

1.4 Working with the Supremum

The least upper bound axiom (Axiom 1.3.1) tells us that suprema exist. To use a supremum, or to recognize one, we need a handle on its second property. As defined (Definition 1.2.4), (S2) quantifies over **all** upper bounds of the set, and in practice there is no way to list them. This section replaces (S2) by an equivalent statement quantified over small positive numbers ε , which can be checked by exhibiting elements. With it we prove the rules for computing suprema of scaled sets and of sums of sets, and meet the first $\varepsilon/2$ argument of the book.

1.4.1 The epsilon characterization

Look back at how (S2) was verified for $(0, 1)$ in Example (The supremum need not belong to the set). We supposed that some $M < 1$ was an upper bound and then produced an element of the set above M . Writing $M = 1 - \varepsilon$, the argument really showed: for every $\varepsilon > 0$, some element of the set exceeds $1 - \varepsilon$. That is a statement about elements of the set, not about upper bounds, and it is exactly equivalent to (S2).

A supremum is an upper bound that the set comes arbitrarily close to.

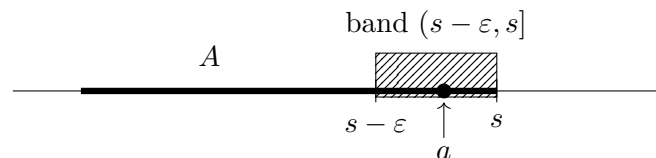
Theorem 1.4.1. Epsilon Characterization of the Supremum

Let $A \subseteq \mathbb{R}$ and let $s \in \mathbb{R}$ be an upper bound of A . Then $s = \sup A$ if and only if **for every** $\varepsilon > 0$ **there exists** $a \in A$ with $a > s - \varepsilon$.

In symbols, for an upper bound s of A :

$$s = \sup A \iff \forall \varepsilon > 0 \exists a \in A : s - \varepsilon < a (\leq s).$$

The part in parentheses is automatic, since s is an upper bound; it is written to show where a lies.



Idea. Draw the band $(s - \varepsilon, s]$ just below s . The theorem says: s is the supremum exactly when every such band, however thin, contains a point of A .

($\Rightarrow$) The number $s - \varepsilon$ is smaller than s , so it cannot be an upper bound, by (S2). Not being an upper bound means precisely that some element of A lies above it.

($\Leftarrow$) We need (S2): every upper bound M satisfies $s \leq M$. Suppose some upper bound M lies below s . Then the band between M and s contains no point of A , and the hypothesis says every band does. So the band of width $s - M$, that is, the hypothesis with $\varepsilon = s - M$, gives the contradiction.

Proof. ($\Rightarrow$) Suppose $s = \sup A$, and let $\varepsilon > 0$. Then $s - \varepsilon < s$ by Theorem 1.1.2 (a). If $s - \varepsilon$ were an upper bound of A , then (S2) would give $s \leq s - \varepsilon$, contradicting $s - \varepsilon < s$ by trichotomy. So $s - \varepsilon$ is not an upper bound of A , which means there is $a \in A$ with $a > s - \varepsilon$.

($\Leftarrow$) Suppose that for every $\varepsilon > 0$ there is $a \in A$ with $a > s - \varepsilon$. Since s is an upper bound of A by hypothesis, (S1) holds, and it remains to check (S2). Let M be an upper bound of A , and suppose, for a contradiction, that $M < s$. Put $\varepsilon := s - M$, which is positive by Theorem 1.1.2 (a). By hypothesis there is $a \in A$ with $a > s - \varepsilon = M$. This contradicts M being an upper bound of A . Hence $s \leq M$ by trichotomy, which is (S2). So $s = \sup A$. ■

The theorem does not use the axiom. It holds in any ordered field, and it **recognizes** a supremum; the axiom is what guarantees that there is one to recognize. The two halves are used differently. The forward direction is how a known supremum is **used**: it hands us elements of A as close to $\sup A$ as we like. The backward direction is how a candidate is **confirmed**: it replaces “every upper bound is at least s ” by “every band below s meets A ”, which can be checked by writing down an element.

Remark. The hypothesis “ s is an upper bound” cannot be dropped. For $A = (0, 1)$ and $s = \frac{1}{2}$, every band $(\frac{1}{2} - \varepsilon, \frac{1}{2}]$ meets A , yet $\frac{1}{2}$ is not the supremum. The band condition alone says only that s is not too big; being an upper bound says it is not too small.

The element a that the theorem produces **depends on** ε , as the order “ $\forall \varepsilon \exists a$ ” allows. For $A = (0, 1)$ and $s = 1$, the choice $a := \max\{\frac{1}{2}, 1 - \frac{\varepsilon}{2}\}$ works: it lies in $(0, 1)$ and exceeds $1 - \varepsilon$. For $\varepsilon = \frac{1}{10}$ it gives $a = \frac{19}{20}$; for $\varepsilon = \frac{1}{1000}$ it must be above $\frac{999}{1000}$, and $\frac{19}{20}$ no longer works. No single element of $(0, 1)$ works for every ε .

Warning. **The witness depends on ε .** The theorem gives, **for each ε , some a** ; it never gives one a that serves every ε at once. A proof that fixes a first and then says “for every $\varepsilon > 0$, $a > s - \varepsilon$ ” has swapped the quantifiers, and for $(0, 1)$ it asserts something false. When you take a witness from the theorem, write “choose $a \in A$ (depending on ε) with ...”.

Quick check. Let s be an upper bound of a set $A \subseteq \mathbb{R}$. Compare

$$(i) \quad \forall \varepsilon > 0 \quad \exists a \in A : a > s - \varepsilon \quad \text{and} \quad (ii) \quad \exists a \in A \quad \forall \varepsilon > 0 : a > s - \varepsilon.$$

Statement (i) says $s = \sup A$. What does (ii) say, and when is it true?

Solution. Fix $a \in A$. The condition “ $a > s - \varepsilon$ for every $\varepsilon > 0$ ” forces $a \geq s$: if $a < s$, then $\varepsilon := s - a > 0$ would give $a > s - (s - a) = a$, which is absurd. Since s is an upper bound, also $a \leq s$, so $a = s$. Conversely, if $s \in A$, then $a = s$ satisfies $s > s - \varepsilon$ for every $\varepsilon > 0$.

So (ii) says exactly that $s \in A$, that is, s is the **maximum** of A . Swapping the quantifiers turned “supremum” into “maximum”. For $A = (0, 1)$ and $s = 1$, (i) is true and (ii) is false.

Here is the theorem at work on a set where the witness must be found by a short computation. The Idea finds it backwards; the solution writes it forwards.

Example. Confirming a supremum with epsilon

Prove that $\sup\{x^2 : x \in (0, 1)\} = 1$.

Solution. Idea. The upper bound part is easy. For the band condition, fix $\varepsilon > 0$; we need $x \in (0, 1)$ with $x^2 > 1 - \varepsilon$. Try $x = 1 - h$ with $h > 0$ small. Then $x^2 = 1 - 2h + h^2 > 1 - 2h$, so it suffices that $1 - 2h \geq 1 - \varepsilon$, that is, $h \leq \frac{\varepsilon}{2}$. But we also need $x \in (0, 1)$, that is, $0 < h < 1$, and $\frac{\varepsilon}{2}$ can be large. Capping it fixes this: take $h = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$.

Proof. Let $A = \{x^2 : x \in (0, 1)\}$. For $x \in (0, 1)$, Theorem 1.1.2 (g) gives $x^2 < 1^2 = 1$, so 1 is an upper bound of A .

Let $\varepsilon > 0$, and let $h := \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$, which depends only on ε . Then $0 < h \leq \frac{1}{2}$, so $x := 1 - h$ lies in $[\frac{1}{2}, 1) \subseteq (0, 1)$, and $x^2 \in A$. Since $h > 0$, $h^2 > 0$, and so

$$x^2 = 1 - 2h + h^2 > 1 - 2h \geq 1 - \varepsilon,$$

where the last step uses $h \leq \frac{\varepsilon}{2}$. By Theorem 1.4.1, $\sup A = 1$.

The $\frac{1}{2}$ in the minimum is arbitrary: any cap below 1 keeps x positive. What matters is that a constraint discovered late in the Idea (here $h < 1$) is built into the constant from the start of the proof.

1.4.2 Sup approximation

The forward direction of Theorem 1.4.1 is used so often with $\varepsilon = \frac{1}{n}$ that it deserves its own statement.

Corollary 1.4.2. Sup Approximation

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. For every $n \in \mathbb{Z}_{>0}$ there is $a_n \in A$ with

$$\sup A - \frac{1}{n} < a_n \leq \sup A.$$

Proof. By Axiom 1.3.1, $\sup A$ exists. Let $n \in \mathbb{Z}_{>0}$. Then $n > 0$, and $\frac{1}{n} > 0$ by Theorem 1.1.2 (f). By Theorem 1.4.1 ($\Rightarrow$) with $\varepsilon = \frac{1}{n}$, there is $a_n \in A$ with $a_n > \sup A - \frac{1}{n}$, and $a_n \leq \sup A$ by (S1). ■

The corollary gives, for each n separately, an element a_n . Choosing one for every n at once, so as to form a sequence (a_n) , is a use of the axiom of countable choice; we flag it and move on. Once limits are defined in Chapter 2, this says that $\sup A$ is the limit of a sequence from A . We call the move **sup approximation**: “ $\sup A - \frac{1}{n}$ is not an upper bound, so some $a_n \in A$ exceeds it.”

1.4.3 Monotonicity and scaling

We now compute suprema of sets built from other sets. The first rule says that enlarging a set cannot lower its supremum.

Theorem 1.4.3. Monotonicity of the Supremum

Let $A \subseteq B \subseteq \mathbb{R}$, where A is nonempty and B is bounded above. Then $\sup A$ and $\sup B$ exist, and $\sup A \leq \sup B$.

Proof. Since A is nonempty and $A \subseteq B$, the set B is nonempty; as it is bounded above, $\sup B$ exists by Axiom 1.3.1. Every $a \in A$ lies in B , so $a \leq \sup B$ by (S1) for B . Hence $\sup B$ is an upper bound of A . So A is nonempty and bounded above, $\sup A$ exists by Axiom 1.3.1, and $\sup A \leq \sup B$ by (S2) for A . ■

Strict inclusion does not give strict inequality: $(0, 1) \subsetneq (0, 1]$, and both have supremum 1. The hypothesis that A is nonempty cannot be dropped, since $\emptyset \subseteq B$ always and $\emptyset$ has no supremum.

For scaling we need the scaled set. For $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, let

$$cA := \{ca : a \in A\}.$$

For example $2(0, 1] = (0, 2]$ and $(-1)A = -A$.

Theorem 1.4.4. Scaling the Supremum

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$.

- (a) If $c > 0$, then cA is nonempty and bounded above, and $\sup(cA) = c \sup A$.
- (b) If $c < 0$, then cA is nonempty and bounded below, and $\inf(cA) = c \sup A$.

Idea. For (a), prove the equality as two inequalities. Multiplying the inequality $a \leq \sup A$ by $c > 0$ shows that $c \sup A$ is an upper bound of cA , which gives $\leq$. For $\geq$, undo the scaling: dividing by c turns an upper bound of cA into an upper bound of A . No ε is needed, because multiplication by $c > 0$ is invertible and preserves order. For (b), a negative factor reverses order, so write $c = -|c|$ and let Theorem 1.3.2 handle the sign.

Proof. Let $s := \sup A$, which exists by Axiom 1.3.1. Since A is nonempty, so is cA .

- (a) Let $c > 0$. For every $a \in A$, $a \leq s$ by (S1), so $ca \leq cs$ by Theorem 1.1.2 (c) (or trivially if $a = s$). Hence cs is an upper bound of cA , so $\sup(cA)$ exists by Axiom 1.3.1, and $t := \sup(cA) \leq cs$ by (S2). For the reverse inequality, let $a \in A$. Then $ca \in cA$, so $ca \leq t$. Multiplying by $\frac{1}{c} > 0$ (Theorem 1.1.2 (f) and (c)) gives $a \leq \frac{t}{c}$. Hence $\frac{t}{c}$ is an upper bound of A , and (S2) gives $s \leq \frac{t}{c}$. Multiplying by $c > 0$ gives $cs \leq t$. By trichotomy, $\sup(cA) = cs$.
- (b) Let $c < 0$, so $-c > 0$. The set $-(cA) = \{-ca : a \in A\}$ equals $(-c)A$, which by (a) is nonempty and bounded above with $\sup((-c)A) = -cs$. By Theorem 1.3.2 (a),

$cA = -((-c)A)$ is bounded below, and by Theorem 1.3.2,

$$\inf(cA) = -\sup(-cA) = -\sup((-c)A) = -(-cs) = cs.$$

This proves the theorem. ■

For $c = 0$ the set cA is $\{0\}$, and $\sup(0A) = 0 = 0 \cdot \sup A$ as well. Part (a) genuinely needs $c > 0$: with $c = -1$ and $A = (0, 1]$, the set $cA = [-1, 0)$ has supremum 0, not $c\sup A = -1$. Part (b) says what happens instead: a negative factor exchanges supremum and infimum, and indeed $\inf[-1, 0) = -1$.

1.4.4 The supremum of a sum

For $A, B \subseteq \mathbb{R}$, let

$$A + B := \{a + b : a \in A, b \in B\}.$$

For instance $(0, 1) + \{5\} = (5, 6)$ and $[0, 1] + [0, 2] = [0, 3]$. It is natural to expect $\sup(A + B) = \sup A + \sup B$. This time the reverse inequality cannot be obtained by undoing an operation: a single element of $A + B$ mixes one element of each set. We need the ε characterization.

Theorem 1.4.5. Supremum of a Sum

Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded above. Then $A + B$ is nonempty and bounded above, and

$$\sup(A + B) = \sup A + \sup B.$$

Try it first (ten minutes): the inequality $\leq$ takes one line. Try to prove $\geq$.

Idea. The attempt that comes first to mind takes $a = \sup A$ and $b = \sup B$: then $a + b \in A + B$, and (S1) for $\sup(A + B)$ gives $\geq$ at once. It fails because a supremum need not belong to its set: for $A = B = (0, 1)$, the number 2 is not in $A + B = (0, 2)$. (The last exercise of this section takes that attempt apart.) The repair is to use elements that are only **close** to the suprema, which is what [Theorem 1.4.1](#) ($\Rightarrow$) supplies.

So we prove the equality as **two inequalities**.

($\leq$) Every $a + b$ is at most $\sup A + \sup B$, so the right side is an upper bound, and the supremum is at most any upper bound.

($\geq$) Work backwards. To show that $\sup(A + B)$ is at least $\sup A + \sup B$, it suffices to find, for each $\varepsilon > 0$, an element of $A + B$ above $\sup A + \sup B - \varepsilon$. The natural candidate is $a + b$ with a close to $\sup A$ and b close to $\sup B$. If a falls short of $\sup A$ by less than $\varepsilon/2$, and b falls short of $\sup B$ by less than $\varepsilon/2$, the total shortfall is less than ε . So we split the budget ε into two halves, one for each set, and ask [Theorem 1.4.1](#) for an element within $\varepsilon/2$ of each supremum. This is the $\varepsilon/2$ **argument**. The number $\varepsilon/2$ depends only on ε ; the elements a and b depend on ε . *Before reading on: the proof asks for elements within $\varepsilon/2$ of each supremum. What would it give if it asked for ε instead, and would that still be enough?*

Proof. By Axiom 1.3.1, $\sup A$ and $\sup B$ exist. Choosing $a_0 \in A$ and $b_0 \in B$ gives $a_0 + b_0 \in A + B$, so $A + B$ is nonempty.

($\leq$) Let $a \in A$ and $b \in B$. By (S1), $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$ by Theorem 1.1.2 (b). Hence $\sup A + \sup B$ is an upper bound of $A + B$. By Axiom 1.3.1, $\sup(A + B)$ exists, and by (S2), $\sup(A + B) \leq \sup A + \sup B$.

($\geq$) Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. By Theorem 1.4.1 ($\Rightarrow$), there are $a \in A$ and $b \in B$ (both depending on ε) with

$$a > \sup A - \frac{\varepsilon}{2} \quad \text{and} \quad b > \sup B - \frac{\varepsilon}{2}.$$

Adding, by Theorem 1.1.2 (b), gives $a + b > \sup A + \sup B - \varepsilon$. Since $a + b \in A + B$, (S1) gives

$$\sup(A + B) \geq a + b > \sup A + \sup B - \varepsilon.$$

This holds for every $\varepsilon > 0$. If we had $\sup(A + B) < \sup A + \sup B$, then the choice $\varepsilon := \sup A + \sup B - \sup(A + B) > 0$ would give $\sup(A + B) > \sup(A + B)$, which is absurd. Hence $\sup(A + B) \geq \sup A + \sup B$.

The two inequalities and trichotomy give $\sup(A + B) = \sup A + \sup B$. ■

The answer to the prompt: with ε in place of $\varepsilon/2$ the proof gives $\sup(A + B) > \sup A + \sup B - 2\varepsilon$ for every $\varepsilon > 0$, and that is still enough, since 2ε runs through all positive numbers as ε does. Halving is a convenience: it makes the final inequality come out with exactly ε , which is the form Theorem 1.4.1 ($\Leftarrow$) asks for.

The halves of the proof use the two halves of the definition. The first inequality used only that each supremum is an upper bound (S1) and that $\sup(A + B)$ is least (S2). The second used that each of $\sup A$ and $\sup B$ is least, in its ε form, and that $\sup(A + B)$ is an upper bound. We call the pattern **equality by two inequalities**: prove $\leq$ by exhibiting an upper bound, and $\geq$ with an ε .

Every hypothesis is needed. If $B = \emptyset$, then $A + B = \emptyset$ has no supremum. If B is not bounded above, neither is $A + B$ (Exercise (C2: Exit check: sums of sets)). And the analogous rule for differences is **not** $\sup(A - B) = \sup A - \sup B$: the correct version subtracts an infimum (Exercise (B3: Supremum of a difference)).

Warning. Proving one inequality only. Here is a tempting “proof” of Theorem 1.4.5: “For $a \in A$ and $b \in B$, $a + b \leq \sup A + \sup B$. Therefore $\sup(A + B) = \sup A + \sup B$.” The first sentence proves only that $\sup A + \sup B$ is **an** upper bound of $A + B$, which gives $\leq$. It says nothing about being the **least** one: the same sentence with $\sup A + \sup B + 1$ is equally true, and would “prove” $\sup(A + B) = \sup A + \sup B + 1$. Whenever a supremum is claimed to equal something, both (S1) and (S2), or both inequalities, must appear.

1.4.5 Exercises

1.4.5.1 A. Check your understanding

Exercise. A1

- Use Theorem 1.4.1 to confirm that $\sup[0, 5) = 5$. Give the witness a as a formula in ε .
- Use Theorem 1.4.1 to confirm that $\sup\{3\} = 3$. Does your witness depend on ε ? Why not?
- Use Theorem 1.4.1 to confirm that $\sup((-1, 1) \cup (2, 4)) = 4$.
- Let s be an upper bound of A . Write in symbols, using the ε characterization, the statement “ s is **not** the supremum of A ”.
- Run the $(\geq)$ half of the proof of Theorem 1.4.5 on $A = (0, 1)$, $B = \{5\}$, and $\varepsilon = \frac{1}{10}$: give an a and a b the proof could use, and check the inequality it derives.
- In the proof of Theorem 1.4.5, which line can no longer be justified if B is allowed to be empty?

Solution.

- 5 is an upper bound of $[0, 5)$. Given $\varepsilon > 0$, let $a := \max\{0, 5 - \frac{\varepsilon}{2}\}$. Then $a \geq 0$, and $a < 5$ because both 0 and $5 - \frac{\varepsilon}{2}$ are less than 5; so $a \in [0, 5)$. Also $a \geq 5 - \frac{\varepsilon}{2} > 5 - \varepsilon$. By Theorem 1.4.1, $\sup[0, 5) = 5$.
- 3 is an upper bound of $\{3\}$, and for every $\varepsilon > 0$ the element $a = 3$ satisfies $3 > 3 - \varepsilon$. The witness does not depend on ε because 3 belongs to the set: by the Quick check above, a single witness for all ε exists exactly when the supremum is a maximum.
- Every element is less than 4, so 4 is an upper bound. Given $\varepsilon > 0$, let $a := \max\{3, 4 - \frac{\varepsilon}{2}\}$. Then $3 \leq a < 4$, so $a \in (2, 4)$, and $a \geq 4 - \frac{\varepsilon}{2} > 4 - \varepsilon$. By Theorem 1.4.1 the supremum is 4.
- $\exists \varepsilon > 0 \quad \forall a \in A : a \leq s - \varepsilon$. In words: some band below s contains no point of A , so $s - \varepsilon$ is a smaller upper bound.
- Here $\sup A = 1$, $\sup B = 5$, and $\frac{\varepsilon}{2} = \frac{1}{20}$. The proof needs $a \in (0, 1)$ with $a > 1 - \frac{1}{20} = \frac{19}{20}$, for instance $a = \frac{24}{25}$, and $b \in B$ with $b > 5 - \frac{1}{20}$, which forces $b = 5$. Then $a + b = \frac{149}{25} = 5.96 > 5.9 = \sup A + \sup B - \varepsilon$. Any $a \in (\frac{19}{20}, 1)$ works; the one b works for every ε because 5 is the maximum of B .
- The first line, “By Axiom 1.3.1, $\sup A$ and $\sup B$ exist”. The axiom needs B nonempty, and $\emptyset$ has no supremum. The line that chooses $b \in B$ in the $(\geq)$ half fails too, since there is nothing to choose.

1.4.5.2 B. Practice
Exercise. B1: A computed supremum

Prove that $\sup\{x^2 : x \in (-1, 2)\} = 4$. Hence show that the set has no maximum.

Solution. Let $A = \{x^2 : x \in (-1, 2)\}$. *Upper bound.* Let $x \in (-1, 2)$. If $x \geq 0$, then $x < 2$ gives $x^2 < 4$ by Theorem 1.1.2 (g). If $-1 < x < 0$, then $|x| < 1$ by Proposition 1.1.3 (d), so $x^2 = |x|^2 < 1 < 4$ by Theorem 1.1.2 (g). So 4 is an upper bound, and $4 \notin A$.

Band condition. Working backwards as in Example (Confirming a supremum with epsilon): with $x = 2 - h$ we have $x^2 = 4 - 4h + h^2 > 4 - 4h$, so $h \leq \frac{\varepsilon}{4}$ suffices, provided $0 < h < 3$ to

keep $x \in (-1, 2)$. So let $\varepsilon > 0$ and $h := \min\{1, \frac{\varepsilon}{4}\}$, which depends only on ε . Then $0 < h \leq 1$, so $x := 2 - h \in [1, 2) \subseteq (-1, 2)$, and since $h^2 > 0$,

$$x^2 = 4 - 4h + h^2 > 4 - 4h \geq 4 - \varepsilon.$$

By [Theorem 1.4.1](#), $\sup A = 4$. Since $4 \notin A$, the set has no maximum, by Exercise (B1: Maxima are suprema) (a).

Exercise. B2: Supremum of a union

Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded above. Prove that $A \cup B$ is bounded above and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Solution. Let $m := \max\{\sup A, \sup B\}$; both suprema exist by Axiom 1.3.1. If $x \in A \cup B$, then $x \leq \sup A \leq m$ or $x \leq \sup B \leq m$, so m is an upper bound of $A \cup B$, which is nonempty. By symmetry we may assume $m = \sup A$. Given $\varepsilon > 0$, [Theorem 1.4.1](#) ($\Rightarrow$) gives $a \in A$ with $a > \sup A - \varepsilon = m - \varepsilon$, and $a \in A \cup B$. By [Theorem 1.4.1](#) ($\Leftarrow$), $\sup(A \cup B) = m$.

Exercise. B3: Supremum of a difference

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and $B \subseteq \mathbb{R}$ nonempty and bounded below. Prove that

$$\sup\{a - b : a \in A, b \in B\} = \sup A - \inf B.$$

Give an example showing that the right side cannot be replaced by $\sup A - \sup B$.

Solution. The set in question is $A + (-B)$. By Theorem 1.3.2 (a), $-B$ is bounded above, and it is nonempty. By Theorem 1.3.3, $\inf B = -\sup(-B)$, that is, $\sup(-B) = -\inf B$. So [Theorem 1.4.5](#) applies to A and $-B$:

$$\sup(A + (-B)) = \sup A + \sup(-B) = \sup A - \inf B.$$

For the example, take $A = \{0\}$ and $B = [0, 1]$. The differences $0 - b$ fill $[-1, 0]$, whose supremum is $0 = \sup A - \inf B$, whereas $\sup A - \sup B = -1$.

1.4.5.3 C. Going deeper

Exercise. C1: Products of sets

For $A, B \subseteq \mathbb{R}$ let $AB := \{ab : a \in A, b \in B\}$.

- Show that $\sup(AB) = \sup A \cdot \sup B$ can fail for nonempty bounded sets A, B .
- Prove that it holds when A and B are nonempty, bounded above, and contained in $(0, \infty)$.

Solution (Hint 1). For (a), try sets of negative numbers. For (b), the inequality $\leq$ is routine; for $\geq$, fix one element $b \in B$ and compare the set bA with AB .

Solution (Hint 2). For $\geq$ in (b): fix $b \in B$. Which theorem computes $\sup(bA)$, and which compares the suprema of two sets, one inside the other? What does the inequality you get say about b , once b varies?

Solution.

- (a) Take $A = B = [-2, -1]$. Then $\sup A \cdot \sup B = (-1)(-1) = 1$, but $4 = (-2)(-2) \in AB$, so $\sup(AB) \geq 4 \neq 1$ by (S1).
- (b) Let $s := \sup A$ and $t := \sup B$, which exist by Axiom 1.3.1. Since A contains a positive element and s is an upper bound, $s > 0$; likewise $t > 0$.

($\leq$) For $a \in A$ and $b \in B$, we have $0 < a \leq s$ and $0 < b \leq t$, so $ab \leq sb \leq st$ by Theorem 1.1.2 (c) (with trivial equality cases). So st is an upper bound of the nonempty set AB , $\sup(AB)$ exists, and $\sup(AB) \leq st$ by (S2).

($\geq$) Fix $b \in B$. Then $bA \subseteq AB$ and $b > 0$, so Theorem 1.4.4 and Theorem 1.4.3 give $bs = \sup(bA) \leq \sup(AB)$. Multiplying by $\frac{1}{s} > 0$ gives $b \leq \frac{\sup(AB)}{s}$. As $b \in B$ was arbitrary, $\frac{\sup(AB)}{s}$ is an upper bound of B , so $t \leq \frac{\sup(AB)}{s}$ by (S2), that is, $st \leq \sup(AB)$.

By trichotomy, $\sup(AB) = st$. Positivity was used in ($\leq$), to multiply inequalities, and in ($\geq$), to scale by b and divide by s ; part (a) shows it cannot be dropped.

Exercise. C2: Exit check: sums of sets

Let $A, B \subseteq \mathbb{R}$ be nonempty.

- (a) Assume A and B are bounded above. Without looking back, prove $\sup(A + B) = \sup A + \sup B$. This time, instead of proving $\geq$ separately, show that $\sup A + \sup B$ satisfies both hypotheses of Theorem 1.4.1 for the set $A + B$.
- (b) Assume A and B are bounded below. State and prove the corresponding formula for $\inf(A + B)$.
- (c) Show that if B is not bounded above, then neither is $A + B$.

Solution.

- (a) Let $s := \sup A + \sup B$, where both suprema exist by Axiom 1.3.1. *Upper bound:* for $a \in A$, $b \in B$, (S1) and Theorem 1.1.2 (b) give $a + b \leq s$. *Band condition:* let $\varepsilon > 0$; then $\frac{\varepsilon}{2} > 0$, and Theorem 1.4.1 ($\Rightarrow$) gives $a \in A$ and $b \in B$, depending on ε , with $a > \sup A - \frac{\varepsilon}{2}$ and $b > \sup B - \frac{\varepsilon}{2}$. Adding, $a + b > s - \varepsilon$, and $a + b \in A + B$. By Theorem 1.4.1 ($\Leftarrow$), $s = \sup(A + B)$.
- (b) $\inf(A + B) = \inf A + \inf B$. Indeed $-(A + B) = (-A) + (-B)$, since $-(a + b) = (-a) + (-b)$. By Theorem 1.3.2 (a), $-A$ and $-B$ are nonempty and bounded above,

so (a) and Theorem 1.3.3 give

$$\begin{aligned}\inf(A + B) &= -\sup((-A) + (-B)) \\ &= -\sup(-A) - \sup(-B) = \inf A + \inf B.\end{aligned}$$

Here $A + B$ is bounded below because $-(A + B)$ is bounded above, by (a) and Theorem 1.3.2 (a).

- (c) Fix $a_0 \in A$ and let $M \in \mathbb{R}$. Since B is not bounded above, $M - a_0$ is not an upper bound of B , so some $b \in B$ has $b > M - a_0$. Then $a_0 + b \in A + B$ and $a_0 + b > M$. As M was arbitrary, $A + B$ is not bounded above.

Marking scheme. (10 points) - 1: (a) says why $\sup A$, $\sup B$ exist and why $A + B$ is nonempty. - 2: (a) shows $\sup A + \sup B$ is an upper bound of $A + B$, adding the two (S1) inequalities. - 3: (a) fixes an arbitrary $\varepsilon > 0$, obtains $a \in A$ and $b \in B$, **depending on** ε , within $\varepsilon/2$ of their suprema, and concludes with [Theorem 1.4.1](#) ($\Leftarrow$). - 2: (b) reduces to (a) through $-(A + B) = (-A) + (-B)$ and Theorem 1.3.3, including why $A + B$ is bounded below. - 2: (c) produces, for an arbitrary M , an element of $A + B$ above M . Fatal errors (0 for the step): taking $a = \sup A \in A$; choosing one a before ε is fixed; proving only $\leq$ in (a). A solution of (a) that proves $\leq$ and $\geq$ separately, as in [Theorem 1.4.5](#), earns the 5 points of the second and third lines if the $\geq$ half is complete. A solution of (b) that repeats the argument of (a) with lower bounds earns the (b) points.

Exercise. C3: Find the wrong line

Each “proof” below contains a line that is false in general. Find it, give a set for which it fails, and decide whether the claim itself is true.

- (a) *Claim.* If $A \subseteq \mathbb{R}$ is nonempty and bounded above and $a < 1$ for every $a \in A$, then $\sup A < 1$. “*Proof.*” (1) Let $s = \sup A$. (2) Since $s \in A$, the hypothesis gives $s < 1$. (3) Hence $\sup A < 1$.
- (b) *Claim.* If $A, B \subseteq \mathbb{R}$ are nonempty and bounded above, then $\sup(A+B) \geq \sup A + \sup B$. “*Proof.*” (1) Let $a = \sup A$ and $b = \sup B$. (2) Then $a \in A$ and $b \in B$, so $a + b \in A + B$. (3) By (S1), $a + b \leq \sup(A + B)$, which is the claim.

Solution.

- (a) Line (2) is wrong: it assumes $\sup A \in A$, which fails for $A = (0, 1)$, where $\sup A = 1 \notin A$. For that set the claim is **false**: every element is less than 1, and $\sup A = 1 \not< 1$. The correct conclusion is $\sup A \leq 1$, which follows from (S2) because 1 is an upper bound.
- (b) Line (2) is wrong for the same reason: for $A = B = (0, 1)$, neither supremum belongs to its set, so $a + b = 2$ need not lie in $A + B = (0, 2)$, and here it does not. The claim is **true**, since it is half of [Theorem 1.4.5](#), but this argument does not prove it; a correct argument for this inequality is the $\varepsilon/2$ argument in the proof of [Theorem 1.4.5](#). The flawed line is valid exactly when A and B have maxima, which is the situation the author of the “proof” silently imagined.