

Number Systems: From $\mathbb{N}$ to $\mathbb{R}$ ended with the real numbers as a complete ordered field, and with the integers modulo n as a commutative ring. This volume takes those two results and builds the algebra that linear algebra and the algebra volumes assume on their first page. It constructs the complex numbers, studies fields as objects of their own, introduces groups through permutations, makes polynomials into formal objects, and proves that polynomials over a field have the arithmetic of the integers. It ends with the Fundamental Theorem of Algebra: every non-constant complex polynomial has a root.

Who it is for. The reader we have in mind has worked through *Logic and Proof*, *Sets, Relations, and Cardinality* and the main line of *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , or has their habits from elsewhere. That means running an induction, checking that an operation on classes is well defined, and proving a statement about every element of a ring from the ring axioms alone. The chapters point back to those volumes wherever they use a definition or a result, with a link, so a reader who is unsure of a step can look it up. Much of the material is familiar in outline from school or from a first linear algebra course; what is new is that every rule is proved, and that nothing is used before it is proved.

What it covers.

Chapter	Title	What it does
1	Complex Numbers	$\mathbb{C}$ as pairs of real numbers, proved a field containing $\mathbb{R}$; conjugate, modulus and the triangle inequality; square roots and the quadratic formula; why $\mathbb{C}$ has no order; polar form, De Moivre's formula and the roots of unity
2	Fields	The field axioms and their consequences; the finite fields $\mathbb{F}_p$; subfields and the subfield test; field homomorphisms; the characteristic and the prime subfield; the binomial theorem, the Frobenius map and Fermat's little theorem
3	Groups, Subgroups, and Homomorphisms	Groups, products of many factors and integer powers; subgroups and the subgroups of $\mathbb{Z}$; homomorphisms, kernels and isomorphisms; permutations, cycle notation and the disjoint cycle decomposition; transpositions generate the symmetric group

Chapter	Title	What it does
4	Formal Polynomials	A polynomial as a sequence of coefficients; the ring $R[x]$ over a commutative ring; degree and integral domains; evaluation and substitution as ring homomorphisms, and why a polynomial is not its function; the formal derivative
5	Polynomials over a Field	Division with remainder, the greatest common divisor and the Euclidean algorithm in $F[x]$; irreducible polynomials and unique factorization; roots, multiplicity and repeated roots; interpolation, the rings $F[x]/\langle m \rangle$ and the Chinese remainder theorem; the Fundamental Theorem of Algebra and the factorization of real polynomials

Chapter 1 gives the volume its most important example, and Chapter 2 names what that example has in common with $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}/p\mathbb{Z}$. Chapter 3 uses Chapters 1 and 2 only in examples, remarks and one review exercise. Chapters 4 and 5 form a pair. Chapter 4 builds polynomials over any commutative ring and asks only what can be done without dividing; Chapter 5 takes the coefficients in a field and runs again, with the degree in place of size, the arithmetic that *Number Systems*, Chapter 3 ran in $\mathbb{Z}$.

How to read it. Read the chapters in order; each uses the ones before it. Every section has Quick checks with folded answers; answer each one before you unfold it. The exercises of a section come in three groups, A to check understanding, B for practice and C to go deeper, and the last section of each chapter adds R items that reach back to earlier chapters. A chapter's opening page lists what it needs, with links, and the time it takes. Second-reading folds hold edge cases and longer justifications. They can stay closed on a first reading, but they are not optional, since later proofs may use what they contain.

One borrowed fact. This volume does not define cosine, sine or π : defining them needs limits and power series, which belong to analysis. Section 1.5 states five facts about them, numbered (T1)–(T5), as an assumption, and *Analysis on the Real Line*, Chapter 10 proves them. They are used only for polar form and for the roots of complex numbers in Sections 1.5 and 1.6, and each proof that uses one names it. Nothing else in the volume depends on *Analysis on the Real Line*. In particular, Chapter 5 proves the Fundamental Theorem of Algebra from the square roots of Section 1.4 and the completeness of $\mathbb{R}$ alone, without the borrowed facts.

Time. The chapter openings add up to about 130 hours on the core track and about 190 hours for everything.

What comes next. The linear algebra volumes, beginning with *Spaces and Maps*, take their scalars from a field and use the complex numbers, polynomials and their roots throughout:

Chapter 2's fields and subfields, Chapter 4's evaluation of a polynomial, Chapter 5's division with remainder and the Fundamental Theorem of Algebra, and Chapter 3's permutations, which *Determinants and the Structure of Operators* needs for the determinant. The algebra volumes start where this one stops. *Groups* continues Chapter 3 with cyclic groups, the sign of a permutation, cosets and Lagrange's theorem; *Rings and Fields* continues Chapters 2, 4 and 5 with fraction fields, polynomials over the integers, Gauss's lemma and field extensions. Many of their first definitions are the ones proved here, under the same names.