above by b_1 , and $\alpha := \sup A$ exists by Axiom 1.3.1. For each n , $a_n \leq \alpha$ because α is an upper bound, and $\alpha \leq b_n$ because b_n is an upper bound and α is the least one. So $\alpha \in I_n$ for every n .
- (b) In an ordered field $2 = 1 + 1 > 0$, so c is defined and $a_n \leq c \leq b_n$ when $a_n \leq b_n$. We prove the three statements together by induction on n . For $n = 1$: $b_1 = M$ is an upper bound, $a_1 = a \in A$, and $b_1 - a_1 = M - a$. Suppose they hold for n . If c is an upper bound of A , then $b_{n+1} = c$ is an upper bound, and $a_{n+1} = a_n$, so the element of A that is $\geq a_n$ still works. Otherwise, c is not an upper bound, so some $y \in A$ has $y > c = a_{n+1}$, and $b_{n+1} = b_n$ is an upper bound. In both cases $b_{n+1} - a_{n+1} = (b_n - a_n)/2 = (M - a)/2^n$. The intervals are nested, since each is one half of the previous one.
- (c) s is an upper bound. Suppose some $y \in A$ has $y > s$. Since $2^{n-1} \geq n$, we have $b_n - a_n \leq (M - a)/n$. By the Archimedean property in F , choose $n > (M - a)/(y - s)$; then $b_n - a_n < y - s$. Since $a_n \leq s$, this gives $b_n < a_n + (y - s) \leq y$. But b_n is an upper bound of A and $y \in A$, a contradiction.

No smaller upper bound. Let $u < s$, and assume for contradiction that u bounds A from above. As before, choose n with $b_n - a_n < s - u$. Since $s \leq b_n$, we get $a_n > b_n - (s - u) \geq s - (s - u) = u$. By (b) some $y \in A$ has $y \geq a_n > u$, contradicting that u is an upper bound.

So $s = \sup A$. Every nonempty subset of F that is bounded above has a supremum, which is the least upper bound axiom for F . The Archimedean property was used twice, to make $b_n - a_n$ smaller than a given positive gap. (If $M = a$, all the intervals are $\{a\}$, and the same argument gives $s = a = \sup A$; the Archimedean step is then not even needed.)

Marking scheme. (10 points) - 2: (a) shows $a_m \leq b_n$ for all m, n , takes $\alpha = \sup\{a_n\}$ with both hypotheses of the axiom checked, and shows $a_n \leq \alpha \leq b_n$ by (S1) and (S2). - 3: (b) proves the three statements of (b) by one induction, treating both cases of the bisection rule, including the element of A that is at least a_{n+1} in the second case. - 2: (c) shows s is an upper bound: a $y \in A$ above s is excluded by choosing n with $b_n - a_n < y - s$, using the Archimedean property. - 2: (c) shows no $u < s$ is an upper bound, choosing n with $b_n - a_n < s - u$ and using the element of A that is at least a_n . - 1: (c) concludes that F has the least upper bound property, and says where each of the two assumed properties was used. Fatal errors (0 for the step): using Axiom 1.3.1, or any supremum, in (b) or (c); choosing

n before the gap $y - s$ or $s - u$ is known; assuming A has a maximum. A proof of (c) that shows $\sup A = s$ through the backward direction of Theorem 1.4.1, rerun in F , instead of (S2) earns the same points; the theorem is stated for $\mathbb{R}$, so citing it without rerunning its proof in F is not enough.

Exercise. C2: Uncountable pieces

- (a) Let $x < y$. Prove that the interval (x, y) is not countable.
- (b) Deduce that (x, y) contains an irrational number, without using $\sqrt{2}$ or any other named irrational. You may use the facts from *Sets, Relations, and Cardinality*, Chapter 3 that $\mathbb{Q}$ is countable, and that a subset of a countable set is countable.
- (c) Let $C \subseteq \mathbb{R}$ be countable. Show that the real numbers not in C are dense: every interval (x, y) with $x < y$ contains a real number that is not in C .

Solution.

- (a) Let f be any function from $\mathbb{N}$ to $\mathbb{R}$, and list its values as $z_n := f(n - 1)$ for $n \geq 1$, as in the proof of Theorem 1.7.2; this list has the same image as f . Run the construction in the proof of Theorem 1.7.2, but start from a closed interval inside (x, y) that avoids z_1 . Let $d := y - x$, and let $L := [x + d/4, x + d/3]$ and $R := [y - d/3, y - d/4]$. They are disjoint, closed, of positive length, and inside (x, y) . Let I_1 be L if $z_1 \notin L$, and R otherwise. Then define $I_2, I_3, \dots$ by the outer-thirds rule from the proof. The intervals are nested, closed, and bounded, $z_n \notin I_n$ for every $n \geq 1$, and by Theorem 1.7.1 some c lies in all of them. Then $c \in I_1 \subseteq (x, y)$, and $c \neq z_n$ for every $n \geq 1$. So no function from $\mathbb{N}$ has image all of (x, y) , and (x, y) , which is nonempty, is not countable.
- (b) Suppose every element of (x, y) were rational. Then $(x, y) \subseteq \mathbb{Q}$, and a subset of the countable set $\mathbb{Q}$ is countable, contradicting (a). So (x, y) contains an irrational number. This reproves Corollary 1.6.5 by counting: there are too many reals in (x, y) for all of them to be rational.
- (c) As in (b): if $(x, y) \subseteq C$, then (x, y) is countable, contradicting (a). Part (b) is the case $C = \mathbb{Q}$.

Exercise. C3: Nested intervals in the rational functions

Work in the ordered field $\mathbb{R}(x)$ of Example (A non-Archimedean field). For $n \in \mathbb{Z}_{>0}$, let $I_n := \{f \in \mathbb{R}(x) : n \leq f \leq x/n\}$.

- (a) Show that $n < x/n$, so I_n is a closed bounded interval of positive length, and that $I_{n+1} \subseteq I_n$.
- (b) Let $f = p/q$ with the leading coefficient of q positive, and suppose $f \in I_n$ for every n . Show that $\deg p > \deg q$.
- (c) Show that $\deg p \geq \deg q + 2$ and $\deg p = \deg q + 1$ both lead to a contradiction, and conclude that $\bigcap_n I_n = \emptyset$.
- (d) Which line of the proof of Theorem 1.7.1 fails in $\mathbb{R}(x)$?

Solution (Hint 1). Positivity in $\mathbb{R}(x)$ is decided by leading coefficients. Write each difference you need to sign, such as $x/n - f$ or $N - f$, as a single fraction with a positive denominator.

Solution (Hint 2). For (b): suppose $\deg p \leq \deg q$, and look for an $N \in \mathbb{N}$ with $f < N$.

Solution. Write lc for the leading coefficient, and recall from Exercise (C2: The order on the rational functions) that for $g = u/w$ with $\text{lc}(w) > 0$, $g > 0$ if and only if $\text{lc}(u) > 0$.

- (a) $x/n - n = (x - n^2)/n$ has numerator with leading coefficient $1 > 0$ and positive denominator, so $n < x/n$. Nesting: $n+1 > n$, and $x/n - x/(n+1) = x/(n(n+1)) > 0$. So $I_{n+1} \subseteq I_n$.
- (b) Since $f \geq 1 > 0$, $\text{lc}(p) > 0$. Suppose $\deg p \leq \deg q$. By Theorem 1.5.1, pick $N \in \mathbb{N}$ with $N > \text{lc}(p)/\text{lc}(q)$, a positive real number, so that $N \in \mathbb{Z}_{>0}$. Then $N - f = (Nq - p)/q$. If $\deg p < \deg q$, the leading coefficient of $Nq - p$ is $N \text{lc}(q) > 0$; if $\deg p = \deg q$, it is $N \text{lc}(q) - \text{lc}(p) > 0$. Either way $f < N$, so $f \notin I_N$, a contradiction. Hence $\deg p > \deg q$.
- (c) For each n ,

$$\frac{x}{n} - f = \frac{xq - np}{nq}, \quad \text{lc}(nq) = n \text{lc}(q) > 0.$$

If $\deg p \geq \deg q + 2$, then $\deg(np) > \deg(xq)$, so the numerator has leading coefficient $-n \text{lc}(p) < 0$. Thus $f > x/n$ for every n , and $f \notin I_1$. If $\deg p = \deg q + 1$, both xq and np have degree $\deg q + 1$, and the numerator has leading coefficient $\text{lc}(q) - n \text{lc}(p)$. Choosing $n > \text{lc}(q)/\text{lc}(p)$, which Theorem 1.5.1 allows, makes it negative, so $f > x/n$ and $f \notin I_n$. Both cases contradict $f \in \bigcap_n I_n$. With (b), no f lies in every I_n , and the intersection is empty.

- (d) The line “ $\alpha := \sup\{a_n\}$ exists by Axiom 1.3.1”. Here the left endpoints are $a_n = n$, and $\{a_n\} = \mathbb{Z}_{>0}$ is bounded above (by $x/1$, for instance) but has no supremum in $\mathbb{R}(x)$, by the argument for $\mathbb{N}$ in the section on the Archimedean property. The claim $a_m \leq b_n$ and every other line survive.

Exercise. C4: A root of a cubic, from half a proof

Let $g(x) := x^3 + x$. Then $g(0) = 0$ and $g(1) = 2$, so g passes the value 1 somewhere in $[0, 1]$. Here is the first half of a proof that it takes the value 1.

Let $S := \{x \in \mathbb{R} : x \geq 0, g(x) < 1\}$. Then $0 \in S$. If $x \geq 1$, then $x^3 \geq 1$ and $g(x) \geq 2 > 1$, so every element of S is less than 1. By Axiom 1.3.1, $s := \sup S$ exists, and $0 \leq s \leq 1$.

Name the move, then complete it. Prove that $g(s) = 1$. Then show that $0.68 < s < 0.69$, and that s is the only real number t with $g(t) = 1$.

Solution (Hint 1). Rule out $g(s) < 1$ and $g(s) > 1$ by nudging s to $s + h$ and $s - h$. For that you need a bound on $g(s + h) - g(s)$, and on $g(s) - g(s - h)$, of the form “a constant times h ”, valid for small h .

Solution (Hint 2). Expanding, $g(s+h) - g(s) = h(3s^2 + 3sh + h^2 + 1)$. With $0 \leq s \leq 1$ and $0 < h \leq 1$ this is at most $8h$.

Solution. The move is **construct a point as a supremum**. The given half defines the set of points where things are still fine ($g(x) < 1$) and takes its supremum; what remains is to rule out each way s could fail.

We use that g is strictly increasing on $[0, \infty)$: if $0 \leq u < v$, then $u^3 < v^3$ by the claim in the proof of Theorem 1.6.2, and adding $u < v$ gives $g(u) < g(v)$ by Theorem 1.1.2 (b). Expanding with Lemma 1.1.1, for every h ,

$$\begin{aligned} g(s+h) - g(s) &= h(3s^2 + 3sh + h^2 + 1), \\ g(s) - g(s-h) &= h(3s^2 - 3sh + h^2 + 1). \end{aligned}$$

Case $g(s) < 1$. Let $h := (1 - g(s))/16$, which depends only on s . Since $0 \leq g(s) < 1$, we have $0 < h \leq \frac{1}{16}$. With $s \leq 1$ and $h \leq 1$, the first identity gives

$$g(s+h) - g(s) \leq h(3 + 3 + 1 + 1) = 8h = \frac{1}{2}(1 - g(s)) < 1 - g(s),$$

so $g(s+h) < 1$. As $s+h \geq 0$, we get $s+h \in S$, and $s+h > s$ contradicts (S1).

Case $g(s) > 1$. Then $s > 0$, since $g(0) = 0$. Let $h := (g(s) - 1)/10 > 0$. Since $s \leq 1$ gives $s^3 \leq s$, we have $g(s) - 1 < g(s) \leq 2s$, so $h < \frac{s}{5} < s$ and $s-h > 0$. As $0 < h < s \leq 1$, we have $-3sh < 0$, and the second identity gives

$$g(s) - g(s-h) \leq h(3 + 1 + 1) = 5h = \frac{1}{2}(g(s) - 1) < g(s) - 1,$$

so $g(s-h) > 1$. Now $s-h$ is an upper bound of S : if $x \in S$ had $x > s-h$, then, as $s-h > 0$, monotonicity would give $g(x) > g(s-h) > 1$, contradicting $x \in S$. So (S2) gives $s \leq s-h$, which is false.

By trichotomy, $g(s) = 1$.

Location. Exact arithmetic gives $g(0.68) = 0.314432 + 0.68 = 0.994432 < 1$ and $g(0.69) = 0.328509 + 0.69 = 1.018509 > 1$. If $s \leq 0.68$, monotonicity would give $g(s) \leq g(0.68) < 1$, and if $s \geq 0.69$, it would give $g(s) > 1$. So $0.68 < s < 0.69$.

Uniqueness. Let $t \in \mathbb{R}$ with $g(t) = 1$. If $t < 0$, then $t^2 > 0$ and $t^3 = t \cdot t^2 < 0$ by Theorem 1.1.2 (c), so $g(t) < 0$; hence $t \geq 0$. On $[0, \infty)$, g is strictly increasing, so $t < s$ or $t > s$ would give $g(t) \neq g(s)$. Hence $t = s$.

This is the intermediate value theorem for one polynomial, proved by the move that proves it in general in Chapter 6.

Marking scheme. (10 points) - 1: names the move, and records that g is strictly increasing on $[0, \infty)$ with a reason. - 3: rules out $g(s) < 1$ with an explicit $h > 0$ depending only on s , a verified bound on $g(s+h) - g(s)$, and a contradiction with (S1). - 3: rules out $g(s) > 1$ with an explicit h , checking $s-h > 0$, that $g(s-h) > 1$, and that $s-h$ is an upper bound, then contradicting (S2). - 1: concludes $g(s) = 1$ by trichotomy. - 2: the location $0.68 < s < 0.69$ from two exact computations and monotonicity, and uniqueness, including the case $t < 0$. Fatal errors (0 for the step): an h that depends on an element of S or is not shown positive;

using that $s - h$ is an upper bound without monotonicity; assuming the intermediate value theorem. Other constants (for instance $h = \min\{1, (1 - g(s))/10\}$) earn full marks when the bound they need is verified.

1.7.5.4 R. Review

Exercise. R1

Let $A \subseteq \mathbb{R}$ be nonempty, and let s be an upper bound of A . Prove that the following are equivalent.

- (i) $s = \sup A$.
- (ii) For every $n \in \mathbb{Z}_{>0}$ there is $a \in A$ with $a > s - \frac{1}{n}$.
- (iii) No real number smaller than s is an upper bound of A .

Solution. By Logic and Proof, Proposition 2.4.6 it suffices to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). This is Corollary 1.4.2, or Theorem 1.4.1 ($\Rightarrow$) with $\varepsilon = \frac{1}{n}$.

(ii) $\Rightarrow$ (iii). Let $u < s$. By Corollary 1.5.2 there is n , depending on $s - u$, with $\frac{1}{n} < s - u$. By (ii) some $a \in A$ has $a > s - \frac{1}{n} > u$, so u is not an upper bound.

(iii) $\Rightarrow$ (i). (S1) holds because s is an upper bound. For (S2), let M be an upper bound. By (iii), $M < s$ is impossible, so $s \leq M$ by trichotomy.

The cycle is the method of *Logic and Proof*, Chapter 2 for equivalences (Logic and Proof, Proposition 2.4.6). Here (ii) $\Rightarrow$ (iii) is the only arrow with content: it is where the Archimedean property turns the special widths $\frac{1}{n}$ into every width.

Exercise. R2

- (a) Prove that if x is a real number and x^2 is irrational, then x is irrational.
- (b) Hence prove that the positive real number t with $t^4 = 2$ is irrational.

Solution.

- (a) By Logic and Proof, Proposition 2.3.2 it suffices to prove: if x is rational, then x^2 is rational. That holds because a product of rationals is rational (Example (The rationals and the reals)).
- (b) The number t exists by Theorem 1.6.2. Then $t^2 > 0$ and $(t^2)^2 = t^4 = 2$, so t^2 is the positive square root of 2, by the uniqueness in Theorem 1.6.1. It is irrational by Logic and Proof, Theorem 2.2.1. By (a), t is irrational.

The direct statement (a) would be awkward to prove as it stands, since “irrational” is a negative property; its contrapositive is one line. That is the reason to reach for the contrapositive (Logic and Proof, Proposition 2.3.2, *Logic and Proof*, Chapter 2).

Exercise. R3

Prove that the real number $\sqrt{2} + \sqrt{3}$ is irrational.

Solution. Here $\sqrt{3}$ and $\sqrt{6}$ are the positive square roots given by Theorem 1.6.2. First, $\sqrt{2}\sqrt{3} = \sqrt{6}$: the left side is positive and its square is $2 \cdot 3 = 6$, so it is the unique positive square root of 6.

Suppose, for a contradiction, that $r := \sqrt{2} + \sqrt{3}$ is rational [assumption for $\neg I$]. Then

$$r^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 5 + 2\sqrt{6},$$

so $\sqrt{6} = (r^2 - 5)/2$ is rational, because the rationals are closed under the field operations (Example (The rationals and the reals)). But no rational number has square 6 (Logic and Proof, Exercise (C1: The square root of six)). Hence r is irrational [$\neg I$].

Logic and Proof, Chapter 2 could prove only the negative half: no rational squares to 6 (Logic and Proof, Exercise (C1: The square root of six)). With Theorem 1.6.2 the number $\sqrt{6}$ exists in $\mathbb{R}$, and the negative statement becomes the positive one, “ $\sqrt{6}$ is irrational”, which is what the argument uses.

Exercise. R4

Prove that $|x - 1| + |x + 1| \geq 2$ for every real x , and find all x for which equality holds.

Solution. The tool is the triangle inequality (Theorem 1.1.4), applied to a sum chosen so that x cancels:

$$2 = |(1 - x) + (x + 1)| \leq |1 - x| + |x + 1| = |x - 1| + |x + 1|,$$

using $|1 - x| = |x - 1|$ (Proposition 1.1.3 (b)). By Exercise (B2: Inserting a third point) (b), equality holds exactly when $(1 - x)(x + 1) \geq 0$, that is, when $1 - x^2 \geq 0$, that is, when $-1 \leq x \leq 1$ (by Theorem 1.1.2 (g) and Proposition 1.1.3 (d), $x^2 \leq 1$ means $|x| \leq 1$). So equality holds exactly on $[-1, 1]$.

The look-alike tools work less well. Proof by cases, as in Logic and Proof, Example (The triangle inequality by cases), succeeds but needs three cases ($x < -1$, $-1 \leq x \leq 1$, $x > 1$) and a formula in each. Squaring both sides leaves the cross term $2|x^2 - 1|$, which still has an absolute value in it, so it only postpones the case split.