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Preface

This volume is analysis on the real line: sequences and series, limits and continuity, the derivative, the integral, and power series. It starts from a short list of axioms for the real numbers, the axioms of a complete ordered field, and proves everything else from them.

Who it is for. The reader we have in mind can already read and write proofs: negate a quantified statement, name a witness, run an induction, chase an element through a preimage. *Logic and Proof* and the first three chapters of *Sets, Relations, and Cardinality* teach exactly these skills, and this volume cites them where it uses them. A reader who comes from elsewhere can test their readiness with Gate 0, the examination at the end of *Sets, Relations, and Cardinality*, Chapter 3. *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* builds $\mathbb{R}$ from the rationals and proves that the axioms hold for it; this volume does not need that construction, since it takes the axioms as its starting point.

What it covers.

Chapter	Title
1	Ordered Fields and the Least Upper Bound Axiom
2	Sequences, Subsequences, and Cauchy Sequences
3	Infinite Series
4	Open, Closed, and Compact Subsets of $\mathbb{R}$
5	Limits of Functions and Continuity
6	Continuous Functions on Compact Sets and Intervals
7	The Derivative and the Mean Value Theorem
8	The Riemann–Darboux Integral
9	Sequences and Series of Functions
10	Power Series, Taylor Series, and the Elementary Functions
11	The Forms of Completeness

Chapter 1 states the axioms and draws the first consequences of the one that separates $\mathbb{R}$ from $\mathbb{Q}$, the least upper bound axiom. Chapters 2–10 build the analysis of one real variable on it. Chapter 11 returns to the axiom and shows which other “no gaps” properties could replace it. Only Chapter 1 is written so far; the others will follow in the order of the table.

How to read it. Read the chapters in order: each uses the ones before it. Every section has Quick checks with folded answers; answer each one before you unfold it. The exercises of a section come in three groups, A to check understanding, B for practice and C to go deeper, and the last section of each chapter adds R items that reach back to earlier chapters. A chapter’s opening page lists what it needs, with links, and the time it takes. Results marked optional can be skipped, and nothing outside them depends on them.

What else you need. Besides the two volumes named above, only school algebra is assumed.

A few facts about fields are quoted from the companion *Linear Algebra* book, each with a link.

Chapter 1

Ordered Fields and the Least Upper Bound Axiom

Calculus is full of theorems that feel too obvious to need proof. A continuous function that is negative at one end of an interval and positive at the other must vanish somewhere in between. An increasing sequence that stays below some fixed number must settle down to a limit. Both statements are **false** if “number” means “rational number”. The polynomial $x^2 - 2$ is negative at 1 and positive at 2, yet no rational x makes it zero, because $\sqrt{2}$ is irrational (Logic and Proof, Theorem 2.2.1). The rational numbers 1, 1.4, 1.41, 1.414, 1.4142, $\dots$, each the largest decimal of its length whose square is below 2, increase and never pass 1.5; yet there is no rational number for them to approach, since its square would have to be 2. The rational line has a gap where $\sqrt{2}$ should be, and calculus falls into it.

This chapter finds the one property that separates the real numbers from the rationals: in $\mathbb{R}$, every nonempty set that is bounded above has a **least upper bound**. We take this property as an axiom and learn to use it. The main tool is a test for recognizing a least upper bound using small positive numbers ε , which will be used throughout the book. From the axiom we derive the first facts that the rationals lack or that need the axiom: the natural numbers are unbounded, the rationals are dense, positive numbers have square roots and n -th roots, nested closed intervals always share a point, and the real numbers cannot be listed. We do not construct $\mathbb{R}$ here. *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , Chapter 5 builds it from $\mathbb{Q}$ by Cauchy sequences and proves the axiom as a theorem, as the optional *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , Chapter 7 does by Dedekind cuts; Chapter 11 of this volume shows that several other “no gaps” properties are equivalent to it.

What you need. From *Logic and Proof*, Chapter 1: proof by cases (Logic and Proof, Definition 1.6.1). From *Logic and Proof*, Chapter 2: the irrationality of $\sqrt{2}$ (Logic and Proof, Theorem 2.2.1), the rules for negation (Logic and Proof, Definition 2.2.1), and the contrapositive (Logic and Proof, Proposition 2.3.2). From *Logic and Proof*, Chapters 3–5, and *Sets, Relations, and Cardinality*, Chapters 1–3: induction (Logic and Proof, Axiom 4.2.1), quantifiers and how to negate them (*Logic and Proof*, Chapter 4), sets and images of sets (*Sets, Relations, and Cardinality*, Chapter 1), the well-ordering of $\mathbb{N}$, definition by recursion, and the arithmetic of $\mathbb{N}$ and $\mathbb{Z}$ (*Sets, Relations, and Cardinality*, Chapter 2), and countable sets (*Sets, Relations, and Cardinality*, Chapter 3). No linear algebra is needed. The elementary rules of field arithmetic are either cited from the linear-algebra companion, Chapter 0, whose field axioms are ours, or proved in the first section.

Time. About 30 hours on the core track (all of A, one B, and one C per section, the exit

checks); about 45 hours for everything.

1.0.1 Roadmap

- **Ordered fields and the gap in the rationals.** The axioms that $\mathbb{Q}$ and $\mathbb{R}$ share, the rules for inequalities they imply, and absolute value with the triangle inequality.
- **Upper bounds, maxima and suprema.** The vocabulary for sets of numbers, and a proof that a bounded set of rationals can have no rational supremum.
- **The least upper bound axiom.** The single assumption that distinguishes $\mathbb{R}$ from $\mathbb{Q}$, and why infima then come free.
- **Working with the supremum.** The ε characterization of the supremum, sup approximation, and the rules for suprema of scaled sets and sums.
- **The Archimedean property.** Where $\mathbb{N}$ sits in $\mathbb{R}$, and the Archimedean property: $\mathbb{N}$ is not bounded above, which fails in some ordered fields.
- **Density of $\mathbb{Q}$ and the existence of roots.** A rational between any two reals, and $\sqrt{2}$ (and every n -th root) constructed as a supremum.
- **Nested intervals, and why $\mathbb{R}$ is uncountable.** Shrinking closed intervals share a point, and no list contains every real number.

1.0.2 Named moves

- **Split the supremum into its two properties.** To prove $s = \sup A$, prove that s is an upper bound, and then that it is the least one.
- **Sup approximation.** “ $\sup A - \varepsilon$ is not an upper bound, so some $a \in A$ exceeds it.” With $\varepsilon = \frac{1}{n}$ this gives $a_n \in A$ with $a_n > \sup A - \frac{1}{n}$.
- **Equality of suprema by two inequalities.** Prove $\leq$ by exhibiting an upper bound, then prove $\geq$ with an ε , splitting the budget as $\varepsilon/2 + \varepsilon/2$ when two sets are involved.
- **Construct a point as a supremum.** Define the set of points where “things are still fine”, take its supremum, and rule out each way it could fail. The existence of $\sqrt{2}$ is the first instance; the intermediate value theorem and the Heine–Borel theorem are proved the same way.

1.1 Ordered Fields and the Gap in the Rationals

Calculus takes for granted that a continuous curve which starts below the axis and ends above it must cross the axis somewhere. Over the rational numbers that is false, and the failure is already visible in the simplest polynomial one can think of. This section shows the failure, then records exactly what the rationals and the reals have in common: the arithmetic of a field and an order compatible with it. Everything in this section holds in both. The one property that separates them is the subject of the rest of the chapter.

1.1.1 A sign change with no zero

Consider $f(x) = x^2 - 2$ on the rational numbers between 1 and 2. At the left end $f(1) = -1 < 0$, and at the right end $f(2) = 2 > 0$. If the rational line had no holes, the graph would have to cross zero somewhere in between. It never does: a rational x with $f(x) = 0$ would be a rational number whose square is 2, and Logic and Proof, Theorem 2.2.1 says there is none.

So the rational numbers have a gap, exactly where $\sqrt{2}$ ought to be. The real numbers are meant to be “the rationals with the gaps filled”, and the first task is to say what that means precisely. Before we can name what $\mathbb{R}$ has and $\mathbb{Q}$ lacks, we must list what the two share. That list is short: four operations with the usual rules, and a way of comparing two numbers that respects those operations.

1.1.2 Ordered fields

A field is a number system in which one can add, subtract, multiply, and divide by anything nonzero, with the usual rules. The rational, real, and complex numbers are fields; so is the arithmetic of remainders modulo a prime. What $\mathbb{Q}$ and $\mathbb{R}$ have beyond this, and $\mathbb{C}$ does not, is an order: any two numbers can be compared, and the comparison behaves well under addition and multiplication. Every inequality we manipulate in analysis rests on four rules about that comparison, and the definition below states them.

An ordered field is a number system where we can do arithmetic and compare, and where comparison respects arithmetic.

Definition 1.1.1. Ordered Field

An **ordered field** consists of a set F , two operations, addition $(a, b) \mapsto a + b$ and multiplication $(a, b) \mapsto ab$, and a relation $<$ on F , such that the following hold.

The **field axioms**: for all $a, b, c \in F$,

- (F1) $a + b = b + a$;
- (F2) $(a + b) + c = a + (b + c)$;
- (F3) there is an element $0 \in F$ with $a + 0 = a$ **for every** $a \in F$;
- (F4) **for every** $a \in F$ there is an element $-a \in F$ with $a + (-a) = 0$;
- (F5) $(ab)c = a(bc)$;
- (F6) $ab = ba$;
- (F7) there is an element $1 \in F$ with $1 \neq 0$ and $a \cdot 1 = a$ **for every** $a \in F$;
- (F8) **for every nonzero** $a \in F$ there is an element $a^{-1} \in F$ with $aa^{-1} = 1$;
- (F9) $a(b + c) = ab + ac$.

The **order axioms**: for all $a, b, c \in F$,

- (O1) (trichotomy) **exactly one** of $a < b$, $a = b$, $b < a$ holds;
- (O2) (transitivity) if $a < b$ and $b < c$, then $a < c$;
- (O3) (addition) if $a < b$, then $a + c < b + c$;
- (O4) (multiplication) if $0 < a$ and $0 < b$, then $0 < ab$.

The field axioms are the ones of the linear-algebra companion, Chapter 0, in the same numbering. We write $a - b := a + (-b)$ and $a/b := ab^{-1}$ for $b \neq 0$. We write $b > a$ to mean $a < b$, and $a \leq b$ to mean “ $a < b$ or $a = b$ ”. An element a is **positive** if $a > 0$ and **negative** if $a < 0$. It is **nonnegative** if $a \geq 0$.

In words, the four order axioms say:

- **(O1)** any two elements can be compared, and the three outcomes exclude one another. In particular $a < a$ never holds.

- **(O2)** comparisons chain.
- **(O3)** adding the same thing to both sides of a strict inequality keeps it strict and keeps its direction.
- **(O4)** the product of two positive elements is positive. This is the only axiom that links the order to multiplication, and every rule about multiplying inequalities will come from it.

From (O1) and (O2), the relation $\leq$ is transitive as well, and a mixed chain $a \leq b < c$ gives $a < c$: if $a = b$ this is the hypothesis $b < c$, and otherwise it is (O2).

Field algebra. The field axioms alone imply the familiar algebraic identities. Their proofs use no order at all. Most of them are already proved in the linear-algebra companion, from the same axioms, and we cite those; the few that are not there are short, and we prove them here.

Lemma 1.1.1. Field Algebra

Let F be an ordered field (only (F1)–(F9) are used) and let $a, b \in F$.

- The elements 0 and 1 are unique, and so are $-a$ and, for $a \neq 0$, a^{-1} . In particular $-(-a) = a$.
- $a \cdot 0 = 0$.
- $(-a)b = -(ab)$ and $(-a)(-b) = ab$.
- If $ab = 0$, then $a = 0$ or $b = 0$.
- $(a + b)^2 = a^2 + 2ab + b^2$, where $x^2 := xx$ and $2 := 1 + 1$.

Idea. Each identity is forced by uniqueness. An element is pinned down by the equation it satisfies, so to show that $(-a)b$ is $-(ab)$, we show that it satisfies the equation defining $-(ab)$, namely that it adds to ab to give 0. Part (a) is the uniqueness, and the other parts use it.

Proof. (a), (b) and (d). Apart from $-(-a) = a$, these are parts of Linear Algebra, Theorem 0.7.1: the uniqueness of 0 and of $-a$; the rule $0 \cdot a = 0$, which gives $a \cdot 0 = 0$ by (F6); and “if $ab = 0$, then $a = 0$ or $b = 0$ ”. The field axioms there are (F1)–(F9) of Definition 1.1.1, in the same numbering, so the proofs there apply verbatim, and we do not repeat them. For $-(-a) = a$: $(-a) + a = 0$ by (F1) and (F4), so a is an additive inverse of $-a$, and $-(-a) = a$ by the uniqueness of additive inverses. The multiplicative halves of (a) are proved the same way. If $1'$ also satisfies $x \cdot 1' = x$ for every x , then $1 = 1 \cdot 1' = 1' \cdot 1 = 1'$, by that property of $1'$, commutativity (F6), and the property of 1. If $ab = 1$ for some b , then $b = b(aa^{-1}) = (ba)a^{-1} = 1 \cdot a^{-1} = a^{-1}$, by associativity and commutativity of multiplication; so a^{-1} is unique.

- By (F6), (F9), (F4) and (b), $ab + (-a)b = (a + (-a))b = 0 \cdot b = 0$, so $(-a)b$ is an additive inverse of ab , and $(-a)b = -(ab)$ by (a). Using this twice, with (F6), and then (a), $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$.
- By (F9) and (F6), $(a + b)(a + b) = a^2 + ba + ab + b^2$, and $ba + ab = ab \cdot 1 + ab \cdot 1 = ab(1 + 1) = 2ab$ by (F6), (F7) and (F9).

■

From now on these identities, and the rules of fraction arithmetic that follow from them in the same way, are used without comment.

Example. The rationals and the reals

Explain why $\mathbb{Q}$ is an ordered field, and what we are assuming when we say $\mathbb{R}$ is one.

Solution. For $\mathbb{Q}$, the field axioms are the rules of fraction arithmetic. The order is $p/q < p'/q'$ (with $q, q' > 0$) exactly when $pq' < p'q$ as integers. Trichotomy, transitivity, (O3), and (O4) then come from the corresponding facts about integers. Checking them is routine and belongs to the construction of $\mathbb{Q}$ from $\mathbb{Z}$; here we accept them.

For $\mathbb{R}$ we have not yet said what a real number *is*, and this chapter deliberately does not. Instead we **assume** that $\mathbb{R}$ is an ordered field containing $\mathbb{Q}$, with its usual operations and order; in particular $\mathbb{R}$ contains $\mathbb{N}$ and $\mathbb{Z}$, with their usual arithmetic. (Later in this chapter we locate $\mathbb{N}$ inside $\mathbb{R}$ using the axioms alone.) A later section of this chapter completes the list of assumptions by adding exactly one more. Everything proved in this section from (F1)–(F9) and (O1)–(O4) therefore holds in $\mathbb{Q}$ and in $\mathbb{R}$ alike.

Example. A minimal change: the integers

The integers $\mathbb{Z}$, with the usual operations and order, satisfy (O1)–(O4) and every field axiom except one. Which one fails, and does the failure matter for inequalities?

Solution. Axiom (F8) fails: $2 \neq 0$, but no integer m has $2m = 1$. Everything else holds, including all four order axioms. So $\mathbb{Z}$ is not an ordered field, although it is an “ordered ring”. The failure matters as soon as we divide: the rule “if $0 < a < b$ then $1/b < 1/a$ ” proved below has no meaning in $\mathbb{Z}$, because $1/a$ is not an integer unless $a = 1$.

The two examples mark the scope of the definition. $\mathbb{Q}$ and $\mathbb{R}$ are the main examples, and the rest of the section is about what the axioms force in both. The complex numbers are the instructive nonexample, and they appear as the first payoff of the next theorem.

Remark. Why state the order axioms with the strict relation $<$ rather than with $\leq$? Either could come first, since each can be defined from the other. We take $<$ because analysis constantly asks whether a strict inequality **survives** an operation. Adding survives (O3); passing to a limit does not: a number that exceeds $1 - \varepsilon$ for every $\varepsilon > 0$ is only guaranteed to be ≥ 1 , not > 1 . Keeping $<$ primary keeps that question in view.

1.1.3 The rules for inequalities

Logic and Proof. Chapters 1 and 2 used facts such as $x^2 \geq 0$ (in Logic and Proof, Example (A negative statement proved by not-introduction)) and the sign rules for products (in Logic and Proof, Example (The absolute value of a product)) as school algebra. We now prove them from (O1)–(O4) and the field axioms. So nothing earlier was circular: those facts were true, and here

is why.

Theorem 1.1.2. Rules for Inequalities

Let F be an ordered field and let $a, b, c, d \in F$.

- (a) $a < b$ exactly when $b - a > 0$, and exactly when $-b < -a$; in particular $a > 0$ if and only if $-a < 0$.
- (b) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$, and the conclusion is strict if at least one hypothesis is strict.
- (c) If $a < b$ and $c > 0$, then $ac < bc$.
- (d) If $a < b$ and $c < 0$, then $ac > bc$.
- (e) If $a \neq 0$, then $a^2 > 0$. Hence $a^2 \geq 0$ for every a , and $1 > 0$.
- (f) If $a > 0$, then $1/a > 0$. If $0 < a < b$, then $0 < 1/b < 1/a$.
- (g) If $a \geq 0$ and $b \geq 0$, then $a < b$ if and only if $a^2 < b^2$.
- (h) $2 := 1 + 1$ is positive, and if $a < b$, then $a < \frac{a+b}{2} < b$.

Idea. Every rule is proved by the same move: turn a comparison into a statement about a single positive element, using (a), and then apply (O3) or (O4) to that element. For instance “ $a < b$ and $c > 0$ ” becomes “ $b - a$ and c are positive”, and (O4) says their product $bc - ac$ is positive. The later parts then lean on the earlier ones, so the proof reads top to bottom.

Proof.

- (a) If $a < b$, then (O3) with $c = -a$ gives $a + (-a) < b + (-a)$, that is, $0 < b - a$. Conversely, if $0 < b - a$, then (O3) with $c = a$ gives $a < b$. For the second statement, $(-a) - (-b) = b - a$, so by what was just shown both $a < b$ and $-b < -a$ are equivalent to $b - a > 0$. Taking $a = 0$ and replacing b by a gives: $0 < a$ if and only if $-a < 0$.
- (b) From $a \leq b$ we get $a + c \leq b + c$: by (O3) if $a < b$, and trivially if $a = b$. In the same way $c \leq d$ gives $b + c \leq b + d$, using (F1) to add b on the left. Chaining the two by transitivity gives $a + c \leq b + d$, and if either link is strict the chain is strict.
- (c) By (a), $b - a > 0$. Since also $c > 0$, (O4) gives $(b - a)c > 0$, that is, $bc - ac > 0$ by (F9). By (a) again, $ac < bc$.
- (d) By (a), $-c > 0$. Applying (c) with the positive element $-c$ gives $a(-c) < b(-c)$, that is, $-(ac) < -(bc)$. By the second statement of (a), this says $bc < ac$.
- (e) Since $a \neq 0$, trichotomy (O1) leaves two cases. *Case 1:* $a > 0$. Then $a^2 = a \cdot a > 0$ by (O4). *Case 2:* $a < 0$. Then $-a > 0$ by (a), and $a^2 = (-a)(-a) > 0$ by (O4). So $a^2 > 0$ whenever $a \neq 0$, and $a^2 = 0$ when $a = 0$; either way $a^2 \geq 0$. Finally $1 = 1^2$ and $1 \neq 0$ by (F7), so $1 > 0$.
- (f) Let $a > 0$. Then $a^{-1} \neq 0$, since otherwise $1 = aa^{-1} = a \cdot 0 = 0$, contradicting (F7). Hence $(a^{-1})^2 > 0$ by (e), and $a^{-1} = a \cdot (a^{-1})^2$ is a product of two positive elements, so $a^{-1} > 0$ by (O4). Now let $0 < a < b$. Then $b > 0$ by (O2), so $a^{-1} > 0$ and $b^{-1} > 0$, and $a^{-1}b^{-1} > 0$ by (O4). Multiplying $a < b$ by this positive element, (c) gives $a \cdot a^{-1}b^{-1} < b \cdot a^{-1}b^{-1}$, which simplifies to $b^{-1} < a^{-1}$.
- (g) ($\Rightarrow$) Suppose $0 \leq a < b$. Then $b > 0$ by the mixed chain $0 \leq a < b$. If $a = 0$, then $a^2 = 0 < b^2$ by (e). If $a > 0$, then (c) applied to $a < b$, once with the positive factor a

and once with the positive factor b , gives $a^2 < ab$ and $ab < b^2$, so $a^2 < b^2$ by (O2). ($\Leftarrow$) Suppose $a^2 < b^2$. By (O1), either $a < b$, or $a = b$, or $b < a$. If $a = b$ then $a^2 = b^2$, and if $0 \leq b < a$ then $b^2 < a^2$ by the forward direction with the roles of a and b swapped. Either conclusion contradicts $a^2 < b^2$ by (O1). Hence $a < b$.

- (h) By (e) and (O3), $1 + 1 > 0 + 1 = 1 > 0$, so $2 > 0$ by (O2); in particular $2 \neq 0$ and $\frac{1}{2} > 0$ by (f). If $a < b$, then (O3) gives $a + a < a + b$ and $a + b < b + b$, that is, $2a < a + b < 2b$, where $a + a = (1 + 1)a = 2a$ by (F9). Multiplying by $\frac{1}{2} > 0$ using (c) gives $a < \frac{a+b}{2} < b$. This proves the theorem. ■

Three more rules of the same kind are [Exercise \(B1: Three more rules\)](#). With (a)–(h) in hand, every inequality manipulation in this chapter is justified by name. Part (h) looks modest, but it already tells us something structural: between any two elements of an ordered field there is a third, so an ordered field has no “next” element after any given one.

The first payoff is a nonexample. The complex numbers form a field, but no relation $<$ on $\mathbb{C}$ satisfies (O1)–(O4). Indeed, if one did, then $i \neq 0$ would give $i^2 > 0$ by (e), that is, $-1 > 0$. But (e) also gives $1 > 0$, and then (a) gives $-1 < 0$. Having both $-1 > 0$ and $-1 < 0$ contradicts trichotomy. So $\mathbb{C}$ is a field that cannot be ordered, and this is why the book compares real numbers and never complex ones.

Warning. Inequalities may be **added** (part (b)) but not freely **multiplied**. From $a \leq b$ and $c \leq d$ it does **not** follow that $ac \leq bd$: take $a = c = -3$ and $b = d = -1$. Then $a \leq b$ and $c \leq d$, but $ac = 9$ and $bd = 1$. Multiplying an inequality needs the sign of the factor, as in (c) and (d), and multiplying two inequalities needs nonnegativity, as in [Exercise \(B1: Three more rules\)](#).

Quick check. The set $\{0, 1\}$, with addition and multiplication modulo 2, is a field (so $1 + 1 = 0$). Can it be made into an ordered field by some choice of $<$?

Solution. No. In any ordered field $1 > 0$ by [Theorem 1.1.2](#) (e), and then (O3) gives $1 + 1 > 0 + 1 = 1$, so $1 + 1 > 0$ by (O2). Here $1 + 1 = 0$, so this says $0 > 0$, which trichotomy forbids. The argument never used which relation was chosen, so no choice works.

1.1.4 Absolute value and distance

In analysis, the question is almost never “is x equal to a ?” but “how far is x from a ?”. An answer should be a nonnegative number that ignores on which side of a the point x lies. The absolute value supplies it.

$|x|$ is the size of x with its sign removed, and $|x - a|$ is the distance from x to a .

Definition 1.1.2. Absolute Value

Let F be an ordered field. The **absolute value** of $x \in F$ is

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The definition is by cases, and trichotomy guarantees that exactly one case applies, so $|x|$ is well defined. For example $|3| = 3$, $|-3| = -(-3) = 3$, and $|0| = 0$; the degenerate point 0 falls in the first case, and it is the only element of absolute value 0. A minimal change shows that the case split is doing real work: the rule “ $|x| = x$ for all x ” agrees with the definition on every nonnegative x and fails at every negative one, where it would make $|-3| = -3 < 0$.

The next proposition collects the facts about absolute value that we use constantly. *Logic and Proof*, Chapter 1 proved some of them by cases (Logic and Proof, Example (The triangle inequality by cases) begins with $a \leq |a|$ and $-a \leq |a|$), taking the sign rules on trust; the sign rules are now [Theorem 1.1.2](#).

Proposition 1.1.3. Rules for Absolute Value

Let F be an ordered field and let $x, y, r \in F$.

- (a) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$.
- (b) $|-x| = |x|$, $|xy| = |x||y|$, and $|x|^2 = x^2$.
- (c) $-|x| \leq x \leq |x|$.
- (d) $|x| \leq r$ if and only if $-r \leq x \leq r$; and $|x| < r$ if and only if $-r < x < r$.

Idea. The definition of $|x|$ has two cases, so each part is proved by checking $x \geq 0$ and $x < 0$ separately and quoting the sign rules of [Theorem 1.1.2](#) in each case. Part (d) is the exception. Once (c) is available, (d) needs no new case split for its forward direction.

Proof.

- (a) If $x \geq 0$ then $|x| = x \geq 0$, with equality exactly when $x = 0$. If $x < 0$ then $|x| = -x > 0$ by [Theorem 1.1.2](#) (a).
- (b) If $x > 0$ then $-x < 0$, so $|-x| = -(-x) = x = |x|$; if $x < 0$ then $-x > 0$, so $|-x| = -x = |x|$; and $|-0| = |0|$. For the product, both $|xy|$ and $|x||y|$ are one of the two elements xy and $-xy$, since each absolute value is its argument up to sign. Both are nonnegative: $|xy| \geq 0$ by (a), and $|x||y| \geq 0$ by (O4) when both factors are positive and trivially when one is 0. If $xy = 0$, both are 0. If $xy \neq 0$, then exactly one of xy and $-xy$ is positive, by trichotomy and [Theorem 1.1.2](#) (a), so both must be that one. Either way $|xy| = |x||y|$. Finally $|x|^2 = (\pm x)^2 = x^2$.
- (c) If $x \geq 0$, then $x = |x|$, and $-|x| = -x \leq 0 \leq x$. If $x < 0$, then $-|x| = x$, and $x < 0 < -x = |x|$.
- (d) $(\Rightarrow)$ Suppose $|x| \leq r$. By (c), $x \leq |x| \leq r$. Also $-r \leq -|x|$ by [Theorem 1.1.2](#) (a), and $-|x| \leq x$ by (c), so $-r \leq x$. $(\Leftarrow)$ Suppose $-r \leq x \leq r$. If $x \geq 0$, then $|x| = x \leq r$. If $x < 0$, then $|x| = -x \leq r$, because $-r \leq x$ gives $-x \leq r$ by [Theorem 1.1.2](#) (a). The strict version is the same argument with $<$ in place of $\leq$ wherever r appears. ■

Part (d), applied to $x - a$ in place of x , is the single most used equivalence in this book. Adding a throughout by (O3), it reads

$$|x - a| < r \iff a - r < x < a + r. \quad (1.1.1)$$

In words: x is within distance r of a exactly when x lies in the open interval of radius r centered at a . Here, for $a \leq b$ in F , the **intervals** are $(a, b) := \{x \in F : a < x < b\}$, $[a, b] := \{x \in F : a \leq x \leq b\}$, and $(a, b]$, $[a, b)$ similarly; in the degenerate case $a = b$, the closed interval $[a, a] = \{a\}$ is a single point and the other three are empty; $(a, \infty) := \{x \in F : x > a\}$, and the other unbounded intervals are defined in the same way. The symbol ∞ here is part of the name of a set, not an element of F . From Chapter 2 on, every statement “ a_n is close to L ” is turned into two inequalities by Eq. 1.1.1, and back.

Example. Reading an absolute-value inequality as an interval

Find all real x with $|4 - 3x| < 2$.

Solution. By Proposition 1.1.3 (d), $|4 - 3x| < 2$ holds if and only if $-2 < 4 - 3x < 2$. Subtracting 4 throughout (O3) gives $-6 < -3x < -2$. Multiplying by the negative number $-\frac{1}{3}$ reverses both inequalities, by Theorem 1.1.2 (d): $2 > x > \frac{2}{3}$. Every step was an equivalence, so the solution set is the interval $(\frac{2}{3}, 2)$.

As a check against Eq. 1.1.1: $|4 - 3x| = |-3||x - \frac{4}{3}| = 3|x - \frac{4}{3}|$ by Proposition 1.1.3 (b), so the condition says $|x - \frac{4}{3}| < \frac{2}{3}$, the interval of radius $\frac{2}{3}$ about $\frac{4}{3}$, which is $(\frac{2}{3}, 2)$ again.

Distance should satisfy one more law: going from x to z directly is never longer than going via a third point y . For numbers, that law is the triangle inequality.

Theorem 1.1.4. Triangle Inequality

Let F be an ordered field. For all $x, y \in F$,

- (a) $|x + y| \leq |x| + |y|$;
- (b) $||x| - |y|| \leq |x - y|$ (the **reverse triangle inequality**).

Idea. Logic and Proof. Chapter 1 proved (a) in Logic and Proof, Example (The triangle inequality by cases) with two case splits: one to show $a \leq |a|$ and $-a \leq |a|$, and then one on the sign of $x + y$. Here the first split is already done, as Proposition 1.1.3 (c), and the second is not needed. By Proposition 1.1.3 (d), the goal $|x + y| \leq |x| + |y|$ is the pair of inequalities $-(|x| + |y|) \leq x + y \leq |x| + |y|$, and these come from adding the two sandwiches $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. For (b), write x as $(x - y) + y$ and apply (a).

Proof.

- (a) By Proposition 1.1.3 (c), $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these by

Theorem 1.1.2 (b) gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

By Proposition 1.1.3 (d) with $r = |x| + |y|$, this says $|x + y| \leq |x| + |y|$.

- (b) By (a) applied to $x - y$ and y , $|x| = |(x - y) + y| \leq |x - y| + |y|$, so $|x| - |y| \leq |x - y|$. Swapping the roles of x and y gives $|y| - |x| \leq |y - x| = |x - y|$, using Proposition 1.1.3 (b). The two inequalities together say $-|x - y| \leq |x| - |y| \leq |x - y|$, which is (b) by Proposition 1.1.3 (d). This proves the theorem. ■

The case-free proof is typical of what the axioms buy: once a fact such as $-|x| \leq x \leq |x|$ is proved, it can be added, multiplied, and chained without redoing sign cases. The triangle inequality is the engine of every estimate in this volume. Its standard use is “insert a third point”: to bound $|x - z|$, write $x - z = (x - y) + (y - z)$ for a well-chosen y , and bound the two pieces separately (Exercise (B2: Inserting a third point)).

Remark. The inequality in (a) can be strict: $|1 + (-1)| = 0 < 2 = |1| + |-1|$. Equality holds exactly when x and y do not have strictly opposite signs, that is, when $xy \geq 0$ (Exercise (B2: Inserting a third point)).

Comparing by squaring. Two absolute values can often be compared without any case split, by squaring. For nonnegative a and b , Theorem 1.1.2 (g) says that $a < b$ holds exactly when $a^2 < b^2$, and Proposition 1.1.3 (b) gives $|u|^2 = u^2$. For example, which x satisfy $|x - 2| < |x|$? Both sides are nonnegative, so the inequality holds if and only if $(x - 2)^2 < x^2$, that is, $x^2 - 4x + 4 < x^2$. By (O3) this is $4 < 4x$, and multiplying by $\frac{1}{4} > 0$ gives $x > 1$. Every step was an equivalence, so the solution set is $(1, \infty)$: the points closer to 2 than to 0.

1.1.5 Where this leaves us

Everything proved so far holds in $\mathbb{Q}$ and in $\mathbb{R}$ equally. So none of it can explain why $x^2 - 2$ has a zero in $[1, 2]$ over $\mathbb{R}$ and not over $\mathbb{Q}$. The difference between the two fields is not arithmetic and not the order rules; it is a property of **sets** of numbers. The next section introduces the language for that property: upper bounds, maxima, and the least upper bound.

1.1.6 Exercises

1.1.6.1 A. Check your understanding

Exercise. A1

- (a) Here is a derivation, in an ordered field, of “if $a > 0$ and $b > 0$ then $a + b > 0$ ”: from $b > 0$ we get $a + b > a + 0$; and $a + 0 = a$; and $a > 0$; hence $a + b > 0$. Name the axiom that justifies each of the four steps.
- (b) True or false, with a reason: in an ordered field, if $a < b$ then $a^2 < b^2$.
- (c) True or false, with a reason: $|x| = \max\{x, -x\}$ for every x in an ordered field. (Here $\max\{a, b\}$ is the larger of a and b : it is a if $a \geq b$, and b otherwise.)

(d) The integers satisfy all four order axioms. Why is $\mathbb{Z}$ not an ordered field?

Solution.

- (a) The first step is (O3), adding a to both sides of $0 < b$ (with (F1) to put a on the left). The second is (F3), the defining property of 0. The third is the hypothesis. The last is transitivity (O2): $0 < a$ and $a < a + b$ give $0 < a + b$.
- (b) False. Take $a = -2$ and $b = 1$: then $a < b$, but $a^2 = 4 > 1 = b^2$. [Theorem 1.1.2](#) (g) needs both numbers nonnegative.
- (c) True. If $x \geq 0$, then $-x \leq 0 \leq x$, so the larger of x and $-x$ is $x = |x|$. If $x < 0$, then $x < 0 < -x$, so the larger is $-x = |x|$.
- (d) Axiom (F8) fails: 2 has no multiplicative inverse in $\mathbb{Z}$, since $2m = 1$ has no integer solution.

1.1.6.2 B. Practice

Exercise. B1: Three more rules

Let F be an ordered field and $a, b, c, d \in F$. Prove the following from the axioms and [Theorem 1.1.2](#).

- (a) If $a > 0$ and $b < 0$, then $ab < 0$.
- (b) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
- (c) If $a > 0$ and $ab > 0$, then $b > 0$.

Solution.

- (a) Since $b < 0$, [Theorem 1.1.2](#) (a) gives $-b > 0$. By (O4), $a(-b) > 0$, that is, $-(ab) > 0$. By [Theorem 1.1.2](#) (a) again, $ab < 0$.
- (b) Since $0 \leq c < d$, we have $d > 0$. From $a < b$ and $d > 0$, [Theorem 1.1.2](#) (c) gives $ad < bd$. From $c < d$ and $a \geq 0$ we get $ac \leq ad$: by [Theorem 1.1.2](#) (c) if $a > 0$, and because both sides are 0 if $a = 0$. Chaining, $ac \leq ad < bd$, so $ac < bd$.
- (c) By trichotomy, $b > 0$, $b = 0$, or $b < 0$. If $b = 0$ then $ab = 0$, and if $b < 0$ then $ab < 0$ by (a). Both contradict $ab > 0$ by (O1). Hence $b > 0$.

Exercise. B2: Inserting a third point

Let F be an ordered field and $x, y, z \in F$.

- (a) Prove that $|x - z| \leq |x - y| + |y - z|$.
- (b) Prove that $|x + y| = |x| + |y|$ if and only if $xy \geq 0$.

Solution.

- (a) Since $x - z = (x - y) + (y - z)$, [Theorem 1.1.4](#) (a) gives $|x - z| \leq |x - y| + |y - z|$.
- (b) Both sides are nonnegative, so by [Theorem 1.1.2](#) (g) they are equal if and only if their

squares are equal (apply (g) in both directions and use trichotomy). Using [Proposition 1.1.3](#) (b),

$$|x + y|^2 = x^2 + 2xy + y^2, \quad (|x| + |y|)^2 = x^2 + 2|xy| + y^2.$$

These are equal if and only if $xy = |xy|$, which by the definition of absolute value happens if and only if $xy \geq 0$.

Exercise. B3: Solving inequalities

Find all real x satisfying each inequality. Hence write each solution set as an interval.

- (a) $|2x - 3| < 5$.
- (b) $|x - 1| < |x + 1|$.

Solution.

- (a) By [Proposition 1.1.3](#) (d), the inequality holds if and only if $-5 < 2x - 3 < 5$. Adding 3 gives $-2 < 2x < 8$, and multiplying by $\frac{1}{2} > 0$ gives $-1 < x < 4$. Each step is reversible, so the solution set is $(-1, 4)$.
- (b) Both sides are nonnegative, so by [Theorem 1.1.2](#) (g) and [Proposition 1.1.3](#) (b) the inequality holds if and only if $(x - 1)^2 < (x + 1)^2$. Expanding, this is $x^2 - 2x + 1 < x^2 + 2x + 1$, which by (O3) is equivalent to $0 < 4x$, and by multiplying with $\frac{1}{4} > 0$ to $x > 0$. The solution set is $(0, \infty)$: the points closer to 1 than to -1 .

1.1.6.3 C. Going deeper

Exercise. C1: Ordered fields contain the counting numbers

- (a) The field $\{0, 1, 2\}$ with arithmetic modulo 3 has $1 + 1 + 1 = 0$. Show that it admits no order satisfying (O1)–(O4).
- (b) In an ordered field F , define $n \cdot 1$ for $n \in \mathbb{N}$ by $0 \cdot 1 := 0$ and $(n + 1) \cdot 1 := n \cdot 1 + 1$. Prove that $n \cdot 1 > 0$ for every $n \in \mathbb{N}$ with $n \geq 1$, and that $m \cdot 1 < n \cdot 1$ whenever $m < n$.
- (c) Deduce that the elements $0, 1, 1 + 1, 1 + 1 + 1, \dots$ of an ordered field are all different, so every ordered field contains a copy of $\mathbb{N}$ (and in particular is infinite).

Hint: for (b), use induction on n (Logic and Proof, Axiom 4.2.1).

Solution.

- (a) Suppose an order existed. By [Theorem 1.1.2](#) (e), $1 > 0$. By (O3), $1 + 1 > 0 + 1 = 1$, so $1 + 1 > 0$ by (O2), and again $1 + 1 + 1 > 1 + 1 > 0$. But $1 + 1 + 1 = 0$, so this says $0 > 0$, contradicting trichotomy (O1).
- (b) We prove $n \cdot 1 > 0$ by induction on n , starting at $n = 1$. For $n = 1$, $1 \cdot 1 = 0 \cdot 1 + 1 = 0 + 1 = 1 > 0$ by [Theorem 1.1.2](#) (e). If $n \cdot 1 > 0$, then by (O3), $(n + 1) \cdot 1 = n \cdot 1 + 1 > n \cdot 1$, and (O2) gives $(n + 1) \cdot 1 > 0$. The same step shows $(k + 1) \cdot 1 > k \cdot 1$ for every k . Now fix m and prove $m \cdot 1 < n \cdot 1$ for every $n > m$ by induction on n , starting at $n = m + 1$: the base case is the step just shown with $k = m$, and if $m \cdot 1 < n \cdot 1$, then

$m \cdot 1 < n \cdot 1 < (n+1) \cdot 1$, so (O2) gives $m \cdot 1 < (n+1) \cdot 1$.

- (c) If $m \neq n$, say $m < n$, then $m \cdot 1 < n \cdot 1$ by (b), so $m \cdot 1 \neq n \cdot 1$ by trichotomy. Hence $n \mapsto n \cdot 1$ is injective from $\mathbb{N}$ into F , and by (b) it respects order. It also respects addition: $(m+n) \cdot 1 = m \cdot 1 + n \cdot 1$, by induction on n with m fixed. For $n = 0$ both sides are $m \cdot 1$, since $0 \cdot 1 = 0$. If it holds for n , then $(m+(n+1)) \cdot 1 = ((m+n)+1) \cdot 1 = (m+n) \cdot 1 + 1 = m \cdot 1 + n \cdot 1 + 1 = m \cdot 1 + (n+1) \cdot 1$, using the definition twice and (F2). This is the sense in which F contains a copy of $\mathbb{N}$. The section on the natural numbers inside $\mathbb{R}$, later in this chapter, starts from exactly this copy.

Exercise. C2: Max and min without cases

Let F be an ordered field and $a, b \in F$. Write $\max\{a, b\}$ for the larger of a and b and $\min\{a, b\}$ for the smaller (as in [Exercise \(A1\)](#)). Prove that

$$\max\{a, b\} = \frac{a+b+|a-b|}{2}, \quad \min\{a, b\} = \frac{a+b-|a-b|}{2}.$$

Hence show that $|\max\{a, b\} - \max\{c, d\}| \leq \max\{|a-c|, |b-d|\}$ for all $a, b, c, d \in F$.

Hint: for the last part, bound $\max\{a, b\}$ above by $\max\{c, d\} + \max\{|a-c|, |b-d|\}$.

Solution. If $a \geq b$, then $|a-b| = a-b$, so $\frac{a+b+|a-b|}{2} = \frac{2a}{2} = a = \max\{a, b\}$ and $\frac{a+b-|a-b|}{2} = b = \min\{a, b\}$. If $a < b$, then $|a-b| = b-a$, and the same computation gives b and a . By trichotomy these cases cover everything.

For the inequality, put $m := \max\{|a-c|, |b-d|\}$. By [Proposition 1.1.3 \(c\)](#), $a \leq c + |a-c| \leq \max\{c, d\} + m$, and likewise $b \leq d + |b-d| \leq \max\{c, d\} + m$. Hence $\max\{a, b\} \leq \max\{c, d\} + m$, that is, $\max\{a, b\} - \max\{c, d\} \leq m$. Swapping the roles of (a, b) and (c, d) , which leaves m unchanged, gives $\max\{c, d\} - \max\{a, b\} \leq m$. By [Proposition 1.1.3 \(d\)](#), the two together say $|\max\{a, b\} - \max\{c, d\}| \leq m$.

1.2 Upper Bounds, Maxima and Suprema

The previous section ended with a puzzle. The rationals and the reals obey the same rules of arithmetic and the same rules of order ([Definition 1.1.1](#)), yet $x^2 - 2$ has a zero in one (as [Theorem 1.6.1](#) will prove) and not in the other. The difference lies in how **sets** of numbers behave: in $\mathbb{Q}$, a set can press against a point that is not there. This section builds the vocabulary to say that precisely. Everything happens in an arbitrary ordered field F , because we need to compare $F = \mathbb{Q}$ with $F = \mathbb{R}$, and the definitions must not already take a side.

1.2.1 A largest element that is not there

Ask for the largest element of the open interval $(0, 1)$ in $\mathbb{R}$. There is none. Whatever $a \in (0, 1)$ you propose, the midpoint $\frac{a+1}{2}$ lies strictly between a and 1 by [Theorem 1.1.2 \(h\)](#), so it belongs to $(0, 1)$ and beats a . The same happens, less visibly, for the set of positive rationals whose square is less than 2; Case 1 of the proof at the end of this section is the reason, and [Exercise \(C1: The two sides of the gap\)](#) writes it out.

And yet both sets are heading somewhere. The interval $(0, 1)$ is pressed against 1: nothing in it reaches 1, and nothing smaller than 1 stays above it. “Largest element” is the wrong notion, because it demands that the target belong to the set. What survives is the smallest number that sits above the whole set. To define it we first need words for “sits above”.

1.2.2 Upper and lower bounds

Definition 1.2.1. Upper and Lower Bounds

Let F be an ordered field and $A \subseteq F$. An element $M \in F$ is an **upper bound** of A if $a \leq M$ **for every** $a \in A$. An element $m \in F$ is a **lower bound** of A if $m \leq a$ **for every** $a \in A$.

In symbols, M is an upper bound of A when

$$\forall a \in A : a \leq M.$$

The bound is an element of F , and nothing requires it to belong to A .

For the interval $(0, 1)$ in $\mathbb{R}$, the numbers 1, $\frac{3}{2}$, and 2026 are all upper bounds, and 0 and -17 are lower bounds. Upper bounds are never unique: if M is one, so is every $M' \geq M$, by transitivity. A minimal change gives a nonexample: $\frac{1}{2}$ is **not** an upper bound of $(0, 1)$, because $\frac{3}{4} \in (0, 1)$ and $\frac{3}{4} > \frac{1}{2}$. One element above M is enough to spoil it. Negating the definition quantifier by quantifier (the negation rules are the subject of *Logic and Proof*, Chapter 4),

$$M \text{ is not an upper bound of } A \iff \exists a \in A : a > M,$$

where we have used trichotomy to rewrite “not $a \leq M$ ” as “ $a > M$ ”.

The empty set is the degenerate case, and it will matter. **Every** $M \in F$ is an upper bound of $\emptyset$: the condition “ $a \leq M$ for every $a \in \emptyset$ ” has no instances to check, so it holds vacuously. Instance by instance it is the implication “if $a \in \emptyset$, then $a \leq M$ ”, whose hypothesis is false, and *Logic and Proof*, Proposition 2.1.2 derives an implication from the negation of its hypothesis. Likewise every element of F is a lower bound of $\emptyset$.

Definition 1.2.2. Bounded Sets

Let F be an ordered field and $A \subseteq F$. The set A is **bounded above** if it has **at least one** upper bound, **bounded below** if it has at least one lower bound, and **bounded** if it is both bounded above and bounded below.

In symbols, A is bounded above when

$$\exists M \in F \quad \forall a \in A : a \leq M.$$

The order of the quantifiers carries the meaning. **One** M is chosen first and must then work **for every** a . Swap them, to “ $\forall a \in A \exists M \in F : a \leq M$ ”, and the statement becomes empty of content: every set satisfies it, since each a can take $M = a$.

Examples in $\mathbb{R}$: the interval $(0, 1)$ is bounded; the empty set is bounded (by anything); the interval $[0, \infty)$ is bounded below by 0 but **not** bounded above. To see the last claim, negate the definition: we must show that for every $M \in \mathbb{R}$ there is $a \in [0, \infty)$ with $a > M$. Given M , the element $a := \max\{M, 0\} + 1$ works, where $\max\{M, 0\}$ is the larger of M and 0 (maxima are defined in general just below): it is at least $1 > 0$, so it lies in $[0, \infty)$, and it exceeds M by (O3). The witness a depends on M , as the order $\forall M \exists a$ allows.

Whether $\mathbb{N}$ is bounded above in $\mathbb{R}$ looks like the same kind of question, but it is not. The answer, “no”, is the Archimedean property, and it is proved later in this chapter from the axiom of the next section. It cannot be proved from the ordered-field axioms alone.

1.2.3 Maximum and minimum

Definition 1.2.3. Maximum and Minimum

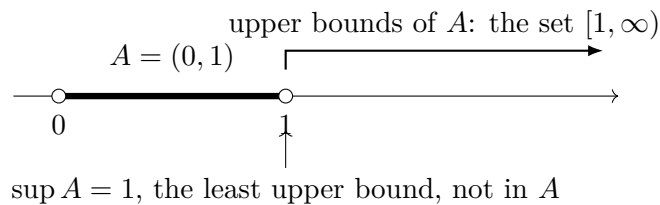
Let F be an ordered field and $A \subseteq F$. An element m is a **maximum** (or greatest element) of A if $m \in A$ **and** m is an upper bound of A . It is a **minimum** (or least element) of A if $m \in A$ **and** m is a lower bound of A .

A set has at most one maximum. If m and m' are both maxima, then $m \in A$ and m' is an upper bound, so $m \leq m'$; by the same argument with the roles swapped, $m' \leq m$; and trichotomy forces $m = m'$. So when it exists we may write $\max A$, and likewise $\min A$.

Examples: the set $\{-1, 3, \frac{7}{2}\}$ has maximum $\frac{7}{2}$, since $\frac{7}{2}$ is in the set and $-1 \leq \frac{7}{2}$, $3 \leq \frac{7}{2}$, $\frac{7}{2} \leq \frac{7}{2}$. The interval $(0, 1]$ has maximum 1. The minimal change from $(0, 1]$ to $(0, 1)$ keeps every upper bound but removes the only candidate: $(0, 1)$ has **no** maximum, as the opening subsection showed. The empty set has no maximum, since it has no elements at all.

1.2.4 The supremum

Here is the notion that survives where “maximum” fails. Look again at $(0, 1)$. Its upper bounds form the set $[1, \infty)$, and **that** set has a least element, namely 1. The number we want is the least element of the set of upper bounds.



The supremum is the upper bound that cannot be lowered.

Definition 1.2.4. Supremum

Let F be an ordered field and $A \subseteq F$. An element $s \in F$ is a **supremum** (or **least upper bound**) of A in F if

- (S1) s is an upper bound of A , and
- (S2) $s \leq M$ **for every** upper bound M of A .

In symbols, s is a supremum of A in F when

$$(\forall a \in A : a \leq s) \text{ and } (\forall M \in F : (\forall a \in A : a \leq M) \Rightarrow s \leq M).$$

In words: (S1) says that s sits above the whole set. (S2) says that no upper bound sits below s ; every upper bound is at least s . So s is an upper bound (S1), and it is the least one (S2). The words “in F ” matter: s must be an element of F , and the upper bounds M in (S2) range over

F . The same set of rationals can have a supremum when viewed in one field and none when viewed in another, and that is precisely the phenomenon this chapter is about.

Before we write “the supremum”, we must know there cannot be two.

Theorem 1.2.1. Uniqueness of the Supremum

Let F be an ordered field and $A \subseteq F$. If s and s' are both suprema of A in F , then $s = s'$.

Proof. By (S1) for s' , the element s' is an upper bound of A . So (S2) for s gives $s \leq s'$. Swapping the roles of s and s' gives $s' \leq s$. By trichotomy, $s = s'$. ■

The proof is the standard uniqueness move: suppose there are two, use the first property of one against the second property of the other, and then swap. From now on, when A has a supremum in F , we call it **the** supremum and write $\sup A$. Existence is a different matter, and nothing so far guarantees it. It can fail in three ways: the set may have no upper bound at all, like $[0, \infty)$ above; it may have no elements; or it may press against a gap in the field. The rest of this section shows the last two.

The mirror-image notion comes with its own definition, since we use it as often.

Definition 1.2.5. Infimum

Let F be an ordered field and $A \subseteq F$. An element $t \in F$ is an **infimum** (or **greatest lower bound**) of A in F if

- (I1) t is a lower bound of A , and
- (I2) $M \leq t$ **for every** lower bound M of A .

The proof of [Theorem 1.2.1](#), with every inequality reversed, shows that a set has at most one infimum, written $\inf A$. The names are Latin: *supremum* is “the highest”, *infimum* “the lowest”.

Example. A finite set, and a half-open interval

Find the supremum of $A = \{-1, 3, \frac{7}{2}\}$ and of $B = (0, 1]$ in $\mathbb{R}$.

Solution. For A : the element $\frac{7}{2}$ is an upper bound, as checked above, so (S1) holds. For (S2), let M be any upper bound of A . Since $\frac{7}{2} \in A$, the definition of upper bound gives $\frac{7}{2} \leq M$. So $\sup A = \frac{7}{2} = \max A$.

For B : the same two lines work with 1 in place of $\frac{7}{2}$. It bounds $(0, 1]$ from above by the definition of that interval, and it is least because $1 \in B$, so every upper bound M satisfies $1 \leq M$. So $\sup B = 1 = \max B$.

The argument used only that the candidate is a maximum. It shows that **a maximum, when it exists, is the supremum** ([Exercise \(B1: Maxima are suprema\)](#)). Every nonempty finite set has a maximum, by an induction on the number of elements of the kind practiced

in *Sets, Relations, and Cardinality*, Chapter 2, so for finite sets the supremum is nothing new.

Example. The supremum need not belong to the set

Show that $\sup(0, 1) = 1$ in $\mathbb{R}$, although $1 \notin (0, 1)$.

Solution. (S1): every $a \in (0, 1)$ satisfies $a < 1$, so 1 is an upper bound.

(S2): let M be an upper bound of $(0, 1)$; we show $1 \leq M$. Suppose instead that $M < 1$. Since $\frac{1}{2} \in (0, 1)$, we have $\frac{1}{2} \leq M$, so $M > 0$. By [Theorem 1.1.2](#) (h), the midpoint $a := \frac{M+1}{2}$ satisfies $M < a < 1$. Then $0 < M < a < 1$, so $a \in (0, 1)$, and $a > M$. This contradicts M being an upper bound. Hence $1 \leq M$, by trichotomy.

So 1 is the supremum. It is not a maximum, since $1 \notin (0, 1)$. Nothing in the argument used anything beyond the ordered-field axioms; the same proof shows that $\{x \in \mathbb{Q} : 0 < x < 1\}$ has supremum 1 in $\mathbb{Q}$.

Example. A supremum we cannot yet prove

Let $A = \{1 - \frac{1}{n} : n \in \mathbb{Z}_{>0}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ in $\mathbb{R}$, where $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ is the set of positive integers (taken informally for now; later in this chapter, once the natural numbers are defined inside $\mathbb{R}$, it becomes a defined subset of $\mathbb{R}$). Show that 1 is an upper bound of A and that A has no maximum.

Solution. Each $n \in \mathbb{Z}_{>0}$ is positive, so $\frac{1}{n} > 0$ by [Theorem 1.1.2](#) (f), and hence $1 - \frac{1}{n} < 1$ by [Theorem 1.1.2](#) (a). So 1 is an upper bound, and $1 \notin A$. No element is a maximum either: $1 - \frac{1}{n} < 1 - \frac{1}{n+1}$, because $\frac{1}{n+1} < \frac{1}{n}$ by [Theorem 1.1.2](#) (f), so every element is beaten by the next one.

It is true that $\sup A = 1$, but (S2) is harder than it looks. It says that no $M < 1$ is an upper bound, that is, for every $M < 1$ some $1 - \frac{1}{n}$ exceeds M . That requires $\frac{1}{n} < 1 - M$ for some n . The existence of such an n says, in effect, that the natural numbers are not bounded above, and that does not follow from the ordered-field axioms. We prove it later in this chapter, from the axiom of the next section, and prove $\sup A = 1$ right after it.

The degenerate case is the empty set. By the vacuous-truth remark above, every element of F is an upper bound of $\emptyset$. If s were a supremum, then $s - 1$ would be an upper bound too, and (S2) would give $s \leq s - 1$, that is, $1 \leq 0$, contradicting $1 > 0$. So **the empty set has no supremum**, although it is bounded above. This is why the axiom of the next section has to say “nonempty”.

A nonexample by minimal change shows why both clauses are needed. Keep $A = (0, 1)$. The number 2 satisfies (S1) but fails (S2): the upper bound 1 lies below it. The number $\frac{1}{2}$ satisfies (S2), since every upper bound is at least $1 \geq \frac{1}{2}$, but it fails (S1). Drop (S1) from the definition and every number below 1, such as $\frac{1}{2}$ or -5 , would qualify. Drop (S2) and every upper bound would. Only the two clauses together pin down a single number.

Warning. The supremum is not the maximum. The supremum of a set need not belong to the set: $\sup(0, 1) = 1 \notin (0, 1)$. The two notions agree exactly when the supremum lies in the set ([Exercise \(B1: Maxima are suprema\)](#)). A proof that writes “let $a = \sup A \in A$ ” has assumed something false in general; the section on working with the supremum contains a proof that fails for exactly this reason.

Warning. A bounded set need not have a maximum. The interval $(0, 1)$ is bounded, and it has no greatest element. “Bounded” promises upper bounds, which live outside the set; it promises nothing about the elements themselves.

Quick check. Let $A = \{\frac{(-1)^n}{n} : n \in \mathbb{Z}_{>0}\} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\}$. Find $\max A$ and $\sup A$ in $\mathbb{R}$, if they exist. Do the same for $\min A$ and $\inf A$.

Solution. For odd n the element is $-\frac{1}{n} < 0$. For even $n \geq 2$ it is $\frac{1}{n} \leq \frac{1}{2}$, by [Theorem 1.1.2 \(f\)](#). So $\frac{1}{2}$ is an upper bound, and it is attained at $n = 2$. Hence $\max A = \frac{1}{2}$, and by the argument of [Example \(A finite set, and a half-open interval\)](#) (a maximum is the supremum), $\sup A = \frac{1}{2}$.

Similarly, every element is at least -1 : even-indexed ones are positive, and odd-indexed ones are $-\frac{1}{n} \geq -1$ because $\frac{1}{n} \leq 1$. The value -1 is attained at $n = 1$. So $\min A = \inf A = -1$. Although A is infinite, both extremes are attained, at its first two elements.

1.2.5 A bounded set of rationals with no rational supremum

We now prove what the opening section promised: in $\mathbb{Q}$, a nonempty set that is bounded above can fail to have a supremum. This is the gap in $\mathbb{Q}$, expressed in the language of this section.

Theorem 1.2.2. No Rational Supremum for the Rationals Below the Square Root of Two

Let $A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$. Then A is nonempty and bounded above in $\mathbb{Q}$, but A has no supremum in $\mathbb{Q}$.

Idea. Suppose a rational s were the supremum. By Logic and Proof, [Theorem 2.2.1](#), $s^2 \neq 2$, so $s^2 < 2$ or $s^2 > 2$, and each case is a disaster.

If $s^2 < 2$, then s is too small: there is room to step right to a rational $s + h$ whose square is still below 2, and then $s + h \in A$ beats s . To find h , work backwards from the goal $(s + h)^2 < 2$. Expanding, $(s + h)^2 = s^2 + 2sh + h^2$. The h^2 is awkward, so bound it by something simpler: if $0 < h \leq 1$, then $h^2 \leq h$, and $(s + h)^2 \leq s^2 + h(2s + 1)$. The room available is $2 - s^2$. Asking $h(2s + 1)$ to use only half of it, $h(2s + 1) = \frac{2 - s^2}{2}$, gives $h = \frac{2 - s^2}{4s + 2}$. We then owe a check of the condition $h \leq 1$ that we assumed along the way.

If $s^2 > 2$, then s is too big: step left to $s - h$, still an upper bound, and then s was not the least one. Now $(s - h)^2 = s^2 - 2sh + h^2 > s^2 - 2sh$, and the right side equals 2 exactly when $h = \frac{s^2 - 2}{2s}$.

In both cases h depends only on s , and it is rational because s is. *Before reading on: where does the proof use that s is rational, and where does it use Logic and Proof, Theorem 2.2.1?*

Proof. Since $1 > 0$ and $1^2 = 1 < 2$, we have $1 \in A$, so A is nonempty. The rational number 2 is an upper bound: if some $x \in A$ had $x > 2$, then $x^2 > 4$ by Theorem 1.1.2 (g), contradicting $x^2 < 2$, so $x \leq 2$ by trichotomy.

Suppose, for a contradiction, that $s \in \mathbb{Q}$ is a supremum of A in $\mathbb{Q}$. Since $1 \in A$, (S1) gives $s \geq 1$; in particular $s > 0$, and $s^2 \geq 1$ by Theorem 1.1.2 (g). By Logic and Proof, Theorem 2.2.1, $s^2 \neq 2$, so by trichotomy either $s^2 < 2$ or $s^2 > 2$.

Case 1: $s^2 < 2$. Let

$$h := \frac{2 - s^2}{4s + 2}.$$

Then $h \in \mathbb{Q}$, and $h > 0$ because $2 - s^2 > 0$ and $4s + 2 > 0$. Moreover $h \leq \frac{1}{6}$, since $2 - s^2 \leq 1$ (as $s^2 \geq 1$) and $4s + 2 \geq 6$ (as $s \geq 1$). Hence $h \leq 1$, and multiplying by $h > 0$ gives $h^2 \leq h$. Therefore

$$\begin{aligned} (s + h)^2 &= s^2 + 2sh + h^2 \\ &\leq s^2 + h(2s + 1) \\ &= s^2 + \frac{2 - s^2}{2} = 1 + \frac{s^2}{2} < 2, \end{aligned}$$

where the last inequality holds because $s^2 < 2$. So $s + h$ is a positive rational with square less than 2, that is, $s + h \in A$. But $s + h > s$, so s is not an upper bound of A , contradicting (S1).

Case 2: $s^2 > 2$. Let

$$h := \frac{s^2 - 2}{2s}.$$

Then $h \in \mathbb{Q}$ and $h > 0$. Also $s - h = \frac{s^2 + 2}{2s} > 0$, and

$$(s - h)^2 = s^2 - 2sh + h^2 = s^2 - (s^2 - 2) + h^2 = 2 + h^2 > 2.$$

We claim that $s - h$ is an upper bound of A . Indeed, if some $x \in A$ had $x > s - h$, then, as $s - h > 0$, Theorem 1.1.2 (g) would give $x^2 > (s - h)^2 > 2$, contradicting $x^2 < 2$. So $s - h$ is a rational upper bound of A with $s - h < s$, contradicting (S2).

Both cases are impossible. Hence A has no supremum in $\mathbb{Q}$. This proves the theorem. ■

Rationality of s was used in exactly two places: to make h , and so $s \pm h$, rational, which is what membership in A and “upper bound in $\mathbb{Q}$ ” require; and to rule out $s^2 = 2$ by Logic and Proof, Theorem 2.2.1. The logical shape is the same as in the irrationality of $\sqrt{2}$: assume the object exists, reach a contradiction, conclude that it does not.

To see the constants at work, start from the rational upper bound $s = \frac{3}{2}$, whose square $\frac{9}{4}$ exceeds 2. Case 2 gives $h = \frac{1}{12}$ and the smaller upper bound $s - h = \frac{17}{12}$, with $(\frac{17}{12})^2 = \frac{289}{144} = 2 + \frac{1}{144}$.

Starting instead from $s = 1 \in A$, Case 1 gives $h = \frac{1}{6}$, and $\frac{7}{6} \in A$ because $\frac{49}{36} < 2$. Each step can be repeated forever from either side, and no rational number is ever reached at which both processes stop.

This is the gap in $\mathbb{Q}$ in its final form. The next section adds to the ordered-field axioms the single statement that such gaps do not occur in $\mathbb{R}$. Later in the chapter, the same two-case argument, with s now a real number supplied by that axiom, proves that $\sqrt{2}$ exists.

1.2.6 Exercises

1.2.6.1 A. Check your understanding

Exercise. A1

In (a)–(d), give the supremum, infimum, maximum, and minimum of the set in $\mathbb{R}$, or say that it does not exist. Justify each answer in a line or two. Parts (e) and (f) are about the proof of [Theorem 1.2.2](#).

- (a) $\{-2, 0, \frac{7}{3}\}$.
- (b) $[-1, 4)$.
- (c) $\{x \in \mathbb{R} : x^2 \leq 4\}$.
- (d) $\emptyset$.
- (e) Run Case 2 of the proof on the rational upper bound $s = \frac{17}{12}$: compute h and $s - h$, and check that $(s - h)^2 > 2$.
- (f) Replace $\mathbb{Q}$ by $\mathbb{R}$ throughout the statement and the proof. Which single step of the proof can no longer be justified, and what possibility does it leave open?

Solution.

- (a) $\max = \sup = \frac{7}{3}$ and $\min = \inf = -2$: both are elements of the set, and comparing with the three elements shows they are an upper and a lower bound. A maximum is the supremum and a minimum is the infimum ([Exercise \(B1: Maxima are suprema\)](#)).
- (b) $\min = \inf = -1$, since -1 belongs to the set and is a lower bound. $\sup = 4$ and there is no maximum: 4 is an upper bound not in the set, and the midpoint argument of [Example \(The supremum need not belong to the set\)](#) shows that no $M < 4$ is an upper bound (if $M < 4$ is an upper bound, then $M \geq -1$ and $\frac{M+4}{2}$ is in the set and exceeds M).
- (c) The set is $[-2, 2]$: by [Theorem 1.1.2](#) (g) applied to $|x|$ and 2 , and [Proposition 1.1.3](#) (b), $x^2 \leq 4$ holds if and only if $|x| \leq 2$, which by [Proposition 1.1.3](#) (d) means $-2 \leq x \leq 2$. So $\max = \sup = 2$ and $\min = \inf = -2$.
- (d) None of the four exists. Every real number is an upper and a lower bound of $\emptyset$, so there is no least upper bound and no greatest lower bound (as shown in the text), and $\emptyset$ has no elements to be a maximum or minimum.
- (e) Here $s^2 = \frac{289}{144} = 2 + \frac{1}{144}$, so

$$h = \frac{s^2 - 2}{2s} = \frac{1/144}{17/6} = \frac{1}{408}, \quad s - h = \frac{17}{12} - \frac{1}{408} = \frac{577}{408}.$$

Then $(s - h)^2 = 2 + h^2 = 2 + \frac{1}{166464} > 2$, as the proof's display predicts; directly,

$577^2 = 332929 = 2 \cdot 408^2 + 1$. So $\frac{577}{408}$ is a smaller rational upper bound, and the step can be repeated from it.

- (f) The step “by Logic and Proof, Theorem 2.2.1, $s^2 \neq 2$ ”. Over $\mathbb{R}$ nothing rules out $s^2 = 2$, so after Cases 1 and 2 the possibility $s^2 = 2$ remains, and no contradiction is reached. Every other step is algebra valid in any ordered field, $\mathbb{R}$ included. This is exactly the opening that the existence proof for $\sqrt{2}$, later in this chapter, walks through.

1.2.6.2 B. Practice

Exercise. B1: Maxima are suprema

Let F be an ordered field and $A \subseteq F$.

- Prove that if A has a maximum m , then m is the supremum of A .
- Prove that if A has a supremum s and $s \in A$, then s is the maximum of A .
- State and prove the corresponding facts for minimum and infimum.

Solution.

- By Definition 1.2.3, m is an upper bound of A , which is (S1). For (S2), let M be any upper bound of A . Since $m \in A$, the definition of upper bound gives $m \leq M$. Hence m is a supremum of A , and by Theorem 1.2.1 it is the supremum.
- By (S1), s is an upper bound of A , and $s \in A$ by hypothesis. That is exactly Definition 1.2.3.
- If A has a minimum m , then $m = \inf A$; and if $\inf A$ exists and belongs to A , it is the minimum. The proofs are those of (a) and (b) with every inequality reversed: m is a lower bound (I1), and any lower bound M satisfies $M \leq m$ because $m \in A$ (I2).

Exercise. B2: Negating the definitions

Let F be an ordered field and $A \subseteq F$.

- Write in symbols the statement “ s is **not** an upper bound of A ”.
- Write in symbols the statement “ s is **not** a supremum of A ”, as a disjunction of two statements.
- Use (b) to show directly that 2 is not the supremum of $(0, 1)$ in $\mathbb{R}$, by exhibiting a witness.

Solution.

- $\exists a \in A : a > s$.
- Negating the conjunction (S1) and (S2) gives: (S1) fails or (S2) fails. So “ s is not a supremum of A ” is

$$(\exists a \in A : a > s) \text{ or } (\exists M \in F : (\forall a \in A : a \leq M) \text{ and } M < s).$$

The second disjunct says: some upper bound lies strictly below s .

- (c) The first disjunct fails for $s = 2$, since every element of $(0, 1)$ is below 2. The second holds with the witness $M = 1$: it is an upper bound of $(0, 1)$, and $1 < 2$. Hence 2 is not the supremum.

Exercise. B3: A supremum that is approached but not attained

Let $A = \{\frac{x}{x+1} : x \in \mathbb{R}, x > 0\}$. Prove that $\sup A = 1$ and that A has no maximum.

Solution. For $x > 0$, the denominator $x+1$ is positive, and $x < x+1$; multiplying by $\frac{1}{x+1} > 0$ (Theorem 1.1.2 (c) and (f)) gives $\frac{x}{x+1} < 1$. So 1 is an upper bound (S1), and $1 \notin A$.

For (S2), let M be an upper bound of A , and suppose $M < 1$. Since $\frac{1}{2} = \frac{1}{1+1} \in A$, we have $M \geq \frac{1}{2} > 0$. Let $x := \frac{M}{1-M} + 1$, which is positive because $M > 0$ and $1 - M > 0$. Then

$$x(1 - M) = M + (1 - M) = 1 > M,$$

so $x > Mx + M = M(x + 1)$, and dividing by $x + 1 > 0$ gives $\frac{x}{x+1} > M$. This element of A exceeds the upper bound M , a contradiction. Hence $M \geq 1$ for every upper bound M , and $\sup A = 1$. Finally, A has no maximum: by Exercise (B1: Maxima are suprema) (a), a maximum would be the supremum 1, but $1 \notin A$.

(The choice of x comes from solving $\frac{x}{x+1} > M$ backwards: it is equivalent to $x(1 - M) > M$, that is, $x > \frac{M}{1-M}$, and adding 1 gives a strict margin.)

1.2.6.3 C. Going deeper

Exercise. C1: The two sides of the gap

- Give a set $A \subseteq \mathbb{Q}$ that is bounded above and has a supremum in $\mathbb{Q}$ that does not belong to A .
- Let $B = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, and let U be the set of **rational** upper bounds of B . Prove that $U = \{u \in \mathbb{Q} : u > 0, u^2 > 2\}$.
- Deduce that every positive rational number lies in exactly one of B and U , that B has no largest element, and that U has no smallest element.

Hint: the two cases in the proof of Theorem 1.2.2 each produce a new rational from an old one.

Solution.

- $A = \{x \in \mathbb{Q} : 0 < x < 1\}$. The argument of Example (The supremum need not belong to the set) works in the ordered field $\mathbb{Q}$, because the midpoint of two rationals is rational; it shows $\sup A = 1$ in $\mathbb{Q}$, and $1 \notin A$.
- ($\subseteq$) Let $u \in U$. Since $1 \in B$, we get $u \geq 1 > 0$. By Logic and Proof, Theorem 2.2.1, $u^2 \neq 2$. If $u^2 < 2$, then $u \in B$, and Case 1 of the proof of Theorem 1.2.2, applied to $s = u$ (which used only $s \in \mathbb{Q}$, $s \geq 1$, and $s^2 < 2$), produces $u + h \in B$ with $u + h > u$, contradicting $u \in U$. Hence $u^2 > 2$. ($\supseteq$) Let $u \in \mathbb{Q}$ with $u > 0$ and $u^2 > 2$. If some

$x \in B$ had $x > u$, then $x^2 > u^2 > 2$ by [Theorem 1.1.2](#) (g), contradicting $x^2 < 2$. So u is an upper bound of B , and $u \in U$.

- (c) Let $q \in \mathbb{Q}$ with $q > 0$. By Logic and Proof, Theorem 2.2.1, $q^2 \neq 2$, so by trichotomy exactly one of $q^2 < 2$ and $q^2 > 2$ holds, that is, q lies in exactly one of B and U , by (b). If B had a largest element b , Case 1 of the proof (with $s = b$, valid since $b \geq 1$ when b is largest, as $1 \in B$) would give a larger element $b + h \in B$. If U had a smallest element u , Case 2 of the proof (with $s = u$, which needs only $s > 0$, $s \in \mathbb{Q}$, and $s^2 > 2$) would give $u - h \in U$, smaller. So the positive rationals split into a lower piece with no top and an upper piece with no bottom. Nothing rational sits between the two pieces; that missing point is the gap.

Exercise. C2: The least element of the upper bounds

Let F be an ordered field, $A \subseteq F$, and let $U(A)$ be the set of all upper bounds of A in F .

- Prove that s is a supremum of A if and only if s is the minimum of $U(A)$. (This explains the name “least upper bound”.)
- Prove that if $A \subseteq B \subseteq F$, then $U(B) \subseteq U(A)$.
- Describe $U(\emptyset)$ and $U(F)$, and use (a) to decide whether $\emptyset$ and F have suprema.

Solution.

- $(\Rightarrow)$ If s is a supremum, then $s \in U(A)$ by (S1), and $s \leq M$ for every $M \in U(A)$ by (S2). So s is an element of $U(A)$ and a lower bound of $U(A)$, which is [Definition 1.2.3](#)’s definition of minimum. $(\Leftarrow)$ If $s = \min U(A)$, then $s \in U(A)$, which is (S1), and $s \leq M$ for every $M \in U(A)$, which is (S2).
- Let $M \in U(B)$. For every $a \in A$ we have $a \in B$, so $a \leq M$. Hence $M \in U(A)$.
- $U(\emptyset) = F$, since every element is an upper bound of the empty set vacuously; and F has no minimum, because $m - 1 < m$ for every $m \in F$. So $\emptyset$ has no supremum. $U(F) = \emptyset$, since for any M the element $M + 1 \in F$ exceeds M ; and $\emptyset$ has no minimum. So F has no supremum either. The two degenerate cases fail for opposite reasons: too many upper bounds, and none.

1.3 The Least Upper Bound Axiom

We now know exactly what goes wrong in the rationals. The set of positive rationals whose square is less than 2 is nonempty and bounded above, and yet it has no supremum in $\mathbb{Q}$ ([Theorem 1.2.2](#)). Every failure of calculus over $\mathbb{Q}$ traces back to gaps of this kind. This section adds one axiom to the ordered-field axioms, stating that such gaps do not occur in $\mathbb{R}$. Then it draws the first consequence: infima exist too, for free.

1.3.1 The axiom

Up to now, “ $\mathbb{R}$ is an ordered field containing $\mathbb{Q}$ ” has been our only assumption about the real numbers ([Example \(The rationals and the reals\)](#)). That assumption holds for $\mathbb{Q}$ itself, so it cannot be what makes $\mathbb{R}$ different. We add exactly one more.

Every set of reals that is bounded above has a least upper bound, unless it is empty.

Axiom 1.3.1. Least Upper Bound Axiom

Every **nonempty** subset of $\mathbb{R}$ that is **bounded above** has a supremum in $\mathbb{R}$.

In symbols: for every $A \subseteq \mathbb{R}$,

$$\left(A \neq \emptyset \text{ and } \exists M \in \mathbb{R} \forall a \in A : a \leq M \right) \Rightarrow \exists s \in \mathbb{R} : s = \sup A.$$

Standing assumption. From now on, $\mathbb{R}$ is an ordered field (Definition 1.1.1), containing $\mathbb{Q}$, that satisfies Axiom 1.3.1. Every theorem about real numbers in this book is proved from these assumptions, together with the logic and set theory of *Logic and Proof* and *Sets, Relations, and Cardinality* (including, where it is flagged, a choice principle).

In words, the axiom has two hypotheses and one conclusion. The hypotheses are about the set A : it has at least one element, and at least one upper bound. The conclusion is about $\mathbb{R}$: among the upper bounds of A there is a least one, and it is a real number. By Theorem 1.2.1 it is unique, so the notation $\sup A$ is justified whenever the two hypotheses hold. The axiom is also called the **completeness axiom**; “complete” will acquire a second, related meaning in Chapter 2, and the two are reconciled in Chapter 11.

Each hypothesis is there because without it the conclusion is false.

- **Nonempty.** The empty set is bounded above, by every real number, and it has no supremum (shown in the section on suprema). Without “nonempty”, the axiom would be false.
- **Bounded above.** The interval $[0, \infty)$ contains 0, but no real number sits above all of it, so there is no least such number either. The set $\mathbb{N}$ is a subtler example of the same failure; that it is not bounded above in $\mathbb{R}$ is proved later in this chapter, from this very axiom.

The axiom is a statement about $\mathbb{R}$ and not about ordered fields in general. It fails in $\mathbb{Q}$: some nonempty subset of $\mathbb{Q}$, bounded above in $\mathbb{Q}$, has no supremum in $\mathbb{Q}$. Which result of the previous section shows this, and whether it needs adapting, is Exercise (C1: The axiom in other ordered fields) (a). So the axiom is exactly the property that separates the two fields we have been comparing.

Remark. Why is this an axiom and not a theorem? Because we have not said what a real number is. One can **construct** $\mathbb{R}$ from $\mathbb{Q}$ and then prove the least upper bound property as a theorem. *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , Chapters 5 and 6 take this route with Cauchy sequences of rationals, and show that any two ordered fields satisfying the axiom are the same up to relabeling, so the axiom pins $\mathbb{R}$ down completely; the optional *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , Chapter 7 builds $\mathbb{R}$ by Dedekind cuts instead. This volume does not use those chapters: everything we need follows from the axiom.

Example. A set of reals that must have a supremum

Let $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\}$. Show that $\sup S$ exists in $\mathbb{R}$. What can be said about it so far?

Solution. We check the two hypotheses of [Axiom 1.3.1](#). S is nonempty, since $1 > 0$ and $1^2 < 2$. S is bounded above by 2: if $x \in S$ had $x > 2$, then $x^2 > 4$ by [Theorem 1.1.2 \(g\)](#), contradicting $x^2 < 2$. So the axiom gives a real number $s = \sup S$.

So far we know only what (S1) and (S2) give: $1 \leq s \leq 2$, since $1 \in S$ and 2 is an upper bound. Compare this with [Theorem 1.2.2](#). The rational version of this set has no supremum in $\mathbb{Q}$, and the proof of that theorem showed that a rational candidate s with $s^2 < 2$ or $s^2 > 2$ cannot be the supremum. Neither case used rationality except to keep $s \pm h$ rational. So we can guess the answer: $s^2 = 2$. Proving it, and so proving that $\sqrt{2}$ exists, is done later in this chapter.

The example shows the pattern of every use of the axiom. First check the two hypotheses, which is usually routine. Then the axiom hands over a number s , and all we know about s is (S1) and (S2). Everything else must be extracted from those two properties.

Warning. **Do not forget “nonempty”.** The empty set is bounded above and has no supremum, so an argument that applies the axiom to a set without first producing an element of that set is incomplete. The failure is easy to commit when the set is defined by a condition, such as $\{x \in \mathbb{R} : x^2 < c\}$: for $c \leq 0$ that set is empty.

Warning. **The axiom gives existence, not a formula.** It says that $\sup A$ exists; it does not say what it is, and it is usually **not** the upper bound M used to check the hypothesis. For S above, the hypothesis was checked with $M = 2$, and $\sup S \neq 2$: the smaller number $\frac{3}{2}$ is also an upper bound, since any $x > \frac{3}{2}$ has $x^2 > \frac{9}{4} > 2$ by [Theorem 1.1.2 \(g\)](#). To identify a supremum we must still verify (S1) and (S2) for a specific candidate.

Quick check. Let $A = \{n \in \mathbb{Z} : n \leq 3.5\} = \{\dots, 1, 2, 3\}$. Does A satisfy the hypotheses of [Axiom 1.3.1](#)? What is $\sup A$, and is it a maximum?

Solution. Yes: $3 \in A$, so A is nonempty, and 3.5 is an upper bound by definition of A . So $\sup A$ exists. It is 3, not 3.5: the element 3 is an upper bound, because an integer greater than 3 is at least 4, which exceeds 3.5 (arithmetic of the integers, *Sets, Relations, and Cardinality*, Chapter 2), and it belongs to A , so it is the maximum, and a maximum is the supremum ([Exercise \(B1: Maxima are suprema\)](#)). The bound used to check the hypothesis was not the supremum.

1.3.2 Infima come free

The axiom is stated for suprema only. We do not need a second axiom for infima: reflecting the line through 0 turns lower bounds into upper bounds and “greatest” into “least”. To say this precisely we need the reflected set.

For $A \subseteq \mathbb{R}$, let

$$-A := \{-a : a \in A\}.$$

So $x \in -A$ if and only if $-x \in A$, and $-(-A) = A$. For example, $-(0, 1] = [-1, 0)$, and $- \{-1, 3\} = \{1, -3\}$. The definition makes sense in any ordered field, and the next theorem holds in any ordered field; the axiom enters only afterwards.

Theorem 1.3.2. Infimum by Reflection

Let F be an ordered field, $A \subseteq F$, and $m, t \in F$.

- (a) m is a lower bound of A if and only if $-m$ is an upper bound of $-A$.
- (b) t is an infimum of A if and only if $-t$ is a supremum of $-A$.

In particular, if $-A$ has a supremum, then A has an infimum, and $\inf A = -\sup(-A)$.

Idea. Reflection $x \mapsto -x$ reverses every inequality (Theorem 1.1.2 (a)), so it should carry each property of an infimum to the matching property of a supremum. The proof checks this property by property: (I1) against (S1), and (I2) against (S2), which is the “split into two properties” move for suprema.

Proof.

- (a) By Theorem 1.1.2 (a), $m \leq a$ if and only if $-a \leq -m$. So $m \leq a$ for every $a \in A$ if and only if $-a \leq -m$ for every $a \in A$. The elements $-a$ with $a \in A$ are exactly the elements of $-A$, so the last statement says that $-m$ is an upper bound of $-A$.
- (b) $(\Rightarrow)$ Suppose t is an infimum of A . By (I1), t is a lower bound of A , so by (a), $-t$ is an upper bound of $-A$: this is (S1) for $-t$. For (S2), let M be an upper bound of $-A$. By (a) applied to $m = -M$, the element $-M$ is a lower bound of A . By (I2), $-M \leq t$, and so $-t \leq M$ by Theorem 1.1.2 (a). Hence $-t$ is a supremum of $-A$.
- $(\Leftarrow)$ Suppose $-t$ is a supremum of $-A$. By (S1) and (a), t is a lower bound of A : this is (I1). For (I2), let m be a lower bound of A . By (a), $-m$ is an upper bound of $-A$, so (S2) gives $-t \leq -m$, that is, $m \leq t$. Hence t is an infimum of A .

The final sentence follows from (b) with $t := -\sup(-A)$, since $-t = \sup(-A)$.

■

Now the axiom can be applied to $-A$.

Theorem 1.3.3. Greatest Lower Bound Property

Every **nonempty** subset A of $\mathbb{R}$ that is **bounded below** has an infimum in $\mathbb{R}$, and

$$\inf A = -\sup(-A).$$

Proof. Since A is nonempty, so is $-A$. Let m be a lower bound of A . By Theorem 1.3.2 (a), $-m$ is an upper bound of $-A$, so $-A$ is bounded above. By Axiom 1.3.1, $-A$ has a supremum, and by Theorem 1.3.2, A has the infimum $-\sup(-A)$.

■

Alternatively, one can take the supremum of the lower bounds. Let L be the set of lower bounds of A . It is nonempty, because A is bounded below, and any element of A , of which there is at least one because A is nonempty, sits above all of L . So $\ell := \sup L$ exists by [Axiom 1.3.1](#). Each $a \in A$ is an upper bound of L , so $\ell \leq a$ by (S2); hence ℓ is a lower bound of A . And every lower bound of A lies in L , so it is at most ℓ by (S1). Thus $\ell = \inf A$. In this route the two hypotheses on A swap roles: “bounded below” makes L nonempty, and “nonempty” makes L bounded above.

The hypotheses of [Theorem 1.3.3](#) are needed for the same reasons as those of the axiom. The empty set is bounded below and has no infimum, since every real number is a lower bound of it and $t + 1$ beats any candidate t . The interval $(-\infty, 0]$ is nonempty but not bounded below, and has no infimum.

So in $\mathbb{R}$, suprema and infima exist under mirror-image hypotheses, and each can be computed from the other. The mirror identity $\sup A = -\inf(-A)$, for A nonempty and bounded above, is [Exercise \(B1: The mirror identity\)](#).

1.3.3 Unbounded and empty sets

It is sometimes convenient to give $\sup A$ a meaning even when the axiom does not apply. For a nonempty set A that is not bounded above one writes $\sup A = +\infty$, and one sets $\sup \emptyset = -\infty$; symmetrically, $\inf A = -\infty$ for a nonempty set not bounded below, and $\inf \emptyset = +\infty$. The strange-looking value $\sup \emptyset = -\infty$ is chosen so that “ $A \subseteq B$ implies $\sup A \leq \sup B$ ” stays true when $A = \emptyset$.

Remark. These are conventions of notation, and $\pm\infty$ are **not** real numbers: they obey none of the field axioms, and no rule of [Theorem 1.1.2](#) applies to them. The series uses them from *Measure and Integration* on, where measure theory needs suprema of arbitrary sets, and announces them again there. Until then, whenever we write $\sup A$, the set A is nonempty and bounded above, and $\sup A$ is a real number.

1.3.4 What the axiom will do

With the axiom in place, the rest of the chapter extracts its first consequences: a practical test for recognizing a supremum, the Archimedean property, the density of the rationals, the existence of roots, and the nested interval property. The first, the test, holds in every ordered field; it is the tool for the others. Each of the rest fails in $\mathbb{Q}$ or in some other ordered field. Three of them, the Archimedean property, the roots, and the nested interval property, are proved by the same pattern: define a set, check the two hypotheses, take its supremum, and extract what is needed from (S1) and (S2). The density of the rationals then follows from the Archimedean property, with no new supremum.

1.3.5 Exercises

1.3.5.1 A. Check your understanding

Exercise. A1

For each set, decide whether it satisfies both hypotheses of [Axiom 1.3.1](#). If a hypothesis fails, name it; if both hold, find the supremum.

- (a) $\{x \in \mathbb{R} : x^2 < 0\}$.
- (b) $\{x^2 : x \in \mathbb{R}\}$.
- (c) $(-\infty, 0)$.
- (d) $\{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, regarded as a subset of the ordered field $\mathbb{Q}$.

Solution.

- (a) The set is empty, since $x^2 \geq 0$ for every x (Theorem 1.1.2 (e)). “Nonempty” fails.
- (b) “Bounded above” fails. Given $M \in \mathbb{R}$, let $x := |M| + 1$. Then $x \geq 1$, so $x^2 \geq x$ (multiply $x \geq 1$ by $x > 0$), and $x > |M| \geq M$. Hence $x^2 > M$, and M is not an upper bound.
- (c) Both hold: -1 is an element and 0 is an upper bound. The supremum is 0 : it is an upper bound, and if $M < 0$ were an upper bound, then $M \geq -1$ and the midpoint $\frac{M}{2}$ would satisfy $M < \frac{M}{2} < 0$, so it lies in the set and exceeds M .
- (d) Both hypotheses hold in $\mathbb{Q}$ (Theorem 1.2.2), but the axiom is a statement about $\mathbb{R}$, not about $\mathbb{Q}$. In $\mathbb{Q}$ the conclusion fails: the set has no supremum in $\mathbb{Q}$, by the same theorem. This is exactly why $\mathbb{Q}$ does not satisfy the axiom.

1.3.5.2 B. Practice

Exercise. B1: The mirror identity

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Show that $-A$ has a lower bound, and that $\sup A = -\inf(-A)$.

Solution. Since A is nonempty, so is $-A$. Let M be an upper bound of A . For every $x \in -A$ we have $-x \in A$, so $-x \leq M$, and hence $-M \leq x$ by Theorem 1.1.2 (a). So $-M$ is a lower bound of $-A$, and $-A$ is bounded below. By Theorem 1.3.3, $\inf(-A)$ exists and equals $-\sup(-(-A)) = -\sup A$, because $-(-A) = A$. Negating both sides gives $\sup A = -\inf(-A)$.

Exercise. B2: The infimum is below the supremum

Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Prove that $\inf A \leq \sup A$, with equality if and only if A has exactly one element.

Solution. Both exist, by Axiom 1.3.1 and Theorem 1.3.3. Since A is nonempty, choose $a \in A$. By (I1), $\inf A \leq a$, and by (S1), $a \leq \sup A$. Hence $\inf A \leq \sup A$.

If $A = \{a\}$, then a is both the maximum and the minimum of A , so $\sup A = a = \inf A$ by Exercise (B1: Maxima are suprema). Conversely, suppose $\inf A = \sup A = c$. For every $a \in A$, $c \leq a \leq c$, so $a = c$ by trichotomy. Hence $A \subseteq \{c\}$, and since A is nonempty, $A = \{c\}$.

1.3.5.3 C. Going deeper

Exercise. C1: The axiom in other ordered fields

- (a) Say what it would mean for the ordered field $\mathbb{Q}$ to satisfy the least upper bound property, and show that it does not. Is [Theorem 1.2.2](#) exactly a witness, or does it need to be adapted?
- (b) Say that an ordered field F has the **greatest lower bound property** if every nonempty subset of F that is bounded below has an infimum in F . Prove that if F has the greatest lower bound property, then it has the least upper bound property.
- (c) Conclude that, for an ordered field, the two properties are equivalent.

Solution.

- (a) $\mathbb{Q}$ would satisfy it if every nonempty subset of $\mathbb{Q}$ that is bounded above in $\mathbb{Q}$ had a supremum in $\mathbb{Q}$. [Theorem 1.2.2](#) is exactly a witness that it does not: its set is a subset of $\mathbb{Q}$, nonempty, bounded above by the rational 2, and has no supremum in $\mathbb{Q}$. Nothing needs adapting, because the theorem was stated and proved inside the ordered field $\mathbb{Q}$, with the upper bounds in (S2) ranging over $\mathbb{Q}$.
- (b) Let $A \subseteq F$ be nonempty and bounded above, with upper bound M . Then $-A$ is nonempty, and it is bounded below by $-M$, by [Theorem 1.3.2](#) (a) applied with $m = -M$ and the set $-A$ in place of A (so that $-(-A) = A$). By the greatest lower bound property, $-A$ has an infimum t in F . By [Theorem 1.3.2](#) (b), applied to the set $-A$, $-t$ is a supremum of $-(-A) = A$. So A has a supremum in F .
- (c) The converse direction is the proof of [Theorem 1.3.3](#), which used only [Theorem 1.3.2](#) (valid in every ordered field) and the least upper bound property. So in any ordered field each property implies the other.

Exercise. C2: The upper bounds form a ray

- (a) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $U(A)$ be its set of upper bounds. Prove that $U(A) = [\sup A, \infty)$.
- (b) Let $B = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, and let U be its set of upper bounds in $\mathbb{Q}$. Prove that there is no rational q with $U = \{u \in \mathbb{Q} : u \geq q\}$.
- (c) Explain in one sentence how (a) and (b) restate the difference between $\mathbb{R}$ and $\mathbb{Q}$.

Solution.

- (a) ($\subseteq$) If $M \in U(A)$, then $\sup A \leq M$ by (S2), so $M \in [\sup A, \infty)$. ($\supseteq$) If $M \geq \sup A$, then for every $a \in A$, $a \leq \sup A \leq M$ by (S1), so $M \in U(A)$.
- (b) Suppose $U = \{u \in \mathbb{Q} : u \geq q\}$ for some rational q . Then $q \in U$, and $q \leq u$ for every $u \in U$. So q is the minimum of U , that is, a rational upper bound of B below every other rational upper bound. That makes q a supremum of B in $\mathbb{Q}$, which contradicts [Theorem 1.2.2](#).
- (c) In $\mathbb{R}$ the upper bounds of a nonempty bounded-above set always form a closed ray that starts at a point, while in $\mathbb{Q}$ they can form a ray whose starting point is missing.

1.4 Working with the Supremum

The least upper bound axiom (Axiom 1.3.1) tells us that suprema exist. To use a supremum, or to recognize one, we need a handle on its second property. As defined (Definition 1.2.4), (S2) quantifies over **all** upper bounds of the set, and in practice there is no way to list them. This section replaces (S2) by an equivalent statement quantified over small positive numbers ε , which can be checked by exhibiting elements. With it we prove the rules for computing suprema of scaled sets and of sums of sets, and meet the first $\varepsilon/2$ argument of the book.

1.4.1 The epsilon characterization

Look back at how (S2) was verified for $(0, 1)$ in Example (The supremum need not belong to the set). We supposed that some $M < 1$ was an upper bound and then produced an element of the set above M . Writing $M = 1 - \varepsilon$, the argument really showed: for every $\varepsilon > 0$, some element of the set exceeds $1 - \varepsilon$. That is a statement about elements of the set, not about upper bounds, and it is exactly equivalent to (S2).

A supremum is an upper bound that the set comes arbitrarily close to.

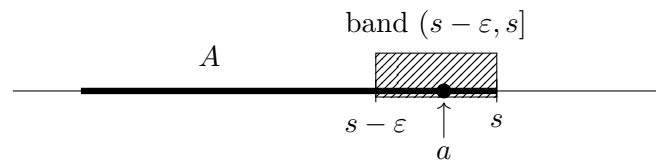
Theorem 1.4.1. Epsilon Characterization of the Supremum

Let $A \subseteq \mathbb{R}$ and let $s \in \mathbb{R}$ be an upper bound of A . Then $s = \sup A$ if and only if **for every** $\varepsilon > 0$ **there exists** $a \in A$ with $a > s - \varepsilon$.

In symbols, for an upper bound s of A :

$$s = \sup A \iff \forall \varepsilon > 0 \exists a \in A : s - \varepsilon < a (\leq s).$$

The part in parentheses is automatic, since s is an upper bound; it is written to show where a lies.



Idea. Draw the band $(s - \varepsilon, s]$ just below s . The theorem says: s is the supremum exactly when every such band, however thin, contains a point of A .

($\Rightarrow$) The number $s - \varepsilon$ is smaller than s , so it cannot be an upper bound, by (S2). Not being an upper bound means precisely that some element of A lies above it.

($\Leftarrow$) We need (S2): every upper bound M satisfies $s \leq M$. Suppose some upper bound M lies below s . Then the band between M and s contains no point of A , and the hypothesis says every band does. So the band of width $s - M$, that is, the hypothesis with $\varepsilon = s - M$, gives the contradiction.

Proof. ($\Rightarrow$) Suppose $s = \sup A$, and let $\varepsilon > 0$. Then $s - \varepsilon < s$ by Theorem 1.1.2 (a). If $s - \varepsilon$ were an upper bound of A , then (S2) would give $s \leq s - \varepsilon$, contradicting $s - \varepsilon < s$ by trichotomy. So $s - \varepsilon$ is not an upper bound of A , which means there is $a \in A$ with $a > s - \varepsilon$.

($\Leftarrow$) Suppose that for every $\varepsilon > 0$ there is $a \in A$ with $a > s - \varepsilon$. Since s is an upper bound of A by hypothesis, (S1) holds, and it remains to check (S2). Let M be an upper bound of A , and suppose, for a contradiction, that $M < s$. Put $\varepsilon := s - M$, which is positive by Theorem 1.1.2 (a). By hypothesis there is $a \in A$ with $a > s - \varepsilon = M$. This contradicts M being an upper bound of A . Hence $s \leq M$ by trichotomy, which is (S2). So $s = \sup A$. ■

The theorem does not use the axiom. It holds in any ordered field, and it **recognizes** a supremum; the axiom is what guarantees that there is one to recognize. The two halves are used differently. The forward direction is how a known supremum is **used**: it hands us elements of A as close to $\sup A$ as we like. The backward direction is how a candidate is **confirmed**: it replaces “every upper bound is at least s ” by “every band below s meets A ”, which can be checked by writing down an element.

Remark. The hypothesis “ s is an upper bound” cannot be dropped. For $A = (0, 1)$ and $s = \frac{1}{2}$, every band $(\frac{1}{2} - \varepsilon, \frac{1}{2}]$ meets A , yet $\frac{1}{2}$ is not the supremum. The band condition alone says only that s is not too big; being an upper bound says it is not too small.

The element a that the theorem produces **depends on** ε , as the order “ $\forall \varepsilon \exists a$ ” allows. For $A = (0, 1)$ and $s = 1$, the choice $a := \max\{\frac{1}{2}, 1 - \frac{\varepsilon}{2}\}$ works: it lies in $(0, 1)$ and exceeds $1 - \varepsilon$. For $\varepsilon = \frac{1}{10}$ it gives $a = \frac{19}{20}$; for $\varepsilon = \frac{1}{1000}$ it must be above $\frac{999}{1000}$, and $\frac{19}{20}$ no longer works. No single element of $(0, 1)$ works for every ε .

Warning. **The witness depends on ε .** The theorem gives, **for each ε , some a** ; it never gives one a that serves every ε at once. A proof that fixes a first and then says “for every $\varepsilon > 0$, $a > s - \varepsilon$ ” has swapped the quantifiers, and for $(0, 1)$ it asserts something false. When you take a witness from the theorem, write “choose $a \in A$ (depending on ε) with ...”.

Quick check. Let s be an upper bound of a set $A \subseteq \mathbb{R}$. Compare

$$(i) \quad \forall \varepsilon > 0 \quad \exists a \in A : a > s - \varepsilon \quad \text{and} \quad (ii) \quad \exists a \in A \quad \forall \varepsilon > 0 : a > s - \varepsilon.$$

Statement (i) says $s = \sup A$. What does (ii) say, and when is it true?

Solution. Fix $a \in A$. The condition “ $a > s - \varepsilon$ for every $\varepsilon > 0$ ” forces $a \geq s$: if $a < s$, then $\varepsilon := s - a > 0$ would give $a > s - (s - a) = a$, which is absurd. Since s is an upper bound, also $a \leq s$, so $a = s$. Conversely, if $s \in A$, then $a = s$ satisfies $s > s - \varepsilon$ for every $\varepsilon > 0$.

So (ii) says exactly that $s \in A$, that is, s is the **maximum** of A . Swapping the quantifiers turned “supremum” into “maximum”. For $A = (0, 1)$ and $s = 1$, (i) is true and (ii) is false.

Here is the theorem at work on a set where the witness must be found by a short computation. The Idea finds it backwards; the solution writes it forwards.

Example. Confirming a supremum with epsilon

Prove that $\sup\{x^2 : x \in (0, 1)\} = 1$.

Solution. Idea. The upper bound part is easy. For the band condition, fix $\varepsilon > 0$; we need $x \in (0, 1)$ with $x^2 > 1 - \varepsilon$. Try $x = 1 - h$ with $h > 0$ small. Then $x^2 = 1 - 2h + h^2 > 1 - 2h$, so it suffices that $1 - 2h \geq 1 - \varepsilon$, that is, $h \leq \frac{\varepsilon}{2}$. But we also need $x \in (0, 1)$, that is, $0 < h < 1$, and $\frac{\varepsilon}{2}$ can be large. Capping it fixes this: take $h = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$.

Proof. Let $A = \{x^2 : x \in (0, 1)\}$. For $x \in (0, 1)$, [Theorem 1.1.2](#) (g) gives $x^2 < 1^2 = 1$, so 1 is an upper bound of A .

Let $\varepsilon > 0$, and let $h := \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$, which depends only on ε . Then $0 < h \leq \frac{1}{2}$, so $x := 1 - h$ lies in $[\frac{1}{2}, 1) \subseteq (0, 1)$, and $x^2 \in A$. Since $h > 0$, $h^2 > 0$, and so

$$x^2 = 1 - 2h + h^2 > 1 - 2h \geq 1 - \varepsilon,$$

where the last step uses $h \leq \frac{\varepsilon}{2}$. By [Theorem 1.4.1](#), $\sup A = 1$.

The $\frac{1}{2}$ in the minimum is arbitrary: any cap below 1 keeps x positive. What matters is that a constraint discovered late in the Idea (here $h < 1$) is built into the constant from the start of the proof.

1.4.2 Sup approximation

The forward direction of [Theorem 1.4.1](#) is used so often with $\varepsilon = \frac{1}{n}$ that it deserves its own statement.

Corollary 1.4.2. Sup Approximation

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. For every $n \in \mathbb{Z}_{>0}$ there is $a_n \in A$ with

$$\sup A - \frac{1}{n} < a_n \leq \sup A.$$

Proof. By [Axiom 1.3.1](#), $\sup A$ exists. Let $n \in \mathbb{Z}_{>0}$. Then $n > 0$, and $\frac{1}{n} > 0$ by [Theorem 1.1.2](#) (f). By [Theorem 1.4.1](#) ($\Rightarrow$) with $\varepsilon = \frac{1}{n}$, there is $a_n \in A$ with $a_n > \sup A - \frac{1}{n}$, and $a_n \leq \sup A$ by (S1). ■

The corollary gives, for each n separately, an element a_n . Choosing one for every n at once, so as to form a sequence (a_n) , is a use of the axiom of countable choice; we flag it and move on. Once limits are defined in Chapter 2, this says that $\sup A$ is the limit of a sequence from A . We call the move **sup approximation**: “ $\sup A - \frac{1}{n}$ is not an upper bound, so some $a_n \in A$ exceeds it.”

1.4.3 Monotonicity and scaling

We now compute suprema of sets built from other sets. The first rule says that enlarging a set cannot lower its supremum.

Theorem 1.4.3. Monotonicity of the Supremum

Let $A \subseteq B \subseteq \mathbb{R}$, where A is nonempty and B is bounded above. Then $\sup A$ and $\sup B$ exist, and $\sup A \leq \sup B$.

Proof. Since A is nonempty and $A \subseteq B$, the set B is nonempty; as it is bounded above, $\sup B$ exists by [Axiom 1.3.1](#). Every $a \in A$ lies in B , so $a \leq \sup B$ by (S1) for B . Hence $\sup B$ is an upper bound of A . So A is nonempty and bounded above, $\sup A$ exists by [Axiom 1.3.1](#), and $\sup A \leq \sup B$ by (S2) for A . ■

Strict inclusion does not give strict inequality: $(0, 1) \subsetneq (0, 1]$, and both have supremum 1. The hypothesis that A is nonempty cannot be dropped, since $\emptyset \subseteq B$ always and $\emptyset$ has no supremum.

For scaling we need the scaled set. For $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, let

$$cA := \{ca : a \in A\}.$$

For example $2(0, 1] = (0, 2]$ and $(-1)A = -A$.

Theorem 1.4.4. Scaling the Supremum

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$.

- (a) If $c > 0$, then cA is nonempty and bounded above, and $\sup(cA) = c \sup A$.
- (b) If $c < 0$, then cA is nonempty and bounded below, and $\inf(cA) = c \sup A$.

Idea. For (a), prove the equality as two inequalities. Multiplying the inequality $a \leq \sup A$ by $c > 0$ shows that $c \sup A$ is an upper bound of cA , which gives $\leq$. For $\geq$, undo the scaling: dividing by c turns an upper bound of cA into an upper bound of A . No ε is needed, because multiplication by $c > 0$ is invertible and preserves order. For (b), a negative factor reverses order, so write $c = -|c|$ and let [Theorem 1.3.2](#) handle the sign.

Proof. Let $s := \sup A$, which exists by [Axiom 1.3.1](#). Since A is nonempty, so is cA .

- (a) Let $c > 0$. For every $a \in A$, $a \leq s$ by (S1), so $ca \leq cs$ by [Theorem 1.1.2](#) (c) (or trivially if $a = s$). Hence cs is an upper bound of cA , so $\sup(cA)$ exists by [Axiom 1.3.1](#), and $t := \sup(cA) \leq cs$ by (S2). For the reverse inequality, let $a \in A$. Then $ca \in cA$, so $ca \leq t$. Multiplying by $\frac{1}{c} > 0$ ([Theorem 1.1.2](#) (f) and (c)) gives $a \leq \frac{t}{c}$. Hence $\frac{t}{c}$ is an upper bound of A , and (S2) gives $s \leq \frac{t}{c}$. Multiplying by $c > 0$ gives $cs \leq t$. By trichotomy, $\sup(cA) = cs$.
- (b) Let $c < 0$, so $-c > 0$. The set $-(cA) = \{-ca : a \in A\}$ equals $(-c)A$, which by (a) is nonempty and bounded above with $\sup((-c)A) = -cs$. By [Theorem 1.3.2](#) (a),

$cA = -((-c)A)$ is bounded below, and by [Theorem 1.3.2](#),

$$\inf(cA) = -\sup(-(-c)A) = -\sup((-c)A) = -(-cs) = cs.$$

This proves the theorem. ■

For $c = 0$ the set cA is $\{0\}$, and $\sup(0A) = 0 = 0 \cdot \sup A$ as well. Part (a) genuinely needs $c > 0$: with $c = -1$ and $A = (0, 1]$, the set $cA = [-1, 0)$ has supremum 0, not $c\sup A = -1$. Part (b) says what happens instead: a negative factor exchanges supremum and infimum, and indeed $\inf[-1, 0) = -1$.

1.4.4 The supremum of a sum

For $A, B \subseteq \mathbb{R}$, let

$$A + B := \{a + b : a \in A, b \in B\}.$$

For instance $(0, 1) + \{5\} = (5, 6)$ and $[0, 1] + [0, 2] = [0, 3]$. It is natural to expect $\sup(A + B) = \sup A + \sup B$. This time the reverse inequality cannot be obtained by undoing an operation: a single element of $A + B$ mixes one element of each set. We need the ε characterization.

Theorem 1.4.5. Supremum of a Sum

Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded above. Then $A + B$ is nonempty and bounded above, and

$$\sup(A + B) = \sup A + \sup B.$$

Try it first (ten minutes): the inequality $\leq$ takes one line. Try to prove $\geq$.

Idea. The attempt that comes first to mind takes $a = \sup A$ and $b = \sup B$: then $a + b \in A + B$, and (S1) for $\sup(A + B)$ gives $\geq$ at once. It fails because a supremum need not belong to its set: for $A = B = (0, 1)$, the number 2 is not in $A + B = (0, 2)$. (The last exercise of this section takes that attempt apart.) The repair is to use elements that are only **close** to the suprema, which is what [Theorem 1.4.1](#) ($\Rightarrow$) supplies.

So we prove the equality as **two inequalities**.

($\leq$) Every $a + b$ is at most $\sup A + \sup B$, so the right side is an upper bound, and the supremum is at most any upper bound.

($\geq$) Work backwards. To show that $\sup(A + B)$ is at least $\sup A + \sup B$, it suffices to find, for each $\varepsilon > 0$, an element of $A + B$ above $\sup A + \sup B - \varepsilon$. The natural candidate is $a + b$ with a close to $\sup A$ and b close to $\sup B$. If a falls short of $\sup A$ by less than $\varepsilon/2$, and b falls short of $\sup B$ by less than $\varepsilon/2$, the total shortfall is less than ε . So we split the budget ε into two halves, one for each set, and ask [Theorem 1.4.1](#) for an element within $\varepsilon/2$ of each supremum. This is the $\varepsilon/2$ **argument**. The number $\varepsilon/2$ depends only on ε ; the elements a and b depend on ε . *Before reading on: the proof asks for elements within $\varepsilon/2$ of each supremum. What would it give if it asked for ε instead, and would that still be enough?*

Proof. By [Axiom 1.3.1](#), $\sup A$ and $\sup B$ exist. Choosing $a_0 \in A$ and $b_0 \in B$ gives $a_0 + b_0 \in A + B$, so $A + B$ is nonempty.

($\leq$) Let $a \in A$ and $b \in B$. By (S1), $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$ by [Theorem 1.1.2](#) (b). Hence $\sup A + \sup B$ is an upper bound of $A + B$. By [Axiom 1.3.1](#), $\sup(A + B)$ exists, and by (S2), $\sup(A + B) \leq \sup A + \sup B$.

($\geq$) Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. By [Theorem 1.4.1](#) ($\Rightarrow$), there are $a \in A$ and $b \in B$ (both depending on ε) with

$$a > \sup A - \frac{\varepsilon}{2} \quad \text{and} \quad b > \sup B - \frac{\varepsilon}{2}.$$

Adding, by [Theorem 1.1.2](#) (b), gives $a + b > \sup A + \sup B - \varepsilon$. Since $a + b \in A + B$, (S1) gives

$$\sup(A + B) \geq a + b > \sup A + \sup B - \varepsilon.$$

This holds for every $\varepsilon > 0$. If we had $\sup(A + B) < \sup A + \sup B$, then the choice $\varepsilon := \sup A + \sup B - \sup(A + B) > 0$ would give $\sup(A + B) > \sup(A + B)$, which is absurd. Hence $\sup(A + B) \geq \sup A + \sup B$.

The two inequalities and trichotomy give $\sup(A + B) = \sup A + \sup B$. ■

The answer to the prompt: with ε in place of $\varepsilon/2$ the proof gives $\sup(A + B) > \sup A + \sup B - 2\varepsilon$ for every $\varepsilon > 0$, and that is still enough, since 2ε runs through all positive numbers as ε does. Halving is a convenience: it makes the final inequality come out with exactly ε , which is the form [Theorem 1.4.1](#) ($\Leftarrow$) asks for.

The halves of the proof use the two halves of the definition. The first inequality used only that each supremum is an upper bound (S1) and that $\sup(A + B)$ is least (S2). The second used that each of $\sup A$ and $\sup B$ is least, in its ε form, and that $\sup(A + B)$ is an upper bound. We call the pattern **equality by two inequalities**: prove $\leq$ by exhibiting an upper bound, and $\geq$ with an ε .

Every hypothesis is needed. If $B = \emptyset$, then $A + B = \emptyset$ has no supremum. If B is not bounded above, neither is $A + B$ ([Exercise \(C2: Exit check: sums of sets\)](#)). And the analogous rule for differences is **not** $\sup(A - B) = \sup A - \sup B$: the correct version subtracts an infimum ([Exercise \(B3: Supremum of a difference\)](#)).

Warning. Proving one inequality only. Here is a tempting “proof” of [Theorem 1.4.5](#): “For $a \in A$ and $b \in B$, $a + b \leq \sup A + \sup B$. Therefore $\sup(A + B) = \sup A + \sup B$.” The first sentence proves only that $\sup A + \sup B$ is **an** upper bound of $A + B$, which gives $\leq$. It says nothing about being the **least** one: the same sentence with $\sup A + \sup B + 1$ is equally true, and would “prove” $\sup(A + B) = \sup A + \sup B + 1$. Whenever a supremum is claimed to equal something, both (S1) and (S2), or both inequalities, must appear.

1.4.5 Exercises

1.4.5.1 A. Check your understanding

Exercise. A1

- (a) Use [Theorem 1.4.1](#) to confirm that $\sup[0, 5) = 5$. Give the witness a as a formula in ε .
- (b) Use [Theorem 1.4.1](#) to confirm that $\sup\{3\} = 3$. Does your witness depend on ε ? Why not?
- (c) Use [Theorem 1.4.1](#) to confirm that $\sup((-1, 1) \cup (2, 4)) = 4$.
- (d) Let s be an upper bound of A . Write in symbols, using the ε characterization, the statement “ s is **not** the supremum of A ”.
- (e) Run the $(\geq)$ half of the proof of [Theorem 1.4.5](#) on $A = (0, 1)$, $B = \{5\}$, and $\varepsilon = \frac{1}{10}$: give an a and a b the proof could use, and check the inequality it derives.
- (f) In the proof of [Theorem 1.4.5](#), which line can no longer be justified if B is allowed to be empty?

Solution.

- (a) 5 is an upper bound of $[0, 5)$. Given $\varepsilon > 0$, let $a := \max\{0, 5 - \frac{\varepsilon}{2}\}$. Then $a \geq 0$, and $a < 5$ because both 0 and $5 - \frac{\varepsilon}{2}$ are less than 5; so $a \in [0, 5)$. Also $a \geq 5 - \frac{\varepsilon}{2} > 5 - \varepsilon$. By [Theorem 1.4.1](#), $\sup[0, 5) = 5$.
- (b) 3 is an upper bound of $\{3\}$, and for every $\varepsilon > 0$ the element $a = 3$ satisfies $3 > 3 - \varepsilon$. The witness does not depend on ε because 3 belongs to the set: by the Quick check above, a single witness for all ε exists exactly when the supremum is a maximum.
- (c) Every element is less than 4, so 4 is an upper bound. Given $\varepsilon > 0$, let $a := \max\{3, 4 - \frac{\varepsilon}{2}\}$. Then $3 \leq a < 4$, so $a \in (2, 4)$, and $a \geq 4 - \frac{\varepsilon}{2} > 4 - \varepsilon$. By [Theorem 1.4.1](#) the supremum is 4.
- (d) $\exists \varepsilon > 0 \ \forall a \in A : a \leq s - \varepsilon$. In words: some band below s contains no point of A , so $s - \varepsilon$ is a smaller upper bound.
- (e) Here $\sup A = 1$, $\sup B = 5$, and $\frac{\varepsilon}{2} = \frac{1}{20}$. The proof needs $a \in (0, 1)$ with $a > 1 - \frac{1}{20} = \frac{19}{20}$, for instance $a = \frac{24}{25}$, and $b \in B$ with $b > 5 - \frac{1}{20}$, which forces $b = 5$. Then $a + b = \frac{149}{25} = 5.96 > 5.9 = \sup A + \sup B - \varepsilon$. Any $a \in (\frac{19}{20}, 1)$ works; the one b works for every ε because 5 is the maximum of B .
- (f) The first line, “By [Axiom 1.3.1](#), $\sup A$ and $\sup B$ exist”. The axiom needs B nonempty, and $\emptyset$ has no supremum. The line that chooses $b \in B$ in the $(\geq)$ half fails too, since there is nothing to choose.

1.4.5.2 B. Practice**Exercise. B1: A computed supremum**

Prove that $\sup\{x^2 : x \in (-1, 2)\} = 4$. Hence show that the set has no maximum.

Solution. Let $A = \{x^2 : x \in (-1, 2)\}$. *Upper bound.* Let $x \in (-1, 2)$. If $x \geq 0$, then $x < 2$ gives $x^2 < 4$ by [Theorem 1.1.2](#) (g). If $-1 < x < 0$, then $|x| < 1$ by [Proposition 1.1.3](#) (d), so $x^2 = |x|^2 < 1 < 4$ by [Theorem 1.1.2](#) (g). So 4 is an upper bound, and $4 \notin A$.

Band condition. Working backwards as in [Example \(Confirming a supremum with epsilon\)](#): with $x = 2 - h$ we have $x^2 = 4 - 4h + h^2 > 4 - 4h$, so $h \leq \frac{\varepsilon}{4}$ suffices, provided $0 < h < 3$ to

keep $x \in (-1, 2)$. So let $\varepsilon > 0$ and $h := \min\{1, \frac{\varepsilon}{4}\}$, which depends only on ε . Then $0 < h \leq 1$, so $x := 2 - h \in [1, 2) \subseteq (-1, 2)$, and since $h^2 > 0$,

$$x^2 = 4 - 4h + h^2 > 4 - 4h \geq 4 - \varepsilon.$$

By [Theorem 1.4.1](#), $\sup A = 4$. Since $4 \notin A$, the set has no maximum, by [Exercise \(B1: Maxima are suprema\)](#) (a).

Exercise. B2: Supremum of a union

Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded above. Prove that $A \cup B$ is bounded above and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Solution. Let $m := \max\{\sup A, \sup B\}$; both suprema exist by [Axiom 1.3.1](#). If $x \in A \cup B$, then $x \leq \sup A \leq m$ or $x \leq \sup B \leq m$, so m is an upper bound of $A \cup B$, which is nonempty. By symmetry we may assume $m = \sup A$. Given $\varepsilon > 0$, [Theorem 1.4.1](#) ($\Rightarrow$) gives $a \in A$ with $a > \sup A - \varepsilon = m - \varepsilon$, and $a \in A \cup B$. By [Theorem 1.4.1](#) ($\Leftarrow$), $\sup(A \cup B) = m$.

Exercise. B3: Supremum of a difference

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and $B \subseteq \mathbb{R}$ nonempty and bounded below. Prove that

$$\sup\{a - b : a \in A, b \in B\} = \sup A - \inf B.$$

Give an example showing that the right side cannot be replaced by $\sup A - \sup B$.

Solution. The set in question is $A + (-B)$. By [Theorem 1.3.2](#) (a), $-B$ is bounded above, and it is nonempty. By [Theorem 1.3.3](#), $\inf B = -\sup(-B)$, that is, $\sup(-B) = -\inf B$. So [Theorem 1.4.5](#) applies to A and $-B$:

$$\sup(A + (-B)) = \sup A + \sup(-B) = \sup A - \inf B.$$

For the example, take $A = \{0\}$ and $B = [0, 1]$. The differences $0 - b$ fill $[-1, 0]$, whose supremum is $0 = \sup A - \inf B$, whereas $\sup A - \sup B = -1$.

1.4.5.3 C. Going deeper

Exercise. C1: Products of sets

For $A, B \subseteq \mathbb{R}$ let $AB := \{ab : a \in A, b \in B\}$.

- Show that $\sup(AB) = \sup A \cdot \sup B$ can fail for nonempty bounded sets A, B .
- Prove that it holds when A and B are nonempty, bounded above, and contained in $(0, \infty)$.

Solution (Hint 1). For (a), try sets of negative numbers. For (b), the inequality $\leq$ is routine; for $\geq$, fix one element $b \in B$ and compare the set bA with AB .

Solution (Hint 2). For $\geq$ in (b): fix $b \in B$. Which theorem computes $\sup(bA)$, and which compares the suprema of two sets, one inside the other? What does the inequality you get say about b , once b varies?

Solution.

(a) Take $A = B = [-2, -1]$. Then $\sup A \cdot \sup B = (-1)(-1) = 1$, but $4 = (-2)(-2) \in AB$, so $\sup(AB) \geq 4 \neq 1$ by (S1).

(b) Let $s := \sup A$ and $t := \sup B$, which exist by [Axiom 1.3.1](#). Since A contains a positive element and s is an upper bound, $s > 0$; likewise $t > 0$.

($\leq$) For $a \in A$ and $b \in B$, we have $0 < a \leq s$ and $0 < b \leq t$, so $ab \leq sb \leq st$ by [Theorem 1.1.2](#)

(c) (with trivial equality cases). So st is an upper bound of the nonempty set AB , $\sup(AB)$ exists, and $\sup(AB) \leq st$ by (S2).

($\geq$) Fix $b \in B$. Then $bA \subseteq AB$ and $b > 0$, so [Theorem 1.4.4](#) and [Theorem 1.4.3](#) give $bs = \sup(bA) \leq \sup(AB)$. Multiplying by $\frac{1}{s} > 0$ gives $b \leq \frac{\sup(AB)}{s}$. As $b \in B$ was arbitrary, $\frac{\sup(AB)}{s}$ is an upper bound of B , so $t \leq \frac{\sup(AB)}{s}$ by (S2), that is, $st \leq \sup(AB)$.

By trichotomy, $\sup(AB) = st$. Positivity was used in ($\leq$), to multiply inequalities, and in ($\geq$), to scale by b and divide by s ; part (a) shows it cannot be dropped.

Exercise. C2: Exit check: sums of sets

Let $A, B \subseteq \mathbb{R}$ be nonempty.

(a) Assume A and B are bounded above. Without looking back, prove $\sup(A + B) = \sup A + \sup B$. This time, instead of proving $\geq$ separately, show that $\sup A + \sup B$ satisfies both hypotheses of [Theorem 1.4.1](#) for the set $A + B$.

(b) Assume A and B are bounded below. State and prove the corresponding formula for $\inf(A + B)$.

(c) Show that if B is not bounded above, then neither is $A + B$.

Solution.

(a) Let $s := \sup A + \sup B$, where both suprema exist by [Axiom 1.3.1](#). *Upper bound:* for $a \in A$, $b \in B$, (S1) and [Theorem 1.1.2](#) (b) give $a + b \leq s$. *Band condition:* let $\varepsilon > 0$; then $\frac{\varepsilon}{2} > 0$, and [Theorem 1.4.1](#) ($\Rightarrow$) gives $a \in A$ and $b \in B$, depending on ε , with $a > \sup A - \frac{\varepsilon}{2}$ and $b > \sup B - \frac{\varepsilon}{2}$. Adding, $a + b > s - \varepsilon$, and $a + b \in A + B$. By [Theorem 1.4.1](#) ($\Leftarrow$), $s = \sup(A + B)$.

(b) $\inf(A + B) = \inf A + \inf B$. Indeed $-(A + B) = (-A) + (-B)$, since $-(a + b) = (-a) + (-b)$. By [Theorem 1.3.2](#) (a), $-A$ and $-B$ are nonempty and bounded above,

so (a) and [Theorem 1.3.3](#) give

$$\begin{aligned}\inf(A + B) &= -\sup((-A) + (-B)) \\ &= -\sup(-A) - \sup(-B) = \inf A + \inf B.\end{aligned}$$

Here $A + B$ is bounded below because $-(A + B)$ is bounded above, by (a) and [Theorem 1.3.2](#) (a).

- (c) Fix $a_0 \in A$ and let $M \in \mathbb{R}$. Since B is not bounded above, $M - a_0$ is not an upper bound of B , so some $b \in B$ has $b > M - a_0$. Then $a_0 + b \in A + B$ and $a_0 + b > M$. As M was arbitrary, $A + B$ is not bounded above.

Marking scheme. (10 points) - 1: (a) says why $\sup A$, $\sup B$ exist and why $A + B$ is nonempty. - 2: (a) shows $\sup A + \sup B$ is an upper bound of $A + B$, adding the two (S1) inequalities. - 3: (a) fixes an arbitrary $\varepsilon > 0$, obtains $a \in A$ and $b \in B$, **depending on** ε , within $\varepsilon/2$ of their suprema, and concludes with [Theorem 1.4.1](#) ($\Leftarrow$). - 2: (b) reduces to (a) through $-(A + B) = (-A) + (-B)$ and [Theorem 1.3.3](#), including why $A + B$ is bounded below. - 2: (c) produces, for an arbitrary M , an element of $A + B$ above M . Fatal errors (0 for the step): taking $a = \sup A \in A$; choosing one a before ε is fixed; proving only $\leq$ in (a). A solution of (a) that proves $\leq$ and $\geq$ separately, as in [Theorem 1.4.5](#), earns the 5 points of the second and third lines if the $\geq$ half is complete. A solution of (b) that repeats the argument of (a) with lower bounds earns the (b) points.

Exercise. C3: Find the wrong line

Each “proof” below contains a line that is false in general. Find it, give a set for which it fails, and decide whether the claim itself is true.

- (a) *Claim.* If $A \subseteq \mathbb{R}$ is nonempty and bounded above and $a < 1$ for every $a \in A$, then $\sup A < 1$. “*Proof.*” (1) Let $s = \sup A$. (2) Since $s \in A$, the hypothesis gives $s < 1$. (3) Hence $\sup A < 1$.
- (b) *Claim.* If $A, B \subseteq \mathbb{R}$ are nonempty and bounded above, then $\sup(A+B) \geq \sup A + \sup B$. “*Proof.*” (1) Let $a = \sup A$ and $b = \sup B$. (2) Then $a \in A$ and $b \in B$, so $a + b \in A + B$. (3) By (S1), $a + b \leq \sup(A + B)$, which is the claim.

Solution.

- (a) Line (2) is wrong: it assumes $\sup A \in A$, which fails for $A = (0, 1)$, where $\sup A = 1 \notin A$. For that set the claim is **false**: every element is less than 1, and $\sup A = 1 \not< 1$. The correct conclusion is $\sup A \leq 1$, which follows from (S2) because 1 is an upper bound.
- (b) Line (2) is wrong for the same reason: for $A = B = (0, 1)$, neither supremum belongs to its set, so $a + b = 2$ need not lie in $A + B = (0, 2)$, and here it does not. The claim is **true**, since it is half of [Theorem 1.4.5](#), but this argument does not prove it; a correct argument for this inequality is the $\varepsilon/2$ argument in the proof of [Theorem 1.4.5](#). The flawed line is valid exactly when A and B have maxima, which is the situation the author of the “proof” silently imagined.

1.5 The Archimedean Property

We now have the least upper bound axiom ([Axiom 1.3.1](#)) and a way to use it, the characterization of the supremum by ε ([Theorem 1.4.1](#)). This section draws the first consequence, one that looks too obvious to need proof: the natural numbers eventually pass every real number. When we first met suprema, we claimed that $\sup\{1 - 1/n : n \in \mathbb{Z}_{>0}\} = 1$ and postponed the proof, because it needs exactly this fact. Here we prove the fact from the axiom, pay off that claim, and then meet an ordered field where the fact is false, so that we can see which line of the proof the axiom is holding up.

1.5.1 Where the natural numbers live

So far $\mathbb{R}$ is an ordered field satisfying one extra axiom. The numbers $0, 1, 2, 3, \dots$ are in it, since 2 is $1 + 1$, 3 is $2 + 1$, and so on. But “and so on” is not a definition. We want the set you get by starting at 0 and adding 1 again and again, and nothing else. The first half of that description is easy to state as a property of a set.

Definition 1.5.1. Inductive set

A subset $S \subseteq \mathbb{R}$ is **inductive** if

- (N1) $0 \in S$, and
- (N2) **for every** $x \in S$, also $x + 1 \in S$.

In words: an inductive set contains 0 , and it is closed under the step “add one”. Nothing stops it from containing other numbers as well.

Some examples, checked clause by clause.

- $\mathbb{R}$ itself is inductive: it contains 0 , and $x + 1$ is a real number whenever x is.
- $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$ is inductive: $0 \geq 0$, and $x \geq 0$ gives $x + 1 \geq 0 + 1 > 0$.
- $\{0\} \cup [1, \infty)$ is inductive. From 0 we step to 1 , which is in $[1, \infty)$, and $[1, \infty)$ is closed under adding 1 .

A nonexample by minimal change: $[0, 5]$ contains 0 , so (N1) holds, but $4.5 \in [0, 5]$ and $5.5 \notin [0, 5]$, so (N2) fails. Chopping the top off $[0, \infty)$ destroys closure under the step.

The inductive sets above are too big: $[0, \infty)$ contains 1.5 and 4.7 . What we want is the *smallest* one. Intersecting all of them throws away everything that some inductive set can do without.

The set of **natural numbers** is

$$\mathbb{N} := \{x \in \mathbb{R} : x \in S \text{ for every inductive set } S \subseteq \mathbb{R}\},$$

that is, the real numbers that belong to every inductive subset of $\mathbb{R}$.

Two things are true of $\mathbb{N}$, and both follow at once from the definition.

- **$\mathbb{N}$ is inductive.** Every inductive set contains 0 , so $0 \in \mathbb{N}$. If $x \in \mathbb{N}$, then x lies in every inductive set S , and each such S then contains $x + 1$ by (N2); so $x + 1 \in \mathbb{N}$.
- **$\mathbb{N}$ is contained in every inductive set**, since it is their intersection.

The second property is the **principle of induction** (Logic and Proof, Axiom 4.2.1) in disguise. To prove that a statement $P(n)$ holds for every $n \in \mathbb{N}$, let $S := \{n \in \mathbb{N} : P(n)\}$. If $P(0)$ holds and $P(n)$ implies $P(n + 1)$, then S is inductive, so $\mathbb{N} \subseteq S$, and $P(n)$ holds for all $n \in \mathbb{N}$.

Example. Every nonzero natural number is at least 1

Show that $n \geq 1$ for every $n \in \mathbb{N}$ with $n \neq 0$, and deduce that for such n , $1/n$ is defined and $0 < 1/n \leq 1$.

Solution. The set $\{0\} \cup [1, \infty)$ is inductive, as checked above. Since $\mathbb{N}$ is contained in every inductive set, $\mathbb{N} \subseteq \{0\} \cup [1, \infty)$, that is, $n \geq 1$ for every $n \in \mathbb{N}$ with $n \neq 0$. In particular such an n is positive, so $1/n$ is defined. From $0 < 1 \leq n$ [Theorem 1.1.2](#) (f) gives $0 < 1/n \leq 1/1 = 1$.

The proof never mentioned a single natural number by name. It found an inductive set containing the property we wanted, and let the definition of $\mathbb{N}$ do the rest. That is the whole technique.

The nonzero natural numbers are the **positive integers**,

$$\mathbb{Z}_{>0} := \{n \in \mathbb{N} : n \neq 0\} = \{n \in \mathbb{N} : n \geq 1\},$$

the two descriptions agreeing by [Example \(Every nonzero natural number is at least 1\)](#). This is the set $\{1, 2, 3, \dots\}$ that indexed our examples informally before; from here on it is a defined subset of $\mathbb{R}$, and $1/n$ is defined and positive for every $n \in \mathbb{Z}_{>0}$.

Remark. *Logic and Proof*, Chapter 3 takes induction as a basic property of $\mathbb{N}$ (*Logic and Proof*, Axiom 4.2.1), and *Sets, Relations, and Cardinality*, Chapter 2 proves well-ordering and the arithmetic below from it. The set defined here is that same $\mathbb{N}$, now located inside $\mathbb{R}$; it is also the copy $0, 1, 1+1, 1+1+1, \dots$ that every ordered field contains ([Exercise \(C1: Ordered fields contain the counting numbers\)](#)), and “ $\mathbb{N}$ is the smallest inductive set” is exactly the induction principle of *Logic and Proof*, Chapter 3, in the form “every inductive subset of $\mathbb{N}$ is all of $\mathbb{N}$ ”. From now on we use the familiar arithmetic of $\mathbb{N}$, each item proved by induction in *Sets, Relations, and Cardinality*, Chapter 2: sums and products of natural numbers are natural numbers; if $n \in \mathbb{N}$ and $n \geq 1$, then $n-1 \in \mathbb{N}$; no natural number lies strictly between n and $n+1$; and every nonempty subset of $\mathbb{N}$ has a least element (**well-ordering**). The integers are $\mathbb{Z} := \mathbb{N} \cup \{-n : n \in \mathbb{N}\}$, and the rationals are $\mathbb{Q} := \{m/n : m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}\}$, with $\mathbb{Z}_{>0}$ the positive integers defined above. We also use, again from *Sets, Relations, and Cardinality*, Chapter 2, that sums, differences, and products of integers are integers. The integers $n > 0$ are exactly the elements of $\mathbb{Z}_{>0}$: such an integer is not $-m$ with $m \in \mathbb{N}$, since $-m \leq 0$, so it is a nonzero natural number; conversely, a nonzero natural number is > 0 by [Example \(Every nonzero natural number is at least 1\)](#). So $\mathbb{Z}_{>0} = \{n \in \mathbb{Z} : n > 0\}$.

1.5.2 The natural numbers are not bounded

Is $\mathbb{N}$ bounded above in $\mathbb{R}$? Everyone’s picture of the number line says no: the marks $1, 2, 3, \dots$ march off to the right forever. But a picture is not a proof, and here the picture hides something. Nothing in the ordered-field axioms ([Definition 1.1.1](#)) forbids a number larger than every natural number. At the end of this section we build an ordered field that contains one. So if $\mathbb{R}$ has no such number, the reason must be the one axiom that $\mathbb{R}$ has and that field lacks.

Theorem 1.5.1. Archimedean Property

The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$; in other words, every real number x is exceeded by

some $n \in \mathbb{N}$.

Idea. Suppose $\mathbb{N}$ had an upper bound. Then the axiom hands us the least one, $s = \sup \mathbb{N}$. Every natural number n has a successor $n + 1$ that is still a natural number, and so still at most s . Moving back one step, $n \leq s - 1$ for every n : the smaller number $s - 1$ is an upper bound too. A least upper bound with a smaller upper bound below it is absurd.

Proof. Suppose, for contradiction, that $\mathbb{N}$ is bounded above. The set $\mathbb{N}$ is nonempty, since $0 \in \mathbb{N}$. By [Axiom 1.3.1](#), $\mathbb{N}$ has a supremum $s := \sup \mathbb{N} \in \mathbb{R}$.

Let $n \in \mathbb{N}$. Since $\mathbb{N}$ is inductive, $n + 1 \in \mathbb{N}$, and since s is an upper bound of $\mathbb{N}$, $n + 1 \leq s$, that is, $n \leq s - 1$. As n was arbitrary, $s - 1$ is an upper bound of $\mathbb{N}$. Property (S2) of the supremum ([Definition 1.2.4](#)) now gives $s \leq s - 1$, that is, $1 \leq 0$, which contradicts $0 < 1$.

Therefore $\mathbb{N}$ is not bounded above. Unwinding the definition ([Definition 1.2.2](#)), no $x \in \mathbb{R}$ is an upper bound of $\mathbb{N}$: for every $x \in \mathbb{R}$ it is false that $n \leq x$ for all $n \in \mathbb{N}$. So for every x some $n \in \mathbb{N}$ has $n \not\leq x$, that is, $n > x$ by trichotomy. This proves the theorem. ■

Each hypothesis of the axiom was used and named: “nonempty” through $0 \in \mathbb{N}$, “bounded above” through the assumption for contradiction. The inductive property of $\mathbb{N}$ supplied the jump $n \mapsto n + 1$.

Remark. The contradiction step is $\neg I$ (Logic and Proof, [Definition 2.2.1](#)), not RAA (Logic and Proof, [Definition 2.5.1](#)), the classical rule of proof by contradiction: we assumed “ $\mathbb{N}$ is bounded above”, reached a contradiction, and concluded its negation. Inside that box every step is direct. The proof shows that $s - 1$ is an upper bound; it does not argue “ $s - 1$ is not an upper bound, so some n exceeds $s - 1$ ”, a step that would itself need the classical law described next. The proof’s one classical step comes after the $\neg I$, in its last paragraph. Passing from “not every n satisfies $n \leq x$ ” to “some n satisfies $n > x$ ” is the quantifier law “not all” gives “some not” of *Logic and Proof*, Chapter 4, and that law is derived there with RAA.

The form we will use most often puts a small number, not a large one, in the role of x .

Corollary 1.5.2

For every $\varepsilon > 0$ there is $n \in \mathbb{Z}_{>0}$ with $1/n < \varepsilon$. In symbols,

$$(\forall \varepsilon > 0) (\exists n \in \mathbb{Z}_{>0}) \frac{1}{n} < \varepsilon.$$

Moreover, every $m \in \mathbb{Z}_{>0}$ with $m \geq n$ then also satisfies $1/m < \varepsilon$.

Proof. Let $\varepsilon > 0$. Then $1/\varepsilon$ is defined and positive. By [Theorem 1.5.1](#) applied to $x = 1/\varepsilon$, there is $n \in \mathbb{N}$ with $n > 1/\varepsilon > 0$; so $n \in \mathbb{Z}_{>0}$. Taking reciprocals reverses a strict inequality

between positive numbers (Theorem 1.1.2 (f)), so $1/n < \varepsilon$. If $m \geq n$, then $1/m \leq 1/n < \varepsilon$ by the same rule. ■

Read the quantifiers in order. The ε is given first and the n is found afterwards, so n **depends on** ε : the proof chose $n > 1/\varepsilon$, and a smaller ε forces a larger n . Swapping the quantifiers gives a false statement. “There is $n \in \mathbb{Z}_{>0}$ such that $1/n < \varepsilon$ for every $\varepsilon > 0$ ” would make $1/n$ smaller than every positive number. Since $1/n > 0$ (Example (Every nonzero natural number is at least 1)), we could take $\varepsilon = 1/n$ itself and get $1/n < 1/n$, which is absurd.

The “moreover” clause says that once one n works, every later one works too. Chapter 2 turns this “from some point on” into the definition of a limit.

Quick check. Find the least $n \in \mathbb{Z}_{>0}$ with $1/n < 0.003$. Then do the same for $\varepsilon = 0.0003$. What does the comparison say about the dependence of n on ε ?

Solution. For positive n , $1/n < 0.003$ is equivalent to $n > 1/0.003 = 333.\bar{3}$. The least positive integer with this property is $n = 334$; indeed $1/333 = 0.003003\dots$ is too big. For $\varepsilon = 0.0003$ the condition is $n > 3333.\bar{3}$, so the least n is 3334. Dividing ε by 10 multiplied the least n by about 10. No single n serves every ε , which is why the quantifiers read “for every ε there is n ” and not the other way round.

1.5.3 Paying an old debt

In Example (A supremum we cannot yet prove) we showed that 1 is an upper bound of $A = \{1 - 1/n : n \in \mathbb{Z}_{>0}\}$ and that A has no maximum. We stopped short of $\sup A = 1$, because (S2) needed an n with $1/n$ below a given positive number. That is exactly what Corollary 1.5.2 supplies, and the ε -characterization tells us what remains to be shown.

Example. The supremum is 1

Let $A := \{1 - 1/n : n \in \mathbb{Z}_{>0}\}$. Prove that $\sup A = 1$.

Solution. The set A is nonempty, since $0 = 1 - 1/1 \in A$.

Upper bound. For every $n \in \mathbb{Z}_{>0}$ we have $1/n > 0$ (Example (Every nonzero natural number is at least 1)), so $1 - 1/n < 1$. Hence 1 is an upper bound of A .

An element in every band below 1. Let $\varepsilon > 0$. By Corollary 1.5.2 there is $n \in \mathbb{Z}_{>0}$ with $1/n < \varepsilon$; this n depends on ε . Then $a := 1 - 1/n \in A$ and $a > 1 - \varepsilon$.

Since 1 is an upper bound of A and every band $(1 - \varepsilon, 1]$ contains an element of A , Theorem 1.4.1 gives $\sup A = 1$.

For a concrete instance, $\varepsilon = 0.01$ needs $n > 100$, and $n = 101$ gives the element $1 - 1/101 = 100/101 > 0.99$. Together with Example (A supremum we cannot yet prove), A is a set whose supremum 1 is not attained.

The pattern of this example, “upper bound, then an element in every band below it”, is the one every supremum computation in this book follows. The only new input here was [Corollary 1.5.2](#), which turns “ $1/n$ is small” into a precise choice of n .

1.5.4 An ordered field without the Archimedean property

We claimed that the Archimedean property does not follow from the ordered-field axioms alone. Now we back that up with a concrete field containing an element larger than every natural number. The field is made of fractions of polynomials, and the element is the variable x itself.

A **rational function** is a quotient p/q of real polynomials p and q with q not the zero polynomial. Two quotients p/q and p'/q' are the same element when $pq' = p'q$, exactly as for fractions of integers. With the usual addition and multiplication of fractions, the rational functions form a field, written $\mathbb{R}(x)$. This is the same construction that builds $\mathbb{Q}$ out of $\mathbb{Z}$, applied to polynomials; it is algebra, and we take it as known without proof here. The real numbers sit inside $\mathbb{R}(x)$ as the constant functions.

The **leading coefficient** of a nonzero polynomial is the coefficient of its highest power of x . For instance, the leading coefficient of $3 - 2x^2$ is -2 .

Call a nonzero $p/q \in \mathbb{R}(x)$ **positive** if the leading coefficients of p and q are both positive or both negative, and write $f < g$ when $g - f$ is positive.

Here is the picture behind this definition. For large real t , the value $p(t)/q(t)$ has the sign of the product of the two leading coefficients, because the top powers dominate. So “ $f > 0$ ” means “ $f(t) > 0$ for all sufficiently large t ”, and $f < g$ means that the graph of g eventually lies above the graph of f . This is only a picture: the definition uses leading coefficients and nothing else.

Positivity does not depend on how the fraction is written. Suppose $p/q = p'/q'$, so that $pq' = p'q$, and let $\alpha, \beta, \alpha', \beta'$ be the leading coefficients of p, q, p', q' . The leading coefficient of a product of polynomials is the product of their leading coefficients, so $\alpha\beta' = \alpha'\beta$. Multiplying both sides by $\beta\beta'$ gives $\alpha\beta \cdot \beta'^2 = \alpha'\beta' \cdot \beta^2$, and $\beta^2, \beta'^2 > 0$. Hence $\alpha\beta$ and $\alpha'\beta'$ have the same sign: p/q is positive exactly when p'/q' is.

With this order $\mathbb{R}(x)$ is an ordered field. The key facts are that sums and products of positive elements are positive, and that for each nonzero f exactly one of f and $-f$ is positive. Checking these is routine, and [Exercise \(C2: The order on the rational functions\)](#) walks through them. On the constants, the new order is the old one: the constant c is positive exactly when $c > 0$. The example below uses that $\mathbb{R}(x)$ is an ordered field, and with it the rules of [Theorem 1.1.2](#), which hold in every ordered field; so it rests on [Exercise \(C2: The order on the rational functions\)](#), where that is proved.

The natural numbers of $\mathbb{R}(x)$, the smallest inductive subset in the sense of [Definition 1.5.1](#), are the constants $0, 1, 2, 3, \dots$. Indeed, the constant functions n with $n \in \mathbb{N}$ form an inductive subset of $\mathbb{R}(x)$: it contains the constant 0, and the constant n plus 1 is the constant $n + 1$. So every natural number of $\mathbb{R}(x)$ is such a constant. Conversely, let I be an inductive subset of $\mathbb{R}(x)$. The set of $n \in \mathbb{N}$ whose constant lies in I contains 0 and passes from n to $n + 1$, so it is an inductive subset of $\mathbb{N}$ and contains all of $\mathbb{N}$. So every constant n lies in every inductive subset of $\mathbb{R}(x)$.

Example. A non-Archimedean field

In $\mathbb{R}(x)$ with the order just described, show that $n < x$ for every $n \in \mathbb{N}$. Deduce that $\mathbb{N}$ is bounded above in $\mathbb{R}(x)$, and that [Corollary 1.5.2](#) fails in $\mathbb{R}(x)$.

Solution. Let $n \in \mathbb{N}$. Then $x - n = (x - n)/1$. The numerator $x - n$ has leading coefficient 1 and the denominator 1 has leading coefficient 1. They have the same sign, so $x - n$ is positive, that is, $n < x$.

Hence x is an upper bound of $\mathbb{N}$ in $\mathbb{R}(x)$, and $\mathbb{N}$ is bounded above there. So the conclusion of [Theorem 1.5.1](#) is false in $\mathbb{R}(x)$.

Now take $\varepsilon := 1/x$. It is positive: numerator 1 and denominator x both have leading coefficient 1. For each $n \in \mathbb{Z}_{>0}$ we have $0 < n < x$, so taking reciprocals gives $1/x < 1/n$. No positive integer n satisfies $1/n < \varepsilon$, and [Corollary 1.5.2](#) fails.

The element $1/x$ is positive yet smaller than every $1/n$: an “infinitely small” number. The real line has none, and that is exactly what [Corollary 1.5.2](#) says.

1.5.5 The line of the proof that fails

Every step in the proof of [Theorem 1.5.1](#), except one, is valid in any ordered field. So let us run the proof in $\mathbb{R}(x)$ and watch where it breaks.

- “Suppose $\mathbb{N}$ is bounded above.” In $\mathbb{R}(x)$ this is true, with upper bound x .
- “ $\mathbb{N}$ is nonempty.” True.
- “By [Axiom 1.3.1](#), $\mathbb{N}$ has a supremum s .” This is the line that fails. In $\mathbb{R}(x)$, $\mathbb{N}$ has no least upper bound at all. Indeed, let M be any upper bound of $\mathbb{N}$ in $\mathbb{R}(x)$. For each $n \in \mathbb{N}$, also $n + 1 \in \mathbb{N}$, so $n + 1 \leq M$, that is, $n \leq M - 1$. Hence $M - 1$ is also an upper bound, and it is strictly smaller than M . No upper bound is the least one.
- The remaining lines, “ $n + 1 \leq s$ for every $n \in \mathbb{N}$, so $s - 1$ is an upper bound, and (S2) gives $s \leq s - 1$ ”, use only the ordered-field axioms and the fact that $\mathbb{N}$ is inductive. They would go through in $\mathbb{R}(x)$ if there were an s to apply them to.

The argument that shows $\mathbb{N}$ has no supremum in $\mathbb{R}(x)$ is the middle paragraph of the proof of [Theorem 1.5.1](#), with M in place of s . That proof showed that a supremum of $\mathbb{N}$ cannot exist in any ordered field; the axiom then says that, in $\mathbb{R}$, a bounded $\mathbb{N}$ would have to have one. In $\mathbb{R}(x)$ there is no axiom to force it, so no contradiction arises. Two conclusions follow:

- $\mathbb{R}(x)$ is an ordered field in which [Axiom 1.3.1](#) fails, with witness the nonempty, bounded-above set $\mathbb{N}$.
- In any ordered field, the least upper bound axiom implies the Archimedean property. That is precisely what our proof showed.

Warning. The Archimedean property is not “obvious”, and it is not a consequence of the ordered-field axioms. It is exactly what $\mathbb{R}(x)$ lacks, where x exceeds every natural number. In $\mathbb{R}$ it holds only because of the least upper bound axiom. A proof that says “choose an integer larger than x ” is using [Theorem 1.5.1](#), and should cite it.

Remark. The converse fails: an ordered field can be Archimedean without satisfying the axiom. The field $\mathbb{Q}$ is one ([Exercise \(C3: Archimedean without the axiom\)](#)), and it fails the axiom by [Theorem 1.2.2](#). So the Archimedean property is strictly weaker than the axiom. Chapter 11 shows which properties, added to the Archimedean property, bring back the full axiom.

1.5.6 Exercises

1.5.6.1 A. Check your understanding

Exercise. A1

- Write the Archimedean property “for every $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ with $n > x$ ” in symbols, and write its negation in symbols.
- Which of the following subsets of $\mathbb{R}$ are inductive: $[\frac{1}{2}, \infty)$, $[0, 10]$, $\mathbb{N} \cup \{\frac{1}{2}\}$, $\mathbb{Z}$? Name the failing clause for each one that is not.
- True or false, with a reason: in [Corollary 1.5.2](#), one n can be chosen that works for every $\varepsilon > 0$.
- In the proof of [Theorem 1.5.1](#), which line uses the least upper bound axiom, and which line uses that $\mathbb{N}$ is inductive?

Solution.

- The property is $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N}) n > x$. Its negation is $(\exists x \in \mathbb{R})(\forall n \in \mathbb{N}) n \leq x$: some real number is an upper bound of $\mathbb{N}$.
- $[\frac{1}{2}, \infty)$ is not: $0 \notin [\frac{1}{2}, \infty)$, so (N1) fails, although $x \geq \frac{1}{2}$ gives $x + 1 \geq \frac{3}{2} \geq \frac{1}{2}$, so (N2) holds. $[0, 10]$ is not: it contains 0, but $10 \in [0, 10]$ and $11 \notin [0, 10]$, so (N2) fails. $\mathbb{N} \cup \{\frac{1}{2}\}$ is not: $\frac{1}{2}$ is in it but $\frac{3}{2}$ is not, since $\frac{3}{2}$ lies strictly between the natural numbers 1 and 2; so (N2) fails. $\mathbb{Z}$ is inductive: $0 \in \mathbb{Z}$, and $k + 1 \in \mathbb{Z}$ for every integer k .
- False. Such an n would satisfy $1/n < \varepsilon$ for every $\varepsilon > 0$. But $1/n > 0$ by [Example \(Every nonzero natural number is at least 1\)](#), so $\varepsilon = 1/n$ is allowed, and it gives $1/n < 1/n$, which is absurd. The n must depend on ε .
- The axiom is used in “ $\mathbb{N}$ has a supremum s ”. That $\mathbb{N}$ is inductive is used in “ $n + 1 \in \mathbb{N}$ ”.

1.5.6.2 B. Practice

Exercise. B1: An infimum

Prove that $\inf\{1/n : n \in \mathbb{Z}_{>0}\} = 0$, and that the set has no minimum.

Solution. Let $B := \{1/n : n \in \mathbb{Z}_{>0}\}$; it is nonempty, since $1 \in B$. By [Example \(Every nonzero natural number is at least 1\)](#), $1/n > 0$ for every n , so 0 is a lower bound of B .

Let L be any lower bound of B ; we show $L \leq 0$. Suppose instead $L > 0$. By [Corollary 1.5.2](#) with $\varepsilon = L$, there is $n \in \mathbb{Z}_{>0}$ with $1/n < L$. This contradicts L being a lower bound of B . Hence $L \leq 0$. So 0 is a lower bound that is greater than or equal to every lower bound, that

is, $\inf B = 0$ (Definition 1.2.5).

Since $0 \notin B$ (every element is positive), the infimum is not attained, and B has no minimum.

Exercise. B2: Scaled Archimedes

Let $a > 0$ and $y \in \mathbb{R}$. Prove that there is $n \in \mathbb{N}$ with $na > y$. What does this say when a is the length of a measuring stick?

Solution. Since $a > 0$, the number y/a is defined. By Theorem 1.5.1 there is $n \in \mathbb{N}$ with $n > y/a$; the n depends on y and a . Multiplying by $a > 0$ preserves the strict inequality (Theorem 1.1.2 (c)), so $na > y$.

In words: laying a stick of length a , however short, end to end enough times carries you past any given point y , however far away. This is the form in which Archimedes used the property.

Exercise. B3: Smaller than every $1/n$

Let $x \in \mathbb{R}$ with $0 \leq x < 1/n$ for every $n \in \mathbb{Z}_{>0}$. Prove that $x = 0$. Then deduce: if $a, b \in \mathbb{R}$ satisfy $|a - b| < 1/n$ for every $n \in \mathbb{Z}_{>0}$, then $a = b$.

Solution. Suppose $x \neq 0$. Since $x \geq 0$, we get $x > 0$. By Corollary 1.5.2 with $\varepsilon = x$, there is $n \in \mathbb{Z}_{>0}$ with $1/n < x$, contradicting the hypothesis $x < 1/n$. Hence $x = 0$.

For the second statement, put $x := |a - b|$. Then $0 \leq x < 1/n$ for every n , so $x = 0$ by the first part, that is, $a = b$.

This is how equality is proved in analysis over and over: show that the distance is nonnegative and smaller than every $1/n$.

Exercise. B4: Putting a proof in order

Here are nine lines. Seven of them, in a suitable order, prove that $\sup A = 1$ for $A := \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Give an admissible order. Then say why each of the other two lines cannot be used.

- (i) Let $\varepsilon > 0$.
- (ii) Hence $a := \frac{n}{n+1}$ is an element of A with $a > 1 - \varepsilon$.
- (iii) For every $n \in \mathbb{N}$, $n < n + 1$ and $n + 1 > 0$, so $\frac{n}{n+1} < 1$. Hence 1 is an upper bound of A .
- (iv) By Corollary 1.5.2 there is $n \in \mathbb{Z}_{>0}$, depending on ε , with $\frac{1}{n} < \varepsilon$.
- (v) Choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \varepsilon$ for every $\varepsilon > 0$.
- (vi) Since $\varepsilon > 0$ was arbitrary, every band $(1 - \varepsilon, 1]$ contains an element of A .
- (vii) Then $\frac{n}{n+1} = 1 - \frac{1}{n+1}$, and $\frac{1}{n+1} < \frac{1}{n} < \varepsilon$.
- (viii) By Theorem 1.4.1, $\sup A = 1$.

(ix) Fix $n \in \mathbb{Z}_{>0}$ and let $\varepsilon := \frac{1}{n}$; then $\frac{n}{n+1} = 1 - \frac{1}{n+1} > 1 - \varepsilon$.

Solution. An admissible order is (iii), (i), (iv), (vii), (ii), (vi), (viii). The line (iii) may stand anywhere before (viii), since it uses nothing but the definition of A . The other lines admit only this order. Inside the block that starts with (i), line (iv) needs the ε of (i), line (vii) needs the n of (iv), line (ii) needs (vii), and (vi) records that ε was arbitrary. Line (viii) needs both (iii) and (vi), the two hypotheses of [Theorem 1.4.1](#).

Line (v) has the quantifiers in the wrong order. It asks for one n that works for every ε , and there is none: $\varepsilon = \frac{1}{n}$ itself would give $\frac{1}{n} < \frac{1}{n}$. [Corollary 1.5.2](#) gives an n for each ε .

Line (ix) is true, but its ε is of the wrong kind: it is chosen to fit n , instead of being given first. It shows only that the bands of the special widths $\frac{1}{n}$ meet A , while [Theorem 1.4.1](#) needs **every** band, of every width $\varepsilon > 0$. Passing from the special widths to all widths is exactly the work of lines (iv) and (vii), which (ix) skips.

1.5.6.3 C. Going deeper

Exercise. C1: Exit check: the Archimedean property, and a circular proof

- Without looking back, prove the Archimedean property from the least upper bound axiom. Mark the line that uses the axiom, and explain why that line fails in $\mathbb{R}(x)$.
- A student offers the following proof. “Let $x \in \mathbb{R}$. Let $n := \lceil x \rceil + 1$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x . Then $n > x$.” Find the gap. Show that in $\mathbb{R}(x)$ the element x has no “least integer greater than or equal to it”.

Solution.

- Suppose $\mathbb{N}$ is bounded above in $\mathbb{R}$. Since $0 \in \mathbb{N}$, the set is nonempty, so by [Axiom 1.3.1](#) it has a supremum s (**this line uses the axiom**). For every $n \in \mathbb{N}$, also $n + 1 \in \mathbb{N}$, since $\mathbb{N}$ is inductive, so $n + 1 \leq s$ by (S1), that is, $n \leq s - 1$. Hence $s - 1$ is an upper bound of $\mathbb{N}$, and (S2) gives $s \leq s - 1$, which is false. So $\mathbb{N}$ is not bounded above, which means that every $x \in \mathbb{R}$ is exceeded by some $n \in \mathbb{N}$.

In $\mathbb{R}(x)$ the set $\mathbb{N}$ is nonempty and bounded above by x ([Example \(A non-Archimedean field\)](#)), but it has no supremum. Any upper bound M gives a smaller one, $M - 1$, because $n + 1 \leq M$ for every $n \in \mathbb{N}$ forces $n \leq M - 1$. So the marked line has nothing to produce, and there the proof stops.

- The gap is the existence of $\lceil x \rceil$. The set $\{k \in \mathbb{Z} : k \geq x\}$ has a least element only if it is nonempty, and it is nonempty exactly when some integer is at least x . That is (a form of) the Archimedean property, the statement being proved. The proof is circular.

In $\mathbb{R}(x)$, take the element x and suppose $k \in \mathbb{Z}$ satisfies $k \geq x$. If $k \leq 0$, then $k \leq 0 < 1 < x$. If $k \geq 1$, then $k \in \mathbb{N}$ and $k < x$ by [Example \(A non-Archimedean field\)](#). Either way $k < x$, a contradiction. So no integer is greater than or equal to x , and $\lceil x \rceil$ does not exist.

Marking scheme. (10 points) - 1: (a) supposes, for a contradiction, that $\mathbb{N}$ is bounded above, and notes $0 \in \mathbb{N}$. - 2: (a) obtains $s = \sup \mathbb{N}$ from [Axiom 1.3.1](#), with both hypotheses checked, and marks this line. - 2: (a) shows that $s - 1$ is an upper bound of $\mathbb{N}$: $n + 1 \in \mathbb{N}$ because $\mathbb{N}$ is inductive, so $n + 1 \leq s$ by (S1), that is, $n \leq s - 1$, for every $n \in \mathbb{N}$. - 1: (a)

applies the “least” property (S2) to the upper bound $s-1$, reaches $s \leq s-1$, a contradiction, and concludes. - 1: (a) explains why the marked line fails in $\mathbb{R}(x)$: every upper bound M of $\mathbb{N}$ there has the smaller upper bound $M-1$. - 2: (b) names the existence of $\lceil x \rceil$ as the gap, and says it already needs an integer at least x , which is what is being proved. - 1: (b) shows that no integer is at least x in $\mathbb{R}(x)$. Fatal errors (0 for the step): taking s to be a natural number, or the maximum of $\mathbb{N}$; omitting “for every n ” (one n with $n \leq s-1$ does not make $s-1$ an upper bound); using $\lceil x \rceil$, $\lfloor x \rfloor$, or the Archimedean property inside (a). A proof of (a) that argues “by (S2), or [Theorem 1.4.1](#) with $\varepsilon = 1$, $s-1$ is not an upper bound, so some $n > s-1$, and then $n+1 > s$ ” is also correct and earns the same 3 points for those steps; it uses one more classical step, inside the box (the remark after [Theorem 1.5.1](#)). A proof of (a) that applies the axiom to a bounded set of the form $\{n \in \mathbb{N} : n \leq x\}$ instead earns full marks for (a) if it reaches a contradiction from the supremum in the same way.

Exercise. C2: The order on the rational functions

Let $P \subseteq \mathbb{R}(x)$ be the set of positive elements defined in this section, and let $f < g$ mean $g - f \in P$.

- (a) Show that every nonzero $f \in \mathbb{R}(x)$ can be written as p/q with the leading coefficient of q positive, and that then $f \in P$ if and only if the leading coefficient of p is positive.
- (b) Show that for every nonzero f , exactly one of $f \in P$ and $-f \in P$ holds.
- (c) Show that if $f, g \in P$, then $fg \in P$ and $f + g \in P$.
- (d) Deduce the order axioms of [Definition 1.1.1](#) for $<$: trichotomy, transitivity, $f < g \Rightarrow f + h < g + h$, and $f, g > 0 \Rightarrow fg > 0$.

Solution (Hint 1). Choose good representatives first. Every fraction can be rewritten so that its denominator has positive leading coefficient, and after that only the numerator decides positivity.

Solution (Hint 2). For the sum in (c), with both denominators normalized, write $f + g = (p_1q_2 + p_2q_1)/(q_1q_2)$ and split into two cases: the two terms of the numerator have different degrees, or the same degree.

Solution. Write $\text{lc}(p)$ for the leading coefficient of a nonzero polynomial p . Recall that $\text{lc}(pq) = \text{lc}(p)\text{lc}(q)$, since the top powers multiply.

- (a) Let $f = p/q$. If $\text{lc}(q) < 0$, then $f = (-p)/(-q)$ and $\text{lc}(-q) = -\text{lc}(q) > 0$. So we may assume $\text{lc}(q) > 0$. Then “ $\text{lc}(p)$ and $\text{lc}(q)$ have the same sign” means exactly $\text{lc}(p) > 0$. By the well-definedness shown in the section, this does not depend on the chosen representation.
- (b) Write $f = p/q$ with $\text{lc}(q) > 0$, as in (a). Then $-f = (-p)/q$ and $\text{lc}(-p) = -\text{lc}(p)$. Since $p \neq 0$, $\text{lc}(p)$ is a nonzero real number, so by trichotomy in $\mathbb{R}$ exactly one of $\text{lc}(p) > 0$ and $-\text{lc}(p) > 0$ holds. By (a), exactly one of $f \in P$ and $-f \in P$ holds.
- (c) Write $f = p_1/q_1$ and $g = p_2/q_2$ with all four leading coefficients positive, using (a). The product is $fg = (p_1p_2)/(q_1q_2)$ with $\text{lc}(p_1p_2) = \text{lc}(p_1)\text{lc}(p_2) > 0$ and $\text{lc}(q_1q_2) > 0$,

so $fg \in P$. The sum is

$$f + g = \frac{p_1q_2 + p_2q_1}{q_1q_2}.$$

Both p_1q_2 and p_2q_1 have positive leading coefficient. If their degrees differ, the leading coefficient of the sum is that of the term of higher degree, which is positive. If their degrees agree, it is the sum of the two leading coefficients, which is positive. In both cases the numerator is nonzero with positive leading coefficient, and $\text{lc}(q_1q_2) > 0$, so $f + g \in P$.

- (d) *Trichotomy.* Given f, g , either $g - f = 0$, or $g - f \neq 0$ and by (b) exactly one of $g - f \in P$ and $f - g = -(g - f) \in P$ holds. So exactly one of $f = g$, $f < g$, $g < f$ holds.

Transitivity. If $f < g$ and $g < h$, then $g - f \in P$ and $h - g \in P$, so $h - f = (h - g) + (g - f) \in P$ by (c); that is, $f < h$.

Addition. If $f < g$, then $(g + h) - (f + h) = g - f \in P$, so $f + h < g + h$.

Products. If $f > 0$ and $g > 0$, then $f = f - 0 \in P$ and $g \in P$, so $fg \in P$ by (c); that is, $fg > 0$.

Exercise. C3: Archimedean without the axiom

- (a) Prove, without using [Axiom 1.3.1](#), that for every $r \in \mathbb{Q}$ there is $n \in \mathbb{N}$ with $n > r$.
 (b) Deduce that the Archimedean property does not imply the least upper bound axiom in an ordered field.

Solution.

- (a) Write $r = m/k$ with $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$. If $m \leq 0$, then $r \leq 0 < 1$, and $n = 1$ works. If $m \geq 1$, then $m \in \mathbb{N}$. Since $k \geq 1$ ([Example \(Every nonzero natural number is at least 1\)](#)) and $m > 0$, we get $r = m/k \leq m < m + 1$, and $n := m + 1 \in \mathbb{N}$ works. No supremum was used: only the arithmetic of integers and the ordered-field rules.
 (b) By (a), $\mathbb{Q}$ is an ordered field in which $\mathbb{N}$ is not bounded above, so $\mathbb{Q}$ has the Archimedean property. By [Theorem 1.2.2](#), the nonempty set $\{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, which is bounded above by 2, has no supremum in $\mathbb{Q}$. So $\mathbb{Q}$ is Archimedean and fails the least upper bound axiom, and the Archimedean property cannot imply the axiom.

1.6 Density of the Rationals and the Existence of Roots

When we first met suprema, [Theorem 1.2.2](#) showed that the set of positive rationals whose square is less than 2 has no supremum in $\mathbb{Q}$. In $\mathbb{R}$ the axiom gives that set a supremum. This section finds out what the supremum is: a positive number whose square is exactly 2. The same template produces every root $a^{1/n}$. With roots in hand, and with the Archimedean property of the previous section, we then answer a question about arrangement: the rationals and the irrationals are both spread through every interval of the line, however short.

1.6.1 Constructing a point as a supremum

Logic and Proof, Chapter 2 showed that no rational number squares to 2 (Logic and Proof, Theorem 2.2.1). Nothing so far says that any real number does. The ordered-field axioms cannot say it, since $\mathbb{Q}$ satisfies them all. So if a square root of 2 exists in $\mathbb{R}$, the least upper bound axiom must produce it.

Theorem 1.6.1. The Square Root of Two Exists

There is exactly one real number $s > 0$ with $s^2 = 2$.

Try it first (ten minutes): the axiom is the only statement we have that produces a real number. To which set would you apply it, and what would you then have to show?

Idea. The obvious attempts do not get started. There is no formula for $\sqrt{2}$ in terms of rationals, and the decimals 1, 1.4, 1.41, ... of the chapter's introduction would need a limit, which we do not have yet. The only statement that produces a real number is [Axiom 1.3.1](#), so $\sqrt{2}$ will have to be a supremum, and [Theorem 1.2.2](#) suggests which one. Collect the positive reals whose square is below 2 into a set S . The candidate is $s = \sup S$. The plan has three steps: ① check that S is nonempty and bounded above, so the axiom supplies s ; ② rule out $s^2 < 2$; ③ rule out $s^2 > 2$. Trichotomy then leaves $s^2 = 2$.

Each case is ruled out by nudging s . If $s^2 < 2$, there is still room below 2, so a slightly larger number $s + h$ is still in S . That is impossible for an upper bound. If $s^2 > 2$, a slightly smaller number $s - h$ still has square above 2, and is therefore still an upper bound. That is impossible for the *least* upper bound.

The nudges are the ones found in [Theorem 1.2.2](#), where they were derived backwards from the goal. To recall the first: we want $(s + h)^2 = s^2 + 2sh + h^2 < 2$. If $h \leq 1$, then $h^2 \leq h$, so $(s + h)^2 \leq s^2 + (2s + 1)h$. Letting $(2s + 1)h$ use half of the room $2 - s^2$ gives $h = (2 - s^2)/(4s + 2)$, and we must then check $h \leq 1$. For the second nudge, $(s - h)^2 > s^2 - 2sh$, and the right side is 2 exactly when $h = (s^2 - 2)/(2s)$. None of this used rationality. What is new is only that s now exists.

Before reading on: which line of the proof uses the axiom, and what would that line say over $\mathbb{Q}$?

Proof. Let $S := \{x \in \mathbb{R} : x > 0, x^2 < 2\}$. By [Example \(A set of reals that must have a supremum\)](#), S is nonempty (it contains 1) and bounded above by 2, so by [Axiom 1.3.1](#) its supremum $s := \sup S$ exists, and $1 \leq s \leq 2$. In particular $s > 0$ and $s^2 \geq 1$. Throughout we use [Theorem 1.1.2](#) (g): for $u, v \geq 0$, $u < v$ if and only if $u^2 < v^2$.

Case 1: $s^2 < 2$. Let

$$h := \frac{2 - s^2}{4s + 2},$$

which depends only on s . Then $h > 0$, because $2 - s^2 > 0$ and $4s + 2 > 0$. Also $h \leq \frac{1}{6}$, because $2 - s^2 \leq 1$ and $4s + 2 \geq 6$. Hence $h \leq 1$, and multiplying by $h > 0$ gives $h^2 \leq h$.

Therefore

$$\begin{aligned}(s+h)^2 &= s^2 + 2sh + h^2 \leq s^2 + (2s+1)h \\ &= s^2 + \frac{2-s^2}{2} = 1 + \frac{s^2}{2} < 2,\end{aligned}$$

where the last inequality holds because $s^2 < 2$. Also $s+h > 0$. Hence $s+h \in S$. But $s+h > s$, which contradicts (S1), that s is an upper bound of S . So Case 1 is impossible.

Case 2: $s^2 > 2$. Let

$$h := \frac{s^2 - 2}{2s},$$

which is defined and positive because $s > 0$ and $s^2 > 2$. Then $s-h = (s^2+2)/(2s) > 0$, and

$$(s-h)^2 = s^2 - 2sh + h^2 = 2 + h^2 > 2.$$

We claim that $s-h$ is an upper bound of S . Let $x \in S$ and suppose $x > s-h$. Since $s-h > 0$, we get $x^2 > (s-h)^2 > 2$, contradicting $x^2 < 2$. Hence $x \leq s-h$ for every $x \in S$. Then (S2) gives $s \leq s-h$, which is false since $h > 0$. So Case 2 is impossible.

By trichotomy, $s^2 = 2$. For uniqueness, let $t > 0$ with $t^2 = 2$. If $t < s$, then $t^2 < s^2$, and if $s < t$, then $s^2 < t^2$; both contradict $t^2 = 2 = s^2$. Hence $t = s$. This proves the theorem. ■

We write $\sqrt{2}$ for this number. By Logic and Proof, Theorem 2.2.1 it is not rational, so $\mathbb{R} \neq \mathbb{Q}$: irrational numbers exist.

Compare this proof with the one of Theorem 1.2.2. The two computations with h are the same, word for word: they are ordered-field algebra, and for a rational s both values of h are rational. What changed is a single line. In $\mathbb{Q}$ there was no s to start from: the argument showed that no rational number can be the supremum, since its square cannot be less than 2, more than 2, or equal to 2. In $\mathbb{R}$ the axiom supplies s , and the very same argument now shows that its square can only be 2.

The proof is an instance of a move this book uses again and again.

To construct a point: collect the points where “things are still fine” into a set, take its supremum, and rule out each way the supremum could fail to be the point you want.

Here “things are still fine” meant $x^2 < 2$. The exercises of this section and of the last section of the chapter ask you to run the move yourself, with less and less of the proof given. In Chapter 6 the Intermediate Value Theorem is proved the same way: the point where a continuous function crosses zero is the supremum of the points where it is still negative. In Chapter 4 the Heine–Borel theorem takes the supremum of the points x for which $[a, x]$ is still covered by finitely many of the given open sets.

1.6.2 Roots of every order

The template does not care that the exponent is 2 or that the number is 2. Only the choice of h needs new work.

Theorem 1.6.2. Existence of Roots

Let $a > 0$ and $n \in \mathbb{Z}_{>0}$. There is exactly one real number $s > 0$ with $s^n = a$.

We write this number $a^{1/n}$, or $\sqrt[n]{a}$. For $a = 0$ the only real $s \geq 0$ with $s^n = 0$ is $s = 0$ (a product of positive numbers is positive), so we also set $\sqrt[n]{0} := 0$; every $a \geq 0$ then has exactly one nonnegative n -th root.

Idea. Take $S = \{x > 0 : x^n < a\}$ and $s = \sup S$, as before, and rule out $s^n < a$ and $s^n > a$. The new difficulty is the estimate. For $n = 2$ we expanded $(s + h)^2$ and bounded h^2 by h . For general n , expanding $(s + h)^n$ is messy. We work backwards from what we need instead.

We need $(s + h)^n - s^n < a - s^n$. A difference of n -th powers factors:

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + x^{n-1}).$$

With $y = s + h$ and $x = s$, the first factor is h , which we control. The second factor is a sum of n terms, each at most $(s + h)^{n-1}$. That still depends on h , and we cannot solve for h while h is on both sides. So we bound it by something a little bigger: if $h < 1$, each term is at most $(s + 1)^{n-1}$. This gives

$$(s + h)^n - s^n \leq h \cdot C, \quad C := n(s + 1)^{n-1}.$$

The point is that C depends only on n and s , and not on h . Now it suffices to have $hC < a - s^n$, and $h = (a - s^n)/(2C)$ does it, with a factor $\frac{1}{2}$ to make the inequality strict. We needed $h < 1$ along the way, so we cap h at $\frac{1}{2}$.

For $s - h$ the same factorization gives $s^n - (s - h)^n \leq h \cdot ns^{n-1} \leq hC$, so the same C , and the same h , serve both cases.

Proof. We first prove a power estimate.

Claim. Let $0 \leq x < y$ and $k \in \mathbb{Z}_{>0}$. Then $x^k < y^k$: the map $x \mapsto x^k$ is strictly increasing on $[0, \infty)$. Consequently $x^m \leq y^m$ for every integer $m \geq 0$, where $u^0 := 1$. Moreover

$$0 < y^k - x^k \leq ky^{k-1}(y - x).$$

Proof of Claim. Powers are defined by recursion (*Sets, Relations, and Cardinality*, Chapter 2): $u^0 := 1$ and $u^{k+1} := u^k u$, so that $u^1 = u$; here only $k \geq 1$ is needed.

We first show $0 \leq x^k < y^k$ by induction on k . For $k = 1$ this is the hypothesis. If $0 \leq x^k < y^k$, then $x^{k+1} = x^k x \geq 0$, and

$$x^{k+1} = x^k x \leq x^k y < y^k y = y^{k+1},$$

where the first step multiplies $x < y$ by $x^k \geq 0$ ([Theorem 1.1.2](#) (c), or equality if $x^k = 0$) and the second multiplies $x^k < y^k$ by $y > 0$. This proves the first statement, and the second follows, since for $m = 0$ both sides are 1.

For the estimate, write $\Sigma := \sum_{j=0}^{k-1} y^j x^{k-1-j}$. Multiplying out,

$$\begin{aligned} (y - x)\Sigma &= \sum_{j=0}^{k-1} y^{j+1} x^{k-1-j} - \sum_{j=0}^{k-1} y^j x^{k-j} \\ &= y^k - x^k, \end{aligned}$$

since every other term appears once in each sum and cancels. Each term of Σ satisfies $0 \leq y^j x^{k-1-j} \leq y^j y^{k-1-j} = y^{k-1}$, by the second statement with $m = k - 1 - j$, multiplied by $y^j > 0$; and the term with $j = k - 1$ equals $y^{k-1} > 0$. Hence $0 < \Sigma \leq ky^{k-1}$. Multiplying by $y - x > 0$ gives the claim. ■

Let $S := \{x \in \mathbb{R} : x > 0, x^n < a\}$.

S is nonempty. Let $x_0 := a/(1 + a)$. Then $0 < x_0 < 1$ and $x_0 < a$. Since $0 < x_0 < 1$, multiplying by x_0 repeatedly gives $x_0^n \leq x_0$ (by induction on n). Hence $x_0^n \leq x_0 < a$, and $x_0 \in S$.

S is bounded above by $1 + a$. If $x \geq 1 + a$, then $x \geq 1$, so $x^n \geq x$ (again by induction on n), and $x^n \geq 1 + a > a$. So $x \notin S$.

By [Axiom 1.3.1](#), $s := \sup S$ exists, and $s \geq x_0 > 0$. Let

$$C := n(s + 1)^{n-1},$$

which depends only on n and s . Suppose $s^n \neq a$, and let

$$h := \min\left\{\frac{1}{2}, \frac{|s^n - a|}{2C}\right\} > 0.$$

Then $hC \leq \frac{1}{2}|s^n - a| < |s^n - a|$.

Case 1: $s^n < a$. Apply the claim with $k = n$, $x = s$, and $y = s + h$. Since $s + h < s + 1$, the claim with $m = n - 1$ gives $(s + h)^{n-1} \leq (s + 1)^{n-1}$ (both sides are 1 when $n = 1$). Therefore

$$(s + h)^n - s^n \leq n(s + h)^{n-1}h \leq Ch < a - s^n,$$

so $(s + h)^n < a$. Hence $s + h \in S$, although $s + h > s$. This contradicts that s is an upper bound of S .

Case 2: $s^n > a$. First, $s - h > 0$. Indeed $(s + 1)^{n-1} \geq s^{n-1}$ by the claim with $m = n - 1$ (both sides are 1 when $n = 1$), so

$$h \leq \frac{s^n - a}{2C} < \frac{s^n}{2C} \leq \frac{s^n}{2ns^{n-1}} = \frac{s}{2n} < s.$$

Apply the claim with $k = n$, $x = s - h$, and $y = s$:

$$s^n - (s - h)^n \leq ns^{n-1}h \leq Ch < s^n - a,$$

so $(s - h)^n > a$. Now $s - h$ is an upper bound of S : if $x \in S$ had $x > s - h$, the claim would give $x^n > (s - h)^n > a$, contradicting $x \in S$. But $s - h < s$, which contradicts that s is the least upper bound.

Both cases are impossible, so $s^n = a$ by trichotomy. If also $t > 0$ and $t^n = a$, then $t = s$, since $t < s$ or $s < t$ would give $t^n \neq s^n$ by the claim. This proves the theorem. ■

For $n = 2$ and $a = 2$ the constant is $C = 2(s + 1)$, slightly larger than the $2s + 1$ that the direct expansion gave. The general proof pays for its generality with a cruder bound. We never needed a good bound, only one that does not depend on h .

Example. Locating a cube root

Show that $1.25 < \sqrt[3]{2} < 1.26$.

Solution. By [Theorem 1.6.2](#), $s := \sqrt[3]{2}$ exists and $s^3 = 2$. Exact arithmetic gives

$$1.25^3 = 1.953125 < 2 < 2.000376 = 1.26^3.$$

The claim in the proof of [Theorem 1.6.2](#) says that $x \mapsto x^3$ is strictly increasing on $[0, \infty)$. If $s \leq 1.25$, then $s^3 \leq 1.25^3 < 2$; and if $s \geq 1.26$, then $s^3 \geq 1.26^3 > 2$. Both contradict $s^3 = 2$, so $1.25 < s < 1.26$.

Remark. With n -th roots we can define $a^{m/n} := (a^{1/n})^m$ for $a > 0$, and check that the usual exponent laws hold for rational exponents. The powers a^x with irrational x wait for the exponential function, which Chapter 10 builds from power series.

1.6.3 The integer part of a real number

Before asking where the rationals are, we need to locate every real number between two consecutive integers. This sounds like school arithmetic, and it is. But its proof uses both the Archimedean property and well-ordering, and in $\mathbb{R}(x)$ it fails for the element x ([Exercise \(C1: Exit check: the Archimedean property, and a circular proof\)](#)).

Lemma 1.6.3. Integer Part

For every $y \in \mathbb{R}$ there is exactly one integer m with $m \leq y < m + 1$.

This integer is the **integer part** or **floor** of y , written $\lfloor y \rfloor$. For instance $\lfloor 2.7 \rfloor = 2$ and $\lfloor -2.7 \rfloor = -3$.

Idea. Picture the integers as marks spaced 1 apart. The Archimedean property puts a mark to the right of y , and a mark to the left of y . Among the marks to the right, well-ordering picks out the first one; the mark just before it is $\lfloor y \rfloor$. Well-ordering is a statement about subsets of $\mathbb{N}$, so we shift the marks so that they start at 0.

Proof. Existence. Let $y \in \mathbb{R}$. By [Theorem 1.5.1](#), there is $p \in \mathbb{N}$ with $p > -y$, and there is $j \in \mathbb{N}$ with $j > y + p$. Hence the set

$$T := \{j \in \mathbb{N} : j - p > y\}$$

is nonempty. By well-ordering (*Sets, Relations, and Cardinality*, Chapter 2), T has a least element j_0 . Let $m := j_0 - p - 1$, an integer because differences of integers are integers (the remark on $\mathbb{N}$ in the previous section). Then $m + 1 = j_0 - p > y$, since $j_0 \in T$. It remains to show $m \leq y$. First, $0 - p = -p < y$ by the choice of p , so $0 \notin T$ and $j_0 \neq 0$. Hence $j_0 \geq 1$ ([Example \(Every nonzero natural number is at least 1\)](#)), so $j_0 - 1 \in \mathbb{N}$ (again from that

remark), and $j_0 - 1 \notin T$ because it is smaller than the least element of T . So $(j_0 - 1) - p \leq y$, that is, $m \leq y$.

Uniqueness. Suppose m and m' are integers with $m \leq y < m + 1$ and $m' \leq y < m' + 1$, and suppose $m < m'$. Then $m' - m$ is a positive integer, hence a nonzero natural number (the remark on $\mathbb{N}$ in the previous section), so $m' - m \geq 1$ by [Example \(Every nonzero natural number is at least 1\)](#); that is, $m' \geq m + 1$. Hence $y \geq m' \geq m + 1 > y$, a contradiction. By symmetry $m' < m$ is impossible too, so $m = m'$. ■

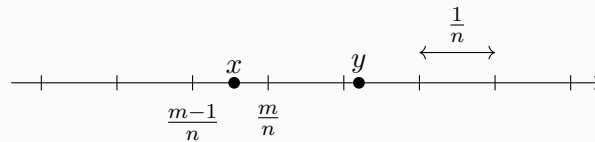
1.6.4 The rationals are dense

We know now that irrational numbers exist. How are they arranged among the rationals? Could there be a whole interval with no rational number in it? No: every interval, however short, contains a rational.

Theorem 1.6.4. Density of the Rationals

Let $x, y \in \mathbb{R}$ with $x < y$. There is $q \in \mathbb{Q}$ with $x < q < y$.

Idea. Think of a ruler whose marks are the fractions k/n , spaced $1/n$ apart. If the spacing is smaller than the gap $y - x$, the ruler cannot jump over the interval (x, y) : the first mark to the right of x is at most one step past x , and so it lands before y .



The plan: ① choose n with $1/n < y - x$, by the Archimedean property; ② let m/n be the first mark strictly to the right of x , that is, $m = \lfloor nx \rfloor + 1$; ③ check that $m/n < y$.

Proof. Let $x < y$. Then $y - x > 0$, so by [Corollary 1.5.2](#) there is $n \in \mathbb{Z}_{>0}$ with $1/n < y - x$; the n depends on x and y , through $y - x$. By [Lemma 1.6.3](#), let $m := \lfloor nx \rfloor + 1$, an integer, since sums of integers are integers (the remark on $\mathbb{N}$ in the previous section). Then $m - 1 \leq nx < m$. Therefore

$$nx < m \leq nx + 1 < nx + n(y - x) = ny,$$

where the last inequality uses $1 < n(y - x)$, which is $1/n < y - x$ multiplied by $n > 0$. Dividing by $n > 0$ gives $x < m/n < y$. Since $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$, $q := m/n$ is rational. This proves the theorem. ■

Example. Running the proof

Follow the proof of [Theorem 1.6.4](#) to find a rational number between $\sqrt{2}$ and $\frac{3}{2}$.

Solution. Step ①: the spacing. We need $1/n < \frac{3}{2} - \sqrt{2}$, that is, $\sqrt{2} < \frac{3}{2} - \frac{1}{n}$. By [Theorem 1.1.2 \(g\)](#), applied to the positive numbers $\sqrt{2}$ and $\frac{3}{2} - \frac{1}{n}$, this holds exactly when $2 < (\frac{3}{2} - \frac{1}{n})^2$. For $n = 12$, $\frac{3}{2} - \frac{1}{12} = \frac{17}{12}$ and $(\frac{17}{12})^2 = \frac{289}{144} > \frac{288}{144} = 2$, so $n = 12$ works. (For $n = 11$, $(\frac{31}{22})^2 = \frac{961}{484} < 2$, so 12 is the least n that works, but any $n \geq 12$ would do.)

Step ②: the first mark past x . We need $\lfloor 12\sqrt{2} \rfloor$. Since $16^2 = 256 \leq 288 = (12\sqrt{2})^2 < 289 = 17^2$, [Theorem 1.1.2 \(g\)](#) gives $16 \leq 12\sqrt{2} < 17$, so $\lfloor 12\sqrt{2} \rfloor = 16$ and $m = 17$.

Step ③: the answer. The proof guarantees $\sqrt{2} < \frac{17}{12} < \frac{3}{2}$. Directly: $(\frac{17}{12})^2 = \frac{289}{144} > 2$ gives the left inequality, and $\frac{17}{12} < \frac{18}{12}$ gives the right one.

The irrationals are just as widely spread. The trick is to shift the rationals by $\sqrt{2}$.

Corollary 1.6.5

Let $x, y \in \mathbb{R}$ with $x < y$. There is an irrational number z with $x < z < y$.

Proof. Since $x - \sqrt{2} < y - \sqrt{2}$, [Theorem 1.6.4](#) gives $q \in \mathbb{Q}$ with $x - \sqrt{2} < q < y - \sqrt{2}$. Let $z := q + \sqrt{2}$. Then $x < z < y$. If z were rational, then $\sqrt{2} = z - q$ would be a difference of rationals, hence rational, contradicting Logic and Proof, Theorem 2.2.1. So z is irrational. ■

This is why the existence of $\sqrt{2}$ came first in this section: the proof of [Corollary 1.6.5](#) needs one irrational number to shift by, and [Theorem 1.6.1](#) supplies it.

Quick check. Name a rational number between $\sqrt{2}$ and 1.415, and an irrational number between 0 and 10^{-6} . Justify both.

Solution. Rational: 1.4143. Its square is $2.00024449 > 2$, so $1.4143 > \sqrt{2}$ by [Theorem 1.1.2 \(g\)](#), and $1.4143 < 1.415$.

Irrational: $z := \sqrt{2} \cdot 10^{-7}$. Since $1 < \sqrt{2} < 2$, we get $0 < z < 2 \cdot 10^{-7} < 10^{-6}$. If z were rational, then $\sqrt{2} = 10^7 z$ would be rational, contradicting Logic and Proof, Theorem 2.2.1.

Warning. “Dense” does not mean “next to each other”. There is no smallest positive rational number and no “next rational after 0”: if $q > 0$ is rational, then $q/2$ is a smaller positive rational. The same holds for irrationals: if $z > 0$ is irrational, so is $z/2$, and it is smaller. Between any two distinct rationals there are infinitely many more, and between any two distinct reals there are rationals and irrationals alike.

Density is what makes decimals, and more generally rational approximation, possible. Every real number can be pinned down by rationals from below as closely as we like ([Exercise \(B1: Every real number is a supremum of rationals\)](#)). In *Metric and Topological Spaces* the word

dense becomes a notion in any metric space: a set that meets every ball. The rationals in the real line are the model example.

1.6.5 Exercises

1.6.5.1 A. Check your understanding

Exercise. A1

True or false? Give a reason or a counterexample for each.

- (a) There is a least positive irrational number.
- (b) Between any two distinct rational numbers there is an irrational number.
- (c) Between any two distinct irrational numbers there is a rational number.
- (d) Every nonempty set of rationals that is bounded above has a supremum in $\mathbb{R}$, and that supremum is rational.
- (e) In the proof of [Theorem 1.6.2](#), the constant C depends on h .
- (f) Run Case 1 of the proof of [Theorem 1.6.1](#) with the value $\frac{7}{5}$ in place of s (so $s^2 = \frac{49}{25} < 2$): compute h , and check that $(s + h)^2 < 2$. Which property of a supremum does this show $\frac{7}{5}$ lacks?

Solution.

- (a) False. If $z > 0$ is irrational, then $z/2$ is positive, smaller than z , and irrational (if $z/2$ were rational, so would be $z = 2 \cdot (z/2)$).
- (b) True, by [Corollary 1.6.5](#) applied to the two rationals.
- (c) True, by [Theorem 1.6.4](#) applied to the two irrationals.
- (d) False. The set $A := \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$ is nonempty and bounded above by 2, so it has a supremum $s \in \mathbb{R}$ by [Axiom 1.3.1](#). If s were rational, it would be a rational upper bound of A that is at most every real upper bound, and in particular at most every rational one. So s would be a supremum of A in $\mathbb{Q}$, contradicting [Theorem 1.2.2](#). (In fact $s = \sqrt{2}$.)
- (e) False. $C = n(s+1)^{n-1}$ depends only on n and s . It has to be independent of h , because h is then chosen in terms of C .
- (f) Here $2 - s^2 = \frac{1}{25}$ and $4s + 2 = \frac{38}{5}$, so $h = \frac{1/25}{38/5} = \frac{1}{190}$ and $s + h = \frac{267}{190}$. Then

$$\left(\frac{267}{190}\right)^2 = \frac{71289}{36100} < \frac{72200}{36100} = 2,$$

and the proof's bound gives the same: $(s + h)^2 \leq 1 + \frac{s^2}{2} = \frac{99}{50} < 2$. So $\frac{267}{190} \in S$ and $\frac{267}{190} > \frac{7}{5}$: the number $\frac{7}{5}$ is not an upper bound of S , so it fails (S1). This is no contradiction, because nothing said that $\frac{7}{5}$ was the supremum. In the proof the same computation contradicts (S1) for $s = \sup S$.

1.6.5.2 B. Practice

Exercise. B1: Every real number is a supremum of rationals

Let $x \in \mathbb{R}$. Prove that $x = \sup\{q \in \mathbb{Q} : q < x\}$.

Solution. Let $A := \{q \in \mathbb{Q} : q < x\}$. By [Theorem 1.6.4](#) applied to $x - 1 < x$, there is a rational in $(x - 1, x)$, so A is nonempty. By definition every element of A is less than x , so x is an upper bound of A .

Let $\varepsilon > 0$. By [Theorem 1.6.4](#) applied to $x - \varepsilon < x$, there is $q \in \mathbb{Q}$ with $x - \varepsilon < q < x$. Then $q \in A$ and $q > x - \varepsilon$. By [Theorem 1.4.1](#), $x = \sup A$.

Exercise. B2: The square root of three, finished

Here is the start of a proof that there is exactly one real $s > 0$ with $s^2 = 3$.

Let $S := \{x \in \mathbb{R} : x > 0, x^2 < 3\}$. Then $1 \in S$, and if $x \geq 2$, then $x^2 \geq 4 > 3$; so S is nonempty and bounded above by 2. By [Axiom 1.3.1](#), $s := \sup S$ exists, and $s \geq 1$.

Case $s^2 < 3$. Let $h := (3 - s^2)/(4s + 2)$. Then $h > 0$. Since $3 - s^2 \leq 2$ and $4s + 2 \geq 6$, we have $h \leq \frac{1}{3}$, so $h^2 \leq h$, and $(s + h)^2 \leq s^2 + (2s + 1)h = \frac{3 + s^2}{2} < 3$. Hence $s + h \in S$, and $s + h > s$ contradicts (S1).

Case $s^2 > 3$

Name the move, then complete it. Give an explicit h for the second case and check that it works; then finish the proof, including uniqueness.

Solution. The move is **construct a point as a supremum**: the points where “things are still fine” ($x^2 < 3$) form S , the axiom supplies $s = \sup S$, and each way s could fail ($s^2 < 3$, $s^2 > 3$) is ruled out. The given text rules out the first; the missing step rules out the second, against (S2) this time.

Case $s^2 > 3$. Let $h := (s^2 - 3)/(2s)$, which is defined and positive because $s \geq 1$ and $s^2 > 3$. Then $s - h = (s^2 + 3)/(2s) > 0$ and

$$(s - h)^2 = s^2 - 2sh + h^2 = 3 + h^2 > 3.$$

If some $x \in S$ had $x > s - h$, then, as $s - h > 0$, [Theorem 1.1.2](#) (g) would give $x^2 > (s - h)^2 > 3$, which is impossible for $x \in S$. So $s - h$ is an upper bound of S , and (S2) gives $s \leq s - h$, which is false since $h > 0$.

Both cases are impossible, so $s^2 = 3$ by trichotomy. For uniqueness, let $t > 0$ with $t^2 = 3$. If $t < s$ or $s < t$, then [Theorem 1.1.2](#) (g) gives $t^2 < s^2$ or $s^2 < t^2$, contradicting $t^2 = 3 = s^2$. Hence $t = s$.

Exercise. B3: The cube root of five

Prove that there is exactly one real $s > 0$ with $s^3 = 5$. Hence show that $1.70 < s < 1.71$.

Solution. By Theorem 1.6.2 with $a = 5$ and $n = 3$, there is exactly one $s > 0$ with $s^3 = 5$.

Exact arithmetic gives $1.70^3 = 4.913 < 5$ and $1.71^3 = 5.000211 > 5$. By the claim in the proof of Theorem 1.6.2, $x \mapsto x^3$ is strictly increasing on $[0, \infty)$. If $s \leq 1.70$, then $s^3 \leq 4.913 < 5$; if $s \geq 1.71$, then $s^3 \geq 5.000211 > 5$. Both contradict $s^3 = 5$, so $1.70 < s < 1.71$.

1.6.5.3 C. Going deeper

Exercise. C1: Exit check: the square root of two, in both fields

- (a) Prove that there is a real number $s > 0$ with $s^2 = 2$.
- (b) Prove that the set $\{x \in \mathbb{Q} : x > 0, x^2 < 2\}$ has no supremum in $\mathbb{Q}$.
- (c) Both proofs use the same two values of h . Explain in two or three sentences why the same computation yields an existence theorem in $\mathbb{R}$ and a nonexistence theorem in $\mathbb{Q}$.

Solution.

- (a) This is Theorem 1.6.1. Let $S := \{x \in \mathbb{R} : x > 0, x^2 < 2\}$; it contains 1 and is bounded above by 2, so $s := \sup S$ exists by Axiom 1.3.1, and $s \geq 1$. If $s^2 < 2$, then $h := (2 - s^2)/(4s + 2)$ satisfies $0 < h \leq \frac{1}{6}$, so $h^2 \leq h$ and $(s + h)^2 \leq s^2 + (2s + 1)h = 1 + s^2/2 < 2$, and $s + h \in S$ exceeds the upper bound s : impossible. If $s^2 > 2$, then $h := (s^2 - 2)/(2s)$ satisfies $s - h = (s^2 + 2)/(2s) > 0$ and $(s - h)^2 = 2 + h^2 > 2$; every $x \in S$ then satisfies $x \leq s - h$, since $x > s - h$ would give $x^2 > 2$. So $s - h < s$ is an upper bound, contradicting (S2). By trichotomy $s^2 = 2$.
- (b) Let $A := \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$, and suppose $r \in \mathbb{Q}$ is a supremum of A in $\mathbb{Q}$, that is, a rational upper bound of A that is at most every rational upper bound. As $1 \in A$, $r \geq 1$. By Logic and Proof, Theorem 2.2.1, $r^2 \neq 2$.
If $r^2 < 2$, let $h := (2 - r^2)/(4r + 2)$. This is rational, and the computation of (a) gives $(r + h)^2 < 2$. So $r + h \in A$ and $r + h > r$, contradicting that r is an upper bound.
If $r^2 > 2$, let $h := (r^2 - 2)/(2r)$. Then $r - h = (r^2 + 2)/(2r)$ is rational and positive, with $(r - h)^2 = 2 + h^2 > 2$. As in (a), every $x \in A$ satisfies $x \leq r - h$, so $r - h$ is a rational upper bound of A smaller than r , contradicting the choice of r .
Both cases are impossible, so A has no supremum in $\mathbb{Q}$.
- (c) The computation shows, in any ordered field, that a supremum of the set cannot have square less than 2 or greater than 2. In $\mathbb{R}$ the axiom guarantees that a supremum exists, so its square must be 2: existence. In $\mathbb{Q}$ no element has square 2 (Logic and Proof, Theorem 2.2.1), so no case is left, and no supremum can exist. For rational r the values of h are rational, which is why the argument stays inside $\mathbb{Q}$.

Marking scheme. (10 points) - 1: (a) defines S , checks that it is nonempty and bounded above, and takes $s = \sup S$ from Axiom 1.3.1. - 2: (a) rules out $s^2 < 2$ with an explicit $h > 0$, verifying $(s + h)^2 < 2$ including any side condition such as $h \leq 1$, and contradicts (S1). - 2: (a) rules out $s^2 > 2$ with an explicit $h > 0$, verifying $s - h > 0$ and that $s - h$ is an upper bound, and contradicts (S2). - 1: (b) supposes a rational supremum r and excludes $r^2 = 2$ by Logic and Proof, Theorem 2.2.1. - 2: (b) in each case checks that the new number is **rational** and derives the contradiction with “upper bound in $\mathbb{Q}$ ” or “least among rational upper bounds”. - 2: (c) says that the computation excludes $s^2 < 2$ and $s^2 > 2$ in any ordered field, that in $\mathbb{R}$ the axiom supplies s so $s^2 = 2$ remains, and that in $\mathbb{Q}$ nothing remains. Fatal errors (0 for the step): using $\sqrt{2}$ to define S or h ; asserting $s + h \in S$ or $(s - h)^2 > 2$ without

the computation; an h that depends on an element of S rather than on s alone. A proof with a different h (for instance $h = \min\{1, (2 - s^2)/6\}$ in Case 1, using $2s + 1 \leq 5$) earns full marks if its bound is verified.

Exercise. C2: Dyadic rationals

A **dyadic rational** is a number of the form $m/2^k$ with $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Prove that between any two real numbers $x < y$ there is a dyadic rational.

Hint: compare 2^k with k .

Solution. First, $2^k > k$ for every $k \in \mathbb{N}$, by induction (Logic and Proof, Axiom 4.2.1; Logic and Proof, Exercise (B1) even gives $2^k \geq k + 1$): $2^0 = 1 > 0$, and if $2^k > k$, then $2^{k+1} = 2^k + 2^k > k + 1$, since $2^k \geq 1$.

Let $x < y$. By [Corollary 1.5.2](#) there is $k \in \mathbb{Z}_{>0}$ with $1/k < y - x$; the k depends on $y - x$. Then $1/2^k < 1/k < y - x$. By [Lemma 1.6.3](#), let $m := \lfloor 2^k x \rfloor + 1$. Then $m - 1 \leq 2^k x < m$, and

$$2^k x < m \leq 2^k x + 1 < 2^k x + 2^k(y - x) = 2^k y,$$

using $1 < 2^k(y - x)$. Dividing by $2^k > 0$ gives $x < m/2^k < y$.

The proof is that of [Theorem 1.6.4](#) with the ruler of step $1/n$ replaced by one of step $1/2^k$. Only the choice of step size changed; any family of steps that can be made smaller than every $\varepsilon > 0$ would do.

Exercise. C3: Find the faulty line

A student proves [Theorem 1.6.4](#) as follows. Let $x < y$.

- (1) Let $R := \{r \in \mathbb{Q} : r > x\}$. By [Theorem 1.5.1](#) some natural number exceeds x , so R is nonempty.
- (2) By well-ordering, R has a least element q .
- (3) By [Corollary 1.5.2](#), choose $n \in \mathbb{Z}_{>0}$ with $1/n < y - x$.
- (4) Since $q - 1/n$ is rational and smaller than the least element q of R , it is not in R . So $q - 1/n \leq x$, and hence $q \leq x + 1/n < y$.
- (5) Therefore $q \in \mathbb{Q}$ and $x < q < y$.

Is the statement the student is proving true? Does the argument prove it? Find the line that fails and say why. Show that for $x = 0$ the set R has no least element. Then repair the proof by changing R so that the faulty line becomes correct.

Solution. The statement is true: it is [Theorem 1.6.4](#). So there is no counterexample to the conclusion, and the only question is whether the argument proves it. It does not.

Line (2) fails. Well-ordering says that a nonempty subset of $\mathbb{N}$ has a least element. The set R is a set of rationals, not of natural numbers, and it need not have a least element. For $x = 0$, R is the set of positive rationals; if $q \in R$, then $q/2 \in R$ and $q/2 < q$. So no element

of R is least. Lines (3)–(5) are correct once a least element is available.

Repair. Fix n first, as in line (3), and replace R by a set of integers: $R_n := \{m \in \mathbb{Z} : m/n > x\}$. By [Lemma 1.6.3](#), its least element is $m := \lfloor nx \rfloor + 1$: indeed $m/n > x$, while $(m-1)/n \leq x$. Now line (4) holds with $q := m/n$: since $(m-1)/n = q - 1/n \leq x$, we get $q \leq x + 1/n < y$. This is the proof of [Theorem 1.6.4](#). The point is that the step $1/n$ is fixed first, so the candidates m/n form a copy of the integers, where “least” makes sense.

1.7 Nested Intervals and the Uncountability of the Reals

Each point we have built so far was a single supremum: the supremum of a set that “reaches toward” the point we want. This section builds points from a different kind of data, a sequence of closed intervals, each inside the one before. We prove that such intervals always have a point in common. In $\mathbb{Q}$ this fails, and so it is one more way of saying that $\mathbb{R}$ has no gaps. The payoff is a proof that the real numbers cannot be listed, one that uses no decimal expansions, only the axiom.

1.7.1 Trapping a point

Here is a way to hunt for $\sqrt{2}$ using nothing but rational arithmetic. Start with the interval $[1, 2]$; since $1^2 < 2 < 2^2$, the number we are after is in there. Cut the interval at its midpoint $\frac{3}{2}$. Since $(\frac{3}{2})^2 = \frac{9}{4} > 2$, keep the left half $[1, \frac{3}{2}]$. Cut again at $\frac{5}{4}$: since $(\frac{5}{4})^2 = \frac{25}{16} < 2$, keep the right half. Continuing, we get

$$\begin{aligned} [1, 2] &\supseteq [1, \tfrac{3}{2}] \supseteq [\tfrac{5}{4}, \tfrac{3}{2}] \\ &\supseteq [\tfrac{11}{8}, \tfrac{3}{2}] \supseteq [\tfrac{11}{8}, \tfrac{23}{16}] \supseteq \cdots, \end{aligned}$$

each interval half as long as the one before, each with rational endpoints, and each containing $\sqrt{2}$. Is there a number lying in all of them?

In $\mathbb{R}$ the answer should be “yes, $\sqrt{2}$ ”. In $\mathbb{Q}$ the answer is “no”: the only candidate is $\sqrt{2}$, and it is not rational. We make both answers precise below. First, a name for the shape of this data. A family of sets $I_1, I_2, I_3, \dots$ is **nested** if $I_{n+1} \subseteq I_n$ for every $n \geq 1$.

Theorem 1.7.1. Nested Interval Theorem

For each $n \geq 1$, let $I_n = [a_n, b_n]$ be a closed bounded interval with $a_n \leq b_n$, and suppose the family is nested: $I_{n+1} \subseteq I_n$ for every $n \geq 1$. Then $\bigcap_{n \geq 1} I_n \neq \emptyset$. More precisely, $\alpha := \sup\{a_n : n \geq 1\}$ exists and lies in every I_n .

If moreover for every $\varepsilon > 0$ there is $n \geq 1$ with $b_n - a_n < \varepsilon$, then $\bigcap_{n \geq 1} I_n = \{\alpha\}$.

Idea. Picture the intervals on the line. As n grows, the left endpoints move right (or stay), the right endpoints move left (or stay), and they never cross: every left endpoint lies below every right endpoint, even for different indices. So the set of left endpoints is bounded above by each b_n . Its supremum α is at least every a_n , because it is an upper bound. It is at most every b_n , because it is the *least* upper bound and each b_n is an upper bound. Hence $\alpha \in [a_n, b_n]$ for every n .

For uniqueness: two different common points $x < y$ would force every interval to have length at least $y - x$, and the shrinking hypothesis with $\varepsilon = y - x$ forbids that.

Proof. We first compare endpoints with different indices.

Claim. $a_m \leq b_n$ for all $m, n \geq 1$.

Proof of Claim. From $I_{j+1} \subseteq I_j$ for every j , induction on $k - m$ gives $I_k \subseteq I_m$ whenever $m \leq k$. Let $m, n \geq 1$ and $k := \max\{m, n\}$. Then $I_k \subseteq I_m$ and $I_k \subseteq I_n$. Since $a_k \in I_k \subseteq I_m$, we have $a_m \leq a_k$. Since $b_k \in I_k \subseteq I_n$, we have $b_k \leq b_n$. Hence $a_m \leq a_k \leq b_k \leq b_n$. ■

Let $A := \{a_n : n \geq 1\}$. It is nonempty, since $a_1 \in A$, and by the claim it is bounded above by b_1 . By [Axiom 1.3.1](#), $\alpha := \sup A$ exists.

Fix $n \geq 1$. Since α is an upper bound of A , $a_n \leq \alpha$. By the claim, b_n is an upper bound of A , so (S2) of [Definition 1.2.4](#) gives $\alpha \leq b_n$. Hence $\alpha \in [a_n, b_n] = I_n$. As n was arbitrary, $\alpha \in \bigcap_n I_n$, and the intersection is nonempty.

Now assume the shrinking hypothesis, and let $x \in \bigcap_n I_n$. Suppose $x \neq \alpha$, and let $u := \min\{x, \alpha\}$ and $v := \max\{x, \alpha\}$, so $u < v$ and both lie in every I_n . For every n , $a_n \leq u < v \leq b_n$, so $b_n - a_n \geq v - u$. But $\varepsilon := v - u > 0$, and by hypothesis some n has $b_n - a_n < v - u$. This is a contradiction, so $x = \alpha$. Hence $\bigcap_n I_n = \{\alpha\}$. ■

Closedness is what lets α belong to I_n . The proof gives only $a_n \leq \alpha \leq b_n$, with nonstrict inequalities, and α may well equal an endpoint. An open interval would exclude it.

Now back to the bisection. Its intervals have length $1/2^{n-1}$, and $2^{n-1} \geq n$ for every $n \geq 1$ (by Logic and Proof, [Axiom 4.2.1](#)), so $1/2^{n-1} \leq 1/n$. Given $\varepsilon > 0$, [Corollary 1.5.2](#) gives n with $1/n < \varepsilon$, and then the n -th interval is shorter than ε . The rule keeps $a_n^2 < 2 < b_n^2$ at every step, so $a_n < \sqrt{2} < b_n$ by [Theorem 1.1.2](#) (g), and every interval contains $\sqrt{2}$. By [Theorem 1.7.1](#) the intersection is exactly $\{\sqrt{2}\}$. The bisection traps $\sqrt{2}$ and nothing else.

1.7.2 What each hypothesis does

[Theorem 1.7.1](#) has four hypotheses: the intervals are closed, they are bounded, they are nested, and they live in $\mathbb{R}$. The uniqueness clause adds a fifth, that the lengths shrink below every ε . Each one is needed, and each has a witness.

Hypothesis dropped	Family	Intersection
closed	$(0, 1/n)$	$\emptyset$
bounded	$[n, \infty)$	$\emptyset$
nested	$[n, n+1]$	$\emptyset$
the field is $\mathbb{R}$	the rational bisection intervals, in $\mathbb{Q}$	$\emptyset$
lengths shrink (uniqueness only)	$[-1/n, 1 + 1/n]$	$[0, 1]$

We check each row.

- **Closed.** The intervals $(0, 1/n)$ are nested and bounded. A real $x \leq 0$ is not in $(0, 1)$. For $x > 0$, [Corollary 1.5.2](#) gives n with $1/n < x$, so $x \notin (0, 1/n)$. Hence no real number is in every interval. Here the supremum of the left endpoints is $\alpha = 0$, and it lies in none of the intervals: exactly the step that closedness secured.
- **Bounded.** The intervals $[n, \infty)$ are closed and nested. For any $x \in \mathbb{R}$, [Theorem 1.5.1](#) gives $n > x$, and then $x \notin [n, \infty)$. The proof breaks earlier here: the left endpoints form $\mathbb{Z}_{>0}$, which has no upper bound, so α does not exist.
- **Nested.** The intervals $[n, n+1]$ are closed and bounded, but $[1, 2] \cap [3, 4] = \emptyset$ already.
- **The field is $\mathbb{R}$.** In $\mathbb{Q}$, a closed interval means $\{q \in \mathbb{Q} : a \leq q \leq b\}$. Take the bisection intervals $[a_n, b_n]$ from the start of this section. They are defined by rational arithmetic alone, because each midpoint c is rational, and $c^2 \neq 2$ by Logic and Proof, Theorem 2.2.1, so the rule “keep the half whose endpoints’ squares straddle 2” always applies. We showed that the real intervals $[a_n, b_n]$ have intersection $\{\sqrt{2}\}$. So their rational parts have intersection $\mathbb{Q} \cap \{\sqrt{2}\} = \emptyset$.
- **Shrinking.** The intervals $[-1/n, 1 + 1/n]$ are closed, bounded, and nested, and their intersection is $[0, 1]$: every point of $[0, 1]$ lies in all of them, and a point outside $[0, 1]$ is excluded from some interval by [Corollary 1.5.2](#) ([Exercise \(B2: An intersection that is not a point\)](#) writes this out). The intersection is nonempty, but it is not a single point. Their lengths $1 + 2/n$ never drop below 1.

Warning. Dropping a hypothesis removes the guarantee. It does not force the conclusion to fail. The open intervals $(-1/n, 1/n)$ are nested, and their intersection is $\{0\}$, which is not empty. A witness that a hypothesis is needed must be a family where the conclusion actually fails, like $(0, 1/n)$.

Quick check. The table’s witness for “closed” uses the open intervals $(0, 1/n)$. Do the half-open intervals $(0, 1/n]$ witness it too? And the half-open intervals $[0, 1/n)$? Give a reason for each.

Solution. The half-open intervals $(0, 1/n]$ are nested and bounded, but no real number lies in all of them: $x \leq 0$ is not in $(0, 1]$, and for $x > 0$ some n has $1/n < x$ ([Corollary 1.5.2](#)). The supremum of the left endpoints is 0, which is excluded from every interval. So $(0, 1/n]$ is a witness too. The intervals $[0, 1/n)$ are not: 0 lies in all of them, so their intersection $\{0\}$ is not empty. Closing the left end puts the supremum 0 of the left endpoints back in.

1.7.3 The real numbers cannot be listed

Sets, Relations, and Cardinality, Chapter 3 calls a set **countable** if its elements can be listed as $x_0, x_1, x_2, \dots$, repetitions allowed; for a nonempty set this means that it is the image of some function from $\mathbb{N}$. That chapter shows that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable. It also shows that $\mathbb{R}$ is not, by Cantor’s diagonal argument with decimal expansions. Decimal expansions were taken there as a fact from school: every real number has one, and two different expansions such as $0.4999\dots$ and $0.5000\dots$ can name the same number. Neither fact has been proved from our axioms.

Here is a proof with no decimals, using only [Theorem 1.7.1](#).

Theorem 1.7.2. Uncountability of the Reals

Let $x_1, x_2, x_3, \dots$ be any list of real numbers, that is, any function $n \mapsto x_n$ from $\mathbb{Z}_{>0}$ to $\mathbb{R}$. Then there is $c \in \mathbb{R}$ with $c \neq x_n$ for every $n \geq 1$. Consequently $\mathbb{R}$ is not countable.

Idea. Build nested closed intervals $I_1 \supseteq I_2 \supseteq \dots$, where I_n is chosen to avoid x_n . A point c common to all of them, which [Theorem 1.7.1](#) provides, avoids every x_n : it lies in I_n , and x_n does not.

To pass from I_n to I_{n+1} , cut I_n into thirds and look at the two outer thirds. They are disjoint closed intervals, so x_{n+1} misses at least one of them; take that one.

Before reading on: why thirds, and not halves?

Proof. We define closed intervals $I_n = [a_n, b_n]$ with $a_n < b_n$ by recursion on n .

Let $I_1 := [x_1 + 1, x_1 + 2]$. Then $x_1 \notin I_1$.

Suppose $I_n = [a_n, b_n]$ with $a_n < b_n$ has been defined. Let $d := b_n - a_n > 0$, and let

$$L := [a_n, a_n + \frac{d}{3}], \quad R := [b_n - \frac{d}{3}, b_n].$$

Both are closed intervals of length $d/3 > 0$ contained in I_n . They are disjoint, because $a_n + d/3 < b_n - d/3$, which is $2d/3 < d$. Define

$$I_{n+1} := \begin{cases} L & \text{if } x_{n+1} \notin L, \\ R & \text{if } x_{n+1} \in L. \end{cases}$$

In the second case $x_{n+1} \in L$, so $x_{n+1} \notin R$, since L and R are disjoint. In both cases $x_{n+1} \notin I_{n+1}$, $I_{n+1} \subseteq I_n$, and I_{n+1} has positive length.

Each I_{n+1} is determined by I_n and x_{n+1} through an explicit rule, so the family is defined by recursion (*Sets, Relations, and Cardinality*, Chapter 2); no choice principle is involved.

The intervals I_n are closed, bounded, and nested. By [Theorem 1.7.1](#), there is $c \in \bigcap_n I_n$. For each $n \geq 1$ we have $c \in I_n$ and $x_n \notin I_n$, so $c \neq x_n$.

Hence no function f from $\mathbb{N}$ to $\mathbb{R}$ has image all of $\mathbb{R}$: the list $x_n := f(n-1)$, for $n \geq 1$, has the same image as f , and it misses c . Since $\mathbb{R}$ is nonempty, it is not countable. This proves the theorem. ■

The answer to the prompt: halves share their midpoint. If x_{n+1} were exactly that midpoint, it would lie in both closed halves, and neither could be chosen to avoid it. Thirds leave a gap between the two outer pieces.

Only the existence half of [Theorem 1.7.1](#) was used; the intervals need not shrink.

Remark. Since $\mathbb{Q}$ is countable, and a union of two countable sets is countable (both from *Sets, Relations, and Cardinality*, Chapter 3), the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable. Otherwise $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be countable. In this sense most real numbers are irrational, although so far we have named only a few of them, such as $\sqrt{2}$ and $\sqrt{2} + q$ for rational q (the proof of [Corollary 1.6.5](#)).

The decimal proof of *Sets, Relations, and Cardinality*, Chapter 3 is secretly a nested interval proof too: the first n digits of a number place it in an interval of length 10^{-n} . The version above avoids the digits, and with them the numbers that have two expansions.

1.7.4 What the axiom bought us

Here is the chapter so far in one table. Each row is a property that $\mathbb{R}$ has, checked against the two ordered fields we have used as foils: $\mathbb{Q}$ and the field $\mathbb{R}(x)$ of [Example \(A non-Archimedean field\)](#).

Property	$\mathbb{R}$	$\mathbb{Q}$	$\mathbb{R}(x)$
nonempty sets bounded above have suprema	yes (axiom)	no: positive q with $q^2 < 2$	no: $\mathbb{N}$
Archimedean property	yes	yes	no: $x > n$ for all n
$\mathbb{Q}$ is dense	yes	yes	no: nothing rational in $(x, x + 1)$
positive elements have square roots	yes	no: 2	no: x
nested closed intervals meet	yes	no: bisection toward $\sqrt{2}$	no: $[n, x/n]$
uncountable	yes	no (<i>Sets, Relations, and Cardinality</i> , Chapter 3)	yes: contains $\mathbb{R}$

The entries for $\mathbb{R}$ are this chapter's theorems. For $\mathbb{Q}$, the failures come from [Theorem 1.2.2](#), Logic and Proof, Theorem 2.2.1, and the bisection above; $\mathbb{Q}$ is Archimedean by [Exercise \(C3: Archimedean without the axiom\)](#); and $\mathbb{Q}$ is dense in itself because the average of two rationals is a rational between them. For $\mathbb{R}(x)$: every rational r is below some natural number ([Exercise \(C3: Archimedean without the axiom\)](#)), hence below x , so none lies in $(x, x + 1)$. If $(p/q)^2 = x$ for polynomials p, q , then $p^2 = xq^2$, whose sides have even and odd degree, which is impossible. That the nested intervals $[n, x/n]$ have no common point is proved in [Exercise \(C3: Nested intervals in the rational functions\)](#). And $\mathbb{R}(x)$ contains the constants, a copy of $\mathbb{R}$, so it cannot be countable, since a subset of a countable set is countable (*Sets, Relations, and Cardinality*, Chapter 3).

The $\mathbb{Q}$ column shows that the Archimedean property and density are far weaker than the axiom; the $\mathbb{R}(x)$ column shows an ordered field where even they fail.

Two questions are left open. Is the least upper bound axiom *equivalent* to some of the other rows? And could a different “no gaps” property serve as the axiom instead? Chapter 11 answers

both. Over an arbitrary ordered field, the least upper bound axiom is equivalent to each of: the monotone convergence of bounded monotone sequences; the Bolzano–Weierstrass property; Cauchy completeness together with the Archimedean property; and the nested interval property together with the Archimedean property. You can prove one arrow of that last equivalence now (Exercise (C1: Exit check: nested intervals and the axiom)).

1.7.5 Exercises

1.7.5.1 A. Check your understanding

Exercise. A1

For each family $(I_n)_{n \geq 1}$, say which hypothesis of Theorem 1.7.1 fails, and find $\bigcap_n I_n$.

- (a) $I_n = [0, 1/n]$.
- (b) $I_n = [n, n + 1/n]$.
- (c) $I_n = (-1/n, 1/n)$.
- (d) $I_n = [-n, n]$.
- (e) Run the construction in the proof of Theorem 1.7.2 on a list that begins $x_1 = 0$, $x_2 = 1$, $x_3 = 2$: find I_1 , I_2 , and I_3 .
- (f) Where does the proof of Theorem 1.7.2 use that the intervals L and R are **closed**, and what could go wrong with open thirds?

Solution.

- (a) Not closed. The intersection is $\{0\}$: 0 lies in every $[0, 1/n]$; a negative number lies in none; and for $x > 0$, Corollary 1.5.2 gives n with $1/n < x$, so $x \notin [0, 1/n]$. The conclusion holds although a hypothesis fails.
- (b) Not nested: $I_2 = [2, \frac{5}{2}] \not\subseteq [1, 2] = I_1$. The intersection is empty, since already $I_1 \cap I_3 = [1, 2] \cap [3, \frac{10}{3}] = \emptyset$.
- (c) Not closed. The intersection is $\{0\}$: 0 lies in every interval, and for $x \neq 0$, Corollary 1.5.2 gives n with $1/n < |x|$, so $x \notin I_n$.
- (d) Not nested in the required direction: $I_{n+1} \supseteq I_n$, and $I_2 = [-2, 2] \not\subseteq [-1, 1] = I_1$. The intersection is $I_1 = [-1, 1]$, since every I_n contains $[-1, 1]$.
- (e) $I_1 = [x_1 + 1, x_1 + 2] = [1, 2]$. For $n = 1$, $d = 1$, $L = [1, \frac{4}{3}]$, and $R = [\frac{5}{3}, 2]$; since $x_2 = 1 \in L$, $I_2 = R = [\frac{5}{3}, 2]$. For $n = 2$, $d = \frac{1}{3}$, $L = [\frac{5}{3}, \frac{16}{9}]$, and $R = [\frac{17}{9}, 2]$; since $x_3 = 2 \notin L$, $I_3 = L = [\frac{5}{3}, \frac{16}{9}]$. Each I_n misses x_n , and $I_3 \subseteq I_2 \subseteq I_1$.
- (f) Only in the last step, where Theorem 1.7.1 is applied: its hypothesis is that the intervals are closed. With open thirds the family would be nested open intervals, and those can have empty intersection, like $(0, \frac{1}{n})$; then there might be no point c at all. The avoidance of x_{n+1} does not need closedness, only that L and R are disjoint.

1.7.5.2 B. Practice

Exercise. B1: Shrinking by a fixed factor

- (a) Let $I_n = [a_n, b_n]$ be nested closed intervals with $b_n - a_n = 2^{-n}$ for every n . Prove that $\bigcap_n I_n$ consists of exactly one point.

- (b) For $n \geq 1$, let $m_n := \lfloor 10^n \sqrt{2} \rfloor$ (Lemma 1.6.3), and let $a_n := m_n/10^n$ and $b_n := a_n + 10^{-n}$, so that a_n is the n -digit decimal truncation of $\sqrt{2}$. Prove that the intervals $[a_n, b_n]$ are nested, and that their intersection is $\{\sqrt{2}\}$. Compute a_1 and a_2 .

Solution.

- (a) By Theorem 1.7.1 it suffices to check the shrinking hypothesis. By induction, $2^n > n$ for every $n \in \mathbb{N}$ (as in Exercise (C2: Dyadic rationals)), so $2^{-n} < 1/n$ for every $n \geq 1$. Given $\varepsilon > 0$, Corollary 1.5.2 gives n with $1/n < \varepsilon$, and then $b_n - a_n = 2^{-n} < \varepsilon$. Hence the intersection is a single point.
- (b) By definition of the floor, $m_n \leq 10^n \sqrt{2} < m_n + 1$, so $a_n \leq \sqrt{2} < b_n$ for every n .

Nested. Multiplying $m_n \leq 10^n \sqrt{2}$ by 10 gives $10m_n \leq 10^{n+1} \sqrt{2}$. An integer $k \leq z$ always satisfies $k \leq \lfloor z \rfloor$, since $k \geq \lfloor z \rfloor + 1$ would give $k > z$. Applied to $k = 10m_n$ and $z = 10^{n+1} \sqrt{2}$, this gives $10m_n \leq m_{n+1}$, that is, $a_n \leq a_{n+1}$. Similarly $m_{n+1} \leq 10^{n+1} \sqrt{2} < 10m_n + 10$, so $m_{n+1} < 10m_n + 10$, and since both sides are integers, $m_{n+1} + 1 \leq 10m_n + 10$. Dividing by 10^{n+1} gives $b_{n+1} \leq b_n$. Hence $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$.

The intersection. The lengths are $10^{-n} \leq 2^{-n}$, so by the argument of (a) the intersection is one point. Since $\sqrt{2}$ lies in every interval, that point is $\sqrt{2}$.

The first terms. $14^2 = 196 \leq 200 < 225 = 15^2$, so $14 \leq 10\sqrt{2} < 15$, $m_1 = 14$, and $a_1 = 1.4$. Likewise $141^2 = 19881 \leq 20000 < 20164 = 142^2$, so $m_2 = 141$ and $a_2 = 1.41$.

Exercise. B2: An intersection that is not a point

Prove that $\bigcap_{n \geq 1} [-1/n, 1 + 1/n] = [0, 1]$. Which hypothesis of the uniqueness clause of Theorem 1.7.1 fails?

Solution. ($\supseteq$) Let $x \in [0, 1]$. For every n , $-1/n < 0 \leq x \leq 1 < 1 + 1/n$, so $x \in [-1/n, 1 + 1/n]$.

($\subseteq$) Let x lie in every $[-1/n, 1 + 1/n]$. Suppose $x < 0$. By Corollary 1.5.2 with $\varepsilon = -x > 0$, there is n with $1/n < -x$, that is, $x < -1/n$, so $x \notin [-1/n, 1 + 1/n]$: a contradiction. Suppose $x > 1$. With $\varepsilon = x - 1 > 0$, there is n with $1/n < x - 1$, that is, $x > 1 + 1/n$: again a contradiction. Hence $0 \leq x \leq 1$.

The lengths are $1 + 2/n > 1$, so for $\varepsilon = 1$ no interval is shorter than ε : the shrinking hypothesis fails. The intervals are closed, bounded, and nested, and the existence part applies, as it must.

1.7.5.3 C. Going deeper

Exercise. C1: Exit check: nested intervals and the axiom

- (a) Without looking back, prove the existence part of Theorem 1.7.1 from the least upper bound axiom.
- (b) Now work in an ordered field F in which the Archimedean property holds (for every $t \in F$ some $n \in \mathbb{N}$ has $n > t$) and every nested family of closed bounded intervals has a common point. Let $A \subseteq F$ be nonempty and bounded above by M , and pick

$a \in A$. Define intervals $[a_n, b_n]$ by $a_1 := a$, $b_1 := M$, and, with $c := (a_n + b_n)/2$, let $[a_{n+1}, b_{n+1}] := [a_n, c]$ if c is an upper bound of A , and $[c, b_n]$ otherwise. Prove that for every n : b_n is an upper bound of A ; some element of A is $\geq a_n$; and $b_n - a_n = (M - a)/2^{n-1}$.

- (c) Let s be a common point of all the $[a_n, b_n]$. Prove that $s = \sup A$. Deduce that in an ordered field, the nested interval property together with the Archimedean property implies the least upper bound axiom.

Solution (Hint 1). In (c), prove the two properties of a supremum separately, each by contradiction. Each contradiction comes from making $b_n - a_n$ smaller than a positive gap you are given, which is where the Archimedean property enters.

Solution (Hint 2). For “ s is an upper bound” in (c), argue by contradiction: what does an element $y \in A$ with $y > s$ give you, and what do you know about b_n ?

Solution.

- (a) Let $I_n = [a_n, b_n]$ be nested. For $m \leq k$, $I_k \subseteq I_m$, by induction from $I_{j+1} \subseteq I_j$. Given m, n , put $k := \max\{m, n\}$; then $a_m \leq a_k \leq b_k \leq b_n$. So $A := \{a_n : n \geq 1\}$ is nonempty and bounded above by b_1 , and $\alpha := \sup A$ exists by [Axiom 1.3.1](#). For each n , $a_n \leq \alpha$ because α is an upper bound, and $\alpha \leq b_n$ because b_n is an upper bound and α is the least one. So $\alpha \in I_n$ for every n .
- (b) In an ordered field $2 = 1 + 1 > 0$, so c is defined and $a_n \leq c \leq b_n$ when $a_n \leq b_n$. We prove the three statements together by induction on n . For $n = 1$: $b_1 = M$ is an upper bound, $a_1 = a \in A$, and $b_1 - a_1 = M - a$. Suppose they hold for n . If c is an upper bound of A , then $b_{n+1} = c$ is an upper bound, and $a_{n+1} = a_n$, so the element of A that is $\geq a_n$ still works. Otherwise, c is not an upper bound, so some $y \in A$ has $y > c = a_{n+1}$, and $b_{n+1} = b_n$ is an upper bound. In both cases $b_{n+1} - a_{n+1} = (b_n - a_n)/2 = (M - a)/2^n$. The intervals are nested, since each is one half of the previous one.
- (c) s is an upper bound. Suppose some $y \in A$ has $y > s$. Since $2^{n-1} \geq n$, we have $b_n - a_n \leq (M - a)/n$. By the Archimedean property in F , choose $n > (M - a)/(y - s)$; then $b_n - a_n < y - s$. Since $a_n \leq s$, this gives $b_n < a_n + (y - s) \leq y$. But b_n is an upper bound of A and $y \in A$, a contradiction.

No smaller upper bound. Let $u < s$, and assume for contradiction that u bounds A from above. As before, choose n with $b_n - a_n < s - u$. Since $s \leq b_n$, we get $a_n > b_n - (s - u) \geq s - (s - u) = u$. By (b) some $y \in A$ has $y \geq a_n > u$, contradicting that u is an upper bound.

So $s = \sup A$. Every nonempty subset of F that is bounded above has a supremum, which is the least upper bound axiom for F . The Archimedean property was used twice, to make $b_n - a_n$ smaller than a given positive gap. (If $M = a$, all the intervals are $\{a\}$, and the same argument gives $s = a = \sup A$; the Archimedean step is then not even needed.)

Marking scheme. (10 points) - 2: (a) shows $a_m \leq b_n$ for all m, n , takes $\alpha = \sup\{a_n\}$ with both hypotheses of the axiom checked, and shows $a_n \leq \alpha \leq b_n$ by (S1) and (S2). - 3: (b) proves the three statements of (b) by one induction, treating both cases of the bisection rule, including the element of A that is at least a_{n+1} in the second case. - 2: (c) shows s is an upper bound: a $y \in A$ above s is excluded by choosing n with $b_n - a_n < y - s$, using

the Archimedean property. - 2: (c) shows no $u < s$ is an upper bound, choosing n with $b_n - a_n < s - u$ and using the element of A that is at least a_n . - 1: (c) concludes that F has the least upper bound property, and says where each of the two assumed properties was used. Fatal errors (0 for the step): using [Axiom 1.3.1](#), or any supremum, in (b) or (c); choosing n before the gap $y - s$ or $s - u$ is known; assuming A has a maximum. A proof of (c) that shows $\sup A = s$ through the backward direction of [Theorem 1.4.1](#), rerun in F , instead of (S2) earns the same points; the theorem is stated for $\mathbb{R}$, so citing it without rerunning its proof in F is not enough.

Exercise. C2: Uncountable pieces

- (a) Let $x < y$. Prove that the interval (x, y) is not countable.
- (b) Deduce that (x, y) contains an irrational number, without using $\sqrt{2}$ or any other named irrational. You may use the facts from *Sets, Relations, and Cardinality*, Chapter 3 that $\mathbb{Q}$ is countable, and that a subset of a countable set is countable.
- (c) Let $C \subseteq \mathbb{R}$ be countable. Show that the real numbers not in C are dense: every interval (x, y) with $x < y$ contains a real number that is not in C .

Solution.

- (a) Let f be any function from $\mathbb{N}$ to $\mathbb{R}$, and list its values as $z_n := f(n - 1)$ for $n \geq 1$, as in the proof of [Theorem 1.7.2](#); this list has the same image as f . Run the construction in the proof of [Theorem 1.7.2](#), but start from a closed interval inside (x, y) that avoids z_1 . Let $d := y - x$, and let $L := [x + d/4, x + d/3]$ and $R := [y - d/3, y - d/4]$. They are disjoint, closed, of positive length, and inside (x, y) . Let I_1 be L if $z_1 \notin L$, and R otherwise. Then define $I_2, I_3, \dots$ by the outer-thirds rule from the proof. The intervals are nested, closed, and bounded, $z_n \notin I_n$ for every $n \geq 1$, and by [Theorem 1.7.1](#) some c lies in all of them. Then $c \in I_1 \subseteq (x, y)$, and $c \neq z_n$ for every $n \geq 1$. So no function from $\mathbb{N}$ has image all of (x, y) , and (x, y) , which is nonempty, is not countable.
- (b) Suppose every element of (x, y) were rational. Then $(x, y) \subseteq \mathbb{Q}$, and a subset of the countable set $\mathbb{Q}$ is countable, contradicting (a). So (x, y) contains an irrational number. This reproves [Corollary 1.6.5](#) by counting: there are too many reals in (x, y) for all of them to be rational.
- (c) As in (b): if $(x, y) \subseteq C$, then (x, y) is countable, contradicting (a). Part (b) is the case $C = \mathbb{Q}$.

Exercise. C3: Nested intervals in the rational functions

Work in the ordered field $\mathbb{R}(x)$ of [Example \(A non-Archimedean field\)](#). For $n \in \mathbb{Z}_{>0}$, let $I_n := \{f \in \mathbb{R}(x) : n \leq f \leq x/n\}$.

- (a) Show that $n < x/n$, so I_n is a closed bounded interval of positive length, and that $I_{n+1} \subseteq I_n$.
- (b) Let $f = p/q$ with the leading coefficient of q positive, and suppose $f \in I_n$ for every n . Show that $\deg p > \deg q$.
- (c) Show that $\deg p \geq \deg q + 2$ and $\deg p = \deg q + 1$ both lead to a contradiction, and conclude that $\bigcap_n I_n = \emptyset$.

(d) Which line of the proof of [Theorem 1.7.1](#) fails in $\mathbb{R}(x)$?

Solution (Hint 1). Positivity in $\mathbb{R}(x)$ is decided by leading coefficients. Write each difference you need to sign, such as $x/n - f$ or $N - f$, as a single fraction with a positive denominator.

Solution (Hint 2). For (b): suppose $\deg p \leq \deg q$, and look for an $N \in \mathbb{N}$ with $f < N$.

Solution. Write lc for the leading coefficient, and recall from [Exercise \(C2: The order on the rational functions\)](#) that for $g = u/w$ with $\text{lc}(w) > 0$, $g > 0$ if and only if $\text{lc}(u) > 0$.

- (a) $x/n - n = (x - n^2)/n$ has numerator with leading coefficient $1 > 0$ and positive denominator, so $n < x/n$. Nesting: $n+1 > n$, and $x/n - x/(n+1) = x/(n(n+1)) > 0$. So $I_{n+1} \subseteq I_n$.
- (b) Since $f \geq 1 > 0$, $\text{lc}(p) > 0$. Suppose $\deg p \leq \deg q$. By [Theorem 1.5.1](#), pick $N \in \mathbb{N}$ with $N > \text{lc}(p)/\text{lc}(q)$, a positive real number, so that $N \in \mathbb{Z}_{>0}$. Then $N - f = (Nq - p)/q$. If $\deg p < \deg q$, the leading coefficient of $Nq - p$ is $N \text{lc}(q) > 0$; if $\deg p = \deg q$, it is $N \text{lc}(q) - \text{lc}(p) > 0$. Either way $f < N$, so $f \notin I_N$, a contradiction. Hence $\deg p > \deg q$.
- (c) For each n ,

$$\frac{x}{n} - f = \frac{xq - np}{nq}, \quad \text{lc}(nq) = n \text{lc}(q) > 0.$$

If $\deg p \geq \deg q + 2$, then $\deg(np) > \deg(xq)$, so the numerator has leading coefficient $-\text{lc}(p) < 0$. Thus $f > x/n$ for every n , and $f \notin I_1$. If $\deg p = \deg q + 1$, both xq and np have degree $\deg q + 1$, and the numerator has leading coefficient $\text{lc}(q) - n \text{lc}(p)$. Choosing $n > \text{lc}(q)/\text{lc}(p)$, which [Theorem 1.5.1](#) allows, makes it negative, so $f > x/n$ and $f \notin I_n$. Both cases contradict $f \in \bigcap_n I_n$. With (b), no f lies in every I_n , and the intersection is empty.

- (d) The line “ $\alpha := \sup\{a_n\}$ exists by [Axiom 1.3.1](#)”. Here the left endpoints are $a_n = n$, and $\{a_n\} = \mathbb{Z}_{>0}$ is bounded above (by $x/1$, for instance) but has no supremum in $\mathbb{R}(x)$, by the argument for $\mathbb{N}$ in the section on the Archimedean property. The claim $a_m \leq b_n$ and every other line survive.

Exercise. C4: A root of a cubic, from half a proof

Let $g(x) := x^3 + x$. Then $g(0) = 0$ and $g(1) = 2$, so g passes the value 1 somewhere in $[0, 1]$. Here is the first half of a proof that it takes the value 1.

Let $S := \{x \in \mathbb{R} : x \geq 0, g(x) < 1\}$. Then $0 \in S$. If $x \geq 1$, then $x^3 \geq 1$ and $g(x) \geq 2 > 1$, so every element of S is less than 1. By [Axiom 1.3.1](#), $s := \sup S$ exists, and $0 \leq s \leq 1$.

Name the move, then complete it. Prove that $g(s) = 1$. Then show that $0.68 < s < 0.69$, and that s is the only real number t with $g(t) = 1$.

Solution (Hint 1). Rule out $g(s) < 1$ and $g(s) > 1$ by nudging s to $s + h$ and $s - h$. For that you need a bound on $g(s + h) - g(s)$, and on $g(s) - g(s - h)$, of the form “a constant times h ”, valid for small h .

Solution (Hint 2). Expanding, $g(s + h) - g(s) = h(3s^2 + 3sh + h^2 + 1)$. With $0 \leq s \leq 1$ and $0 < h \leq 1$ this is at most $8h$.

Solution. The move is **construct a point as a supremum**. The given half defines the set of points where things are still fine ($g(x) < 1$) and takes its supremum; what remains is to rule out each way s could fail.

We use that g is strictly increasing on $[0, \infty)$: if $0 \leq u < v$, then $u^3 < v^3$ by the claim in the proof of [Theorem 1.6.2](#), and adding $u < v$ gives $g(u) < g(v)$ by [Theorem 1.1.2](#) (b). Expanding with [Lemma 1.1.1](#), for every h ,

$$\begin{aligned} g(s + h) - g(s) &= h(3s^2 + 3sh + h^2 + 1), \\ g(s) - g(s - h) &= h(3s^2 - 3sh + h^2 + 1). \end{aligned}$$

Case $g(s) < 1$. Let $h := (1 - g(s))/16$, which depends only on s . Since $0 \leq g(s) < 1$, we have $0 < h \leq \frac{1}{16}$. With $s \leq 1$ and $h \leq 1$, the first identity gives

$$g(s + h) - g(s) \leq h(3 + 3 + 1 + 1) = 8h = \frac{1}{2}(1 - g(s)) < 1 - g(s),$$

so $g(s + h) < 1$. As $s + h \geq 0$, we get $s + h \in S$, and $s + h > s$ contradicts (S1).

Case $g(s) > 1$. Then $s > 0$, since $g(0) = 0$. Let $h := (g(s) - 1)/10 > 0$. Since $s \leq 1$ gives $s^3 \leq s$, we have $g(s) - 1 < g(s) \leq 2s$, so $h < \frac{s}{5} < s$ and $s - h > 0$. As $0 < h < s \leq 1$, we have $-3sh < 0$, and the second identity gives

$$g(s) - g(s - h) \leq h(3 + 1 + 1) = 5h = \frac{1}{2}(g(s) - 1) < g(s) - 1,$$

so $g(s - h) > 1$. Now $s - h$ is an upper bound of S : if $x \in S$ had $x > s - h$, then, as $s - h > 0$, monotonicity would give $g(x) > g(s - h) > 1$, contradicting $x \in S$. So (S2) gives $s \leq s - h$, which is false.

By trichotomy, $g(s) = 1$.

Location. Exact arithmetic gives $g(0.68) = 0.314432 + 0.68 = 0.994432 < 1$ and $g(0.69) = 0.328509 + 0.69 = 1.018509 > 1$. If $s \leq 0.68$, monotonicity would give $g(s) \leq g(0.68) < 1$, and if $s \geq 0.69$, it would give $g(s) > 1$. So $0.68 < s < 0.69$.

Uniqueness. Let $t \in \mathbb{R}$ with $g(t) = 1$. If $t < 0$, then $t^2 > 0$ and $t^3 = t \cdot t^2 < 0$ by [Theorem 1.1.2](#) (c), so $g(t) < 0$; hence $t \geq 0$. On $[0, \infty)$, g is strictly increasing, so $t < s$ or $t > s$ would give $g(t) \neq g(s)$. Hence $t = s$.

This is the intermediate value theorem for one polynomial, proved by the move that proves it in general in Chapter 6.

Marking scheme. (10 points) - 1: names the move, and records that g is strictly increasing on $[0, \infty)$ with a reason. - 3: rules out $g(s) < 1$ with an explicit $h > 0$ depending only on s , a

verified bound on $g(s+h) - g(s)$, and a contradiction with (S1). - 3: rules out $g(s) > 1$ with an explicit h , checking $s-h > 0$, that $g(s-h) > 1$, and that $s-h$ is an upper bound, then contradicting (S2). - 1: concludes $g(s) = 1$ by trichotomy. - 2: the location $0.68 < s < 0.69$ from two exact computations and monotonicity, and uniqueness, including the case $t < 0$. Fatal errors (0 for the step): an h that depends on an element of S or is not shown positive; using that $s-h$ is an upper bound without monotonicity; assuming the intermediate value theorem. Other constants (for instance $h = \min\{1, (1-g(s))/10\}$) earn full marks when the bound they need is verified.

1.7.5.4 R. Review

Exercise. R1

Let $A \subseteq \mathbb{R}$ be nonempty, and let s be an upper bound of A . Prove that the following are equivalent.

- (i) $s = \sup A$.
- (ii) For every $n \in \mathbb{Z}_{>0}$ there is $a \in A$ with $a > s - \frac{1}{n}$.
- (iii) No real number smaller than s is an upper bound of A .

Solution. By Logic and Proof, Proposition 2.4.6 it suffices to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). This is [Corollary 1.4.2](#), or [Theorem 1.4.1](#) ($\Rightarrow$) with $\varepsilon = \frac{1}{n}$.

(ii) $\Rightarrow$ (iii). Let $u < s$. By [Corollary 1.5.2](#) there is n , depending on $s-u$, with $\frac{1}{n} < s-u$. By (ii) some $a \in A$ has $a > s - \frac{1}{n} > u$, so u is not an upper bound.

(iii) $\Rightarrow$ (i). (S1) holds because s is an upper bound. For (S2), let M be an upper bound. By (iii), $M < s$ is impossible, so $s \leq M$ by trichotomy.

The cycle is the method of *Logic and Proof*, Chapter 2 for equivalences (Logic and Proof, Proposition 2.4.6). Here (ii) $\Rightarrow$ (iii) is the only arrow with content: it is where the Archimedean property turns the special widths $\frac{1}{n}$ into every width.

Exercise. R2

- (a) Prove that if x is a real number and x^2 is irrational, then x is irrational.
- (b) Hence prove that the positive real number t with $t^4 = 2$ is irrational.

Solution.

- (a) By Logic and Proof, Proposition 2.3.2 it suffices to prove: if x is rational, then x^2 is rational. That holds because a product of rationals is rational ([Example \(The rationals and the reals\)](#)).
- (b) The number t exists by [Theorem 1.6.2](#). Then $t^2 > 0$ and $(t^2)^2 = t^4 = 2$, so t^2 is the positive square root of 2, by the uniqueness in [Theorem 1.6.1](#). It is irrational by Logic and Proof, Theorem 2.2.1. By (a), t is irrational.

The direct statement (a) would be awkward to prove as it stands, since “irrational” is a nega-

tive property; its contrapositive is one line. That is the reason to reach for the contrapositive (Logic and Proof, Proposition 2.3.2, *Logic and Proof*, Chapter 2).

Exercise. R3

Prove that the real number $\sqrt{2} + \sqrt{3}$ is irrational.

Solution. Here $\sqrt{3}$ and $\sqrt{6}$ are the positive square roots given by [Theorem 1.6.2](#). First, $\sqrt{2}\sqrt{3} = \sqrt{6}$: the left side is positive and its square is $2 \cdot 3 = 6$, so it is the unique positive square root of 6.

Suppose, for a contradiction, that $r := \sqrt{2} + \sqrt{3}$ is rational [assumption for $\neg I$]. Then

$$r^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 5 + 2\sqrt{6},$$

so $\sqrt{6} = (r^2 - 5)/2$ is rational, because the rationals are closed under the field operations ([Example \(The rationals and the reals\)](#)). But no rational number has square 6 (Logic and Proof, Exercise (C1: The square root of six)). Hence r is irrational [$\neg I$].

Logic and Proof, Chapter 2 could prove only the negative half: no rational squares to 6 (Logic and Proof, Exercise (C1: The square root of six)). With [Theorem 1.6.2](#) the number $\sqrt{6}$ exists in $\mathbb{R}$, and the negative statement becomes the positive one, “ $\sqrt{6}$ is irrational”, which is what the argument uses.

Exercise. R4

Prove that $|x - 1| + |x + 1| \geq 2$ for every real x , and find all x for which equality holds.

Solution. The tool is the triangle inequality ([Theorem 1.1.4](#)), applied to a sum chosen so that x cancels:

$$2 = |(1 - x) + (x + 1)| \leq |1 - x| + |x + 1| = |x - 1| + |x + 1|,$$

using $|1 - x| = |x - 1|$ ([Proposition 1.1.3](#) (b)). By [Exercise \(B2: Inserting a third point\)](#) (b), equality holds exactly when $(1 - x)(x + 1) \geq 0$, that is, when $1 - x^2 \geq 0$, that is, when $-1 \leq x \leq 1$ (by [Theorem 1.1.2](#) (g) and [Proposition 1.1.3](#) (d)), $x^2 \leq 1$ means $|x| \leq 1$. So equality holds exactly on $[-1, 1]$.

The look-alike tools work less well. Proof by cases, as in Logic and Proof, Example (The triangle inequality by cases), succeeds but needs three cases ($x < -1$, $-1 \leq x \leq 1$, $x > 1$) and a formula in each. Squaring both sides leaves the cross term $2|x^2 - 1|$, which still has an absolute value in it, so it only postpones the case split.