

Complex Numbers

No real number squares to -1 : in an ordered field a non-zero square is positive (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 2.3.5 (e)), so $x^2 + 1 = 0$ has no solution in $\mathbb{R}$. The usual remedy, “invent a symbol i with $i^2 = -1$ ”, leaves open why such a symbol is safe when “a number z with $0 \cdot z = 1$ ” is not. This chapter does not invent anything. It takes pairs of real numbers, objects the reader already trusts, adds them coordinate by coordinate, and multiplies them by the one formula that $i^2 = -1$ would force. Then it checks, law by law, that nothing breaks: the complex numbers $\mathbb{C}$ form a field in the sense of *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , Chapter 2, and they contain $\mathbb{R}$ as the pairs $(a, 0)$. Every later volume that computes with complex numbers, from inner products in linear algebra to roots of unity in algebra, starts from this field.

Two habits of $\mathbb{R}$ do not survive the passage. No order on $\mathbb{C}$ is compatible with its arithmetic, and no choice of “the” square root of each complex number keeps the rule $\sqrt{x}\sqrt{y} = \sqrt{xy}$. One new habit takes the place of order: measure a complex number by its modulus, a non-negative real, and compute with $|u|^2 = u\bar{u}$, which is algebra. The first four sections need nothing but the real numbers, and they reach far: division, the triangle inequality, and square roots and the quadratic formula without a single angle. The picture of multiplication as a turn and a stretch does need angles, and angles need cosine and sine, which belong to analysis. The chapter borrows exactly five facts about them, says so in one place, and derives everything else from those five: the familiar special values, De Moivre’s formula, the n different n -th roots of every non-zero complex number, and the roots of unity. It ends with a look ahead at the Fundamental Theorem of Algebra, which Chapter 5 proves without the borrowed facts.

What you need. *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* above all, and in it chiefly Chapters 2, 4 and 6. From its Chapter 5: the real numbers as a complete ordered field (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Theorem 5.6.3, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 5.6.1), the standing setting of the whole chapter. From its Chapter 6: square roots of non-negative reals, existing and unique (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 6.1.4), which define the modulus in Section 1.2 and build complex square roots in Section 1.4; the uniqueness of the complete ordered field (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Theorem 6.3.1), which lets Section 1.5 carry the borrowed facts over to our $\mathbb{R}$; the automorphism theorem (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Theorem 6.3.4), used in the solution of review exercise R1 and compared there with the two such maps of $\mathbb{C}$; and the cube roots of its Exercise B2 of Section 6.2 (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Exercise (B2: Cube roots)), whose method Section 1.6 runs for every exponent. From its Chapter 2: the axioms of a field and of an ordered field and their rules (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 2.3.1, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 2.3.2, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 2.3.2, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 2.3.5), used on nearly every page; the finite ring that cannot be ordered (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Example (A field with two elements that cannot be ordered)), compared with in Section 1.4; and the gap at $\sqrt{2}$ in $\mathbb{Q}$ (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Theorem 2.6.1), which shows in Section 1.2 why the modulus needs completeness. From its Chapter 4: the rules for fractions in a field (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Lemma 4.1.1), the absolute value of an ordered field (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 4.1.1, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 4.1.3), which the modulus extends and whose triangle inequality Section 1.3 reproves by a new route, and the natural numbers and powers in an ordered field (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 4.2.1); a remark of Section 1.4 recalls its exercise on fields with a square root of -1 (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Exercise (C1: A field that cannot be ordered)). From its Chapter 3: division with remainder, divisibility and parity in $\mathbb{Z}$

(Number Systems: From $\mathbb{N}$ to $\mathbb{R}$), Theorem 3.1.2, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 3.1.1, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Corollary 3.1.3), in Exercise B1 of Section 1.1 and in Sections 1.5 and 1.6; and the rings $\mathbb{Z}/n\mathbb{Z}$ (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 3.4.2, Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Theorem 3.4.3), in a remark and Exercise C2 of Section 1.4 and in review exercise R2. From its Chapter 1: the citing rule by which induction and recursion apply to $\mathbb{N}$ (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Corollary 1.7.3), and the laws of powers in $\mathbb{N}$ (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Proposition 1.6.4), compared with in Section 1.1. From *Sets, Relations, and Cardinality*: ordered pairs, Cartesian products and injective maps, Chapter 1 (Sets, Relations, and Cardinality, Definition 1.3.1, Sets, Relations, and Cardinality, Definition 1.3.2, Sets, Relations, and Cardinality, Definition 1.4.3); divisibility, Section 2.2 (Sets, Relations, and Cardinality, Definition 2.2.1); the recursion theorem, powers and finite sums, Section 2.3 (Sets, Relations, and Cardinality, Theorem 2.3.1, Sets, Relations, and Cardinality, Definition 2.3.2, Sets, Relations, and Cardinality, Proposition 2.3.4, Sets, Relations, and Cardinality, Definition 2.3.1), whose recursions Section 1.1 reuses in any commutative ring; and finite sets, Section 3.1 (Sets, Relations, and Cardinality, Definition 3.1.2), for counting n -th roots. From *Logic and Proof*: induction, Chapter 4 (Logic and Proof, Axiom 4.2.1); the contrapositive, Chapter 2 (Logic and Proof, Proposition 2.3.2), in Exercise B3 of Section 1.1; and the catalogue of what a proof may not do, Chapter 8 (Logic and Proof, Example (What is not allowed)), in the proof-ordering and find-the-error exercises of Sections 1.4 and 1.6. Section 1.4 also proves the complex square roots that *Logic and Proof*, Section 7.1 took as a school fact. Nothing from linear algebra is used, and nothing from analysis before the box of borrowed facts in Section 1.5.

Which real numbers. The chapter builds $\mathbb{C}$ on the real numbers of *Number Systems: From $\mathbb{N}$ to $\mathbb{R}$* , Chapter 5, whose elements are classes of Cauchy sequences of rationals (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Definition 5.2.1). No proof looks inside a real number: every step uses only that $\mathbb{R}$ is a complete ordered field. By the uniqueness theorem of that volume's Chapter 6 (Number Systems: From $\mathbb{N}$ to $\mathbb{R}$, Theorem 6.3.1), any complete ordered field would give the same $\mathbb{C}$, up to the one isomorphism between the two fields.

What this chapter borrows. Everything in Sections 1.1–1.4, and in Section 1.5 up to its box of borrowed facts, is proved from the real numbers. From that box on, five facts (T1)–(T5) about $\cos$, $\sin$ and π are used without proof: the circle identity, the addition formulas, where the cosine equals 1, that every point of the unit circle is reached, and the sign of the sine on the first half turn. *Analysis on the Real Line*, Chapter 10 defines the two functions and the number and proves the five facts. No other fact about cosine and sine is used, and the results that rest on the five say so in their dependency ledgers.

Time. About 25 hours on the core track (all of A, one B and one C per section, the exit checks); about 35 hours for everything.

1.-1.1 Roadmap

- **The Complex Numbers as Pairs of Reals.** $\mathbb{C}$ defined as $\mathbb{R} \times \mathbb{R}$ with the product that $i^2 = -1$ would force; the real numbers inside $\mathbb{C}$ as the pairs $(a, 0)$, which makes $a + bi$ honest notation and $i^2 = -1$ a computation; a product with one sign changed, which has zero divisors; the laws of arithmetic, which make $\mathbb{C}$ a commutative ring with $1 \neq 0$; powers and their laws in any commutative ring; and the factorization of $z^n - w^n$.
- **Conjugate, Modulus, and Division.** The conjugate and the modulus, a non-negative real square root and the place where the completeness of $\mathbb{R}$ enters; the modulus as an extension of the absolute value of $\mathbb{R}$; the rules for conjugate and modulus, with $|zw| = |z||w|$ proved by squaring; the inverse $\bar{z}/|z|^2$ of a non-zero z ; the theorem that $\mathbb{C}$ is a field in which

$\mathbb{R}$ is closed under the field operations; and division by the conjugate of the denominator.

- **The Triangle Inequality.** Why the trapping proof for ordered fields cannot run in $\mathbb{C}$; comparing squares of non-negative reals; real and imaginary parts at most the modulus; the triangle inequality, by expanding $|z+w|^2$ and recognizing the cross term as a real part; and three variants: the reverse inequality, the inequality for n terms, and a lower bound for the modulus of a sum.
- **Square Roots, Quadratic Equations, and the Missing Order.** An explicit formula for a square root of a complex number, built from two real square roots, and the count of exactly two square roots of every non-zero number; the quadratic formula with complex coefficients; the proof that no order makes $\mathbb{C}$ an ordered field; and why no choice of square roots keeps the product rule $\sqrt{z}\sqrt{w} = \sqrt{zw}$.
- **The Unit Circle and Polar Form.** The unit circle $\mathbb{T}$, closed under products and inverses, and every non-zero number as a length times a direction in exactly one way, all without angles; the five borrowed facts about cosine and sine; $e^{i\theta}$ as an abbreviation for $\cos \theta + i \sin \theta$, multiplication as “multiply moduli, add angles”, and polar form, unique up to a multiple of 2π in the argument; De Moivre’s formula; and the values of cosine and sine at $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and π , derived from the five facts.
- **Higher Roots, Roots of Unity, and a Look Ahead.** The unique non-negative n -th root of a non-negative real, constructed as a supremum; exactly n solutions of $z^n = c$ for every non-zero c , listed by polar form; the roots of unity, the set μ_n , and the sums of their powers; a look ahead at the Fundamental Theorem of Algebra; the conjugate of a solution of an equation with real coefficients as a second solution; and review exercises on the maps of $\mathbb{C}$ to itself that respect sums, products and 1 and send $\mathbb{R}$ into $\mathbb{R}$, on $\mathbb{Z}/n\mathbb{Z}$ and μ_n , on the triangle inequality of an ordered field, and on the equation $\bar{z} = z^2$.

1.-1.2 Named moves

- **Compare real and imaginary parts.** One equation between complex numbers is two equations between real numbers: for real a, b, c, d , $a + bi = c + di$ if and only if $a = c$ and $b = d$. It is named in Section 1.1, where it proves the laws of arithmetic; Section 1.4 uses it to turn $w^2 = z$ into two real equations, and Section 1.5 to turn $e^{i\theta} = 1$ into $\cos \theta = 1$ and $\sin \theta = 0$. The warning of Section 1.1 marks the hypothesis it needs: the four numbers must be real. Exercise C1 of Section 1.1 proves the same uniqueness of parts in any field that contains $\mathbb{R}$ and a square root of -1 .
- **Square the modulus.** The modulus is a square root, and square roots do not expand; its square $|u|^2 = u\bar{u}$ is a product, and products do. So square, expand with conjugates, bound or compare the real numbers that appear, and take the non-negative square root back, which is legitimate because non-negative reals compare as their squares do. Section 1.2 uses it first, for $|zw| = |z||w|$; Section 1.3 names it and runs it in full for the triangle inequality. Exercise B2 of Section 1.3 leaves the last step to you, and Exercise C2 of Section 1.3 the last half.
- **Multiply moduli, add angles.** In polar form a product is the product of the moduli and the sum of the arguments, so an equation $z^n = c$ splits into a real equation $r^n = \rho$ for the length and an angle condition $n\theta \in \alpha + 2\pi\mathbb{Z}$. It is named in Section 1.5, after the multiplication theorem, and gives De Moivre’s formula there; Section 1.6 uses it to solve $z^n = c$ and to count the solutions. The warning of Section 1.6 shows what goes wrong when the angle condition is read as an equation $n\theta = \alpha$: all roots but one are lost.